Estimation of the Average Heat Transfer Coefficient for a Long Horizontal Cylindrical Fin Rod

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Abstract
The physical situation considered in this study is a horizontal, long rod of cylindrical cross-section with one end maintained at a constant elevated temperature. This paper examines the assumption that the heat transfer coefficient along the rod is constant and proposes a simple approach to estimate an average total heat transfer coefficient. The approach makes use of a published correlation for natural convection from a horizontal, isothermal cylinder and a simple model for the radiative heat transfer. A comparison is made to experimental results.

Introduction
The efficient transfer of heat in many devices, such as electronic components and heat exchangers, is an engineering challenge and a topic of extensive study. In many such devices, fins are used to enhance the heat transfer. All fins operate by increasing the surface area from which the heat transfer can take place; however, a wide variety of configurations and operating conditions are possible.

Fins have been the topic of many studies (see e.g. references in [1] and [2]) and a typical analysis is found in many textbooks, e.g. [3] and [4]. Most studies begin with the standard assumptions of Murray and Gardner as summarized in Ref. [1]:

1. The heat flow in the fin and its temperature remain constant in time, i.e. steady-state.
2. The fin material is homogeneous, its thermal conductivity is the same in all directions, and it remains constant.
3. The heat transfer coefficient to the fin is constant and uniform over the entire surface.
4. The temperature of the medium surrounding the fin is uniform and constant.
5. Temperature gradients across the fin cross-section may be neglected, i.e. one-dimensional.
6. The temperature at the base of the fin is uniform.
7. There is no contact resistance where the base of the fin joins the prime surface.
8. There are no heat sources within the fin itself.
9. The heat transferred through the outermost edge of the fin is negligible compared to that leaving the fin through its lateral surface.

10. Heat transfer to or from the fin is proportional to the temperature excess between the fin and the surrounding fluid.

These ten assumptions greatly simplify the analysis.

The physical situation considered in this study is a horizontal cylindrical rod or pin fin. One end of the fin is maintained at a constant elevated temperature, and the fin is sufficiently long so that the other end is at the ambient temperature. Heat is transferred by conduction along the fin and, in accordance with common practice, the temperature at each axial location is assumed to be uniform. Thus, the heat transfer is one dimensional. Heat is removed from the surface of the fin via natural convection and radiation. The natural convection and radiation effects are accounted for in the total heat transfer coefficient. In many studies, in order to obtain an analytical expression for the temperature distribution, the heat transfer coefficient along the fin is assumed to be constant. This assumption is technically not correct as the temperature along the fin varies.

Recently, a laboratory exercise in which students were to design, build, and test an “infinitely long” fin was proposed by Abu-Mulaweh [5]. One common question posed by students is “What value should we use for the heat transfer coefficient?” Apparently, most undergraduate textbooks do not address this issue directly. Thus, the motivation for this work is an attempt to answer that question.

This paper examines the assumption that the heat transfer coefficient is constant and proposes a simple approach to estimate the average total heat transfer coefficient for a long cylindrical fin. The approach makes use of a published correlation for natural convection from a horizontal, isothermal cylinder and a simple model for the radiative heat transfer. A comparison is made to experimental results.

**Governing Equations**

Consider a cylindrical pin fin, as shown in Figure 1, that extends into a fluid of temperature $T_\infty$. The base is maintained at constant temperature $T_0$. With the assumption of one-dimensional heat conduction along the fin and steady-state operation, an energy balance applied to a differential element yields

\[
\hat{Q}_x - \hat{Q}_{x+dx} - d\hat{Q}_{\text{loss}} = 0, \tag{1}
\]

where $\hat{Q}_{\text{loss}}$ accounts for heat transfer due to convection and radiation, i.e.

\[
d\hat{Q}_{\text{loss}} = d\hat{Q}_{\text{conv}} + d\hat{Q}_{\text{rad}}. \tag{2}
\]

From Fourier’s Law, the rate of heat conduction is related to the temperature gradient via

\[
\hat{Q}_x = -kA_c\frac{dT}{dx}, \tag{3}
\]

where the $k$ is the thermal conductivity and $A_c$ is the cross-sectional area. The rate of heat conduction at $x + dx$ can be expressed in terms of the rate at $x$ with the expansion

\[
\hat{Q}_{x+dx} = \hat{Q}_x + \frac{d\hat{Q}_x}{dx} dx + \cdots \tag{4}
\]
The rate of heat transfer through the sides of the differential element can be expressed as

\[ d\dot{Q}_{\text{loss}} = h \, dA_s \, (T - T_\infty) \]  

(5)

where \( dA_s \) is the surface area of the differential element and \( h \) is the total heat transfer coefficient that accounts for both convection and radiation. Substitution of Eqs. (3)–(5) into Eq. (1), division by \( dA_s \), and simplification yields

\[ \frac{d}{dx} \left( A_c \frac{dT}{dx} \right) - \frac{h}{k} \frac{dA_s}{dx} (T - T_\infty) = 0. \]  

(6)

Note that \( A_c, h, \) and \( k \) can, in theory, vary along the length of the fin. In this work, a cylindrical pin fin with constant cross-sectional area is considered so that \( A_c = \pi D^2/4 \) and \( dA_s = P \, dx \), where \( P = \pi D \) is the perimeter. The thermal conductivity \( k \) is assumed to be independent of the temperature and thus constant. The heat transfer coefficient \( h \) is allowed to vary along the fin.

To simplify the analysis and obtain an analytical solution, a suitable average value for the heat transfer coefficient \( \bar{h} \) is often substituted into Eq. (6) yielding

\[ \frac{d^2T}{dx^2} - \frac{4\bar{h}}{kD} (T - T_\infty) = 0. \]  

(7)

Next with the introduction of the excess temperature \( \theta = T - T_\infty \), Eq. (7) becomes

\[ \frac{d^2\theta}{dx^2} - m^2 \theta = 0, \]  

(8)
where

\[ m \equiv \sqrt{\frac{4h}{kD}}. \]  

(9)

For an infinitely long fin, the temperature distribution can be written as

\[ \theta(x) = \theta_0 e^{-mx}, \]  

(10)

where \( \theta_0 = T_0 - T_\infty \) or

\[ T(x) = T_\infty + (T_0 - T_\infty) e^{-mx}. \]  

(11)

The rate of heat transfer along the fin by conduction is given by

\[ \dot{Q}_x = -kA_c \frac{d\theta}{dx} = kA_c m \theta_0 e^{-mx}, \]  

(12)

with the rate of heat transfer at the base given by \( \dot{Q}_0 = kA_c m \theta_0 \). Note that as \( L \to \infty, \dot{Q}_x \to 0 \).

### Heat Transfer Coefficient

The rate of heat transfer from the surface of the differential element due to convection can be expressed as

\[ d\dot{Q}_{\text{conv}} = h_c dA_s (T_s - T_\infty) \]  

(13)

where \( h_c \) is the average value of the convection heat transfer coefficient about the circumference of the fin and \( T_s \) is the surface temperature. The convection heat transfer coefficient is related to the Nusselt number via

\[ \text{Nu}_D = \frac{h_c D}{k_f}, \]  

(14)

where \( k_f \) is the thermal conductivity of the ambient fluid. For a horizontal isothermal cylinder, the Nusselt number correlation recommended by Churchill and Chu [6] for pure natural convection is

\[ \text{Nu}_D = 0.36 + \frac{0.518 \text{Ra}_D^{1/4}}{[1 + (0.559/\text{Pr})^{9/16}]^{4/9}} \quad 10^{-6} < \text{Ra}_D < 10^9, \]  

(15)

where the Rayleigh number is given by

\[ \text{Ra}_D = \frac{g \beta (T_s - T_\infty) D^3}{\nu \alpha}. \]  

(16)

\( g \) is the acceleration due to gravity, \( \beta, \nu, \) and \( \alpha \) are thermodynamic properties of the fluid evaluated at the film temperature, i.e. \( T_f = (T_s + T_\infty)/2 \). \( \beta \) is the thermal expansion coefficient, \( \nu \) is the kinematic viscosity, and \( \alpha \) is the thermal diffusivity.

The rate of heat transfer from the surface of the differential element due to radiation between the fin and surroundings, which are assumed to be at \( T_\infty \), can be expressed as

\[ d\dot{Q}_{\text{rad}} = \epsilon \sigma dA_s (T_s^4 - T_{\text{surr}}^4), \]  

(17)

where \( \epsilon \) is the emissivity, \( \sigma = 5.67 \times 10^{-8} \) W/m²K⁴ is the Stefan-Boltzmann constant, and \( T_s \) and \( T_{\text{surr}} \) are the surface and surroundings temperatures in K. To simplify the analysis, the surroundings
are assumed to be the same temperature as the ambient fluid, i.e. \( T_{\text{surf}} = T_\infty \). Then, with the introduction of the radiation heat transfer coefficient [3], i.e.

\[
h_r = \epsilon \sigma (T_s^2 + T_\infty^2)(T_s + T_\infty),
\]

Eq. (17) can expressed in a form similar to Eq. (13), viz.

\[
d\hat{Q}_{\text{rad}} = h_r dA_s (T_s - T_\infty).
\]

Thus, with the use of Eqs. (13) and (19), the rate of heat loss through the surface of the differential element [see Eq. (5)] can be written as

\[
d\hat{Q}_{\text{loss}} = (h_c + h_r) dA_s (T_s - T_\infty) = h dA_s (T_s - T_\infty),
\]

where the total heat transfer coefficient is given by \( h \equiv h_c + h_r \). In this form, the rate of heat transfer from the fin is expressed in terms of the excess temperature; however, according to Eqs. (15) and (18), the heat transfer coefficient also depends on the surface and ambient temperatures.

To better understand the relative magnitude between natural convection and radiation consider a heated, horizontal aluminum alloy cylinder with a diameter of \( \frac{1}{4}'' = 6.35 \text{ mm} \) in 20°C air. Figure 2 shows \( h_c \) and \( h_r \) for a range of surface temperatures. The emissivity of the aluminum alloy was estimated to be 0.35 using an Omegascope 3000 Infrared Pyrometer. As an example, at a surface temperature of 100°C, \( h_c = 11.95 \text{ W/m}^2\text{°C} \), \( h_r = 2.98 \text{ W/m}^2\text{°C} \), and \( h_c + h_r = 14.93 \text{ W/m}^2\text{°C} \). Note that at this temperature the radiation heat transfer coefficient is approximately 20% of the total.

Figure 2: Effect of surface temperature on the heat transfer coefficients.

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Experimental Temperature Data

The temperature along a 1/4" Al 2024-T4 \((k = 120 \text{ W/m} \cdot \text{°C})\) fin was measured using thermocouples as described in Ref. [5]. The apparatus consisted of a heating plate mounted on a stand to which the rod was attached. The heating plate was used to maintain the base of the fin at a constant temperature. Type T thermocouples were mounted along the fin and connected to a digital readout. Once steady-state conditions were achieved, the temperatures along the fin were recorded. The temperature measurements are given in Table 1.

<table>
<thead>
<tr>
<th>(x) (m)</th>
<th>0</th>
<th>0.046</th>
<th>0.097</th>
<th>0.145</th>
<th>0.196</th>
<th>0.247</th>
<th>0.347</th>
<th>0.497</th>
<th>0.677</th>
</tr>
</thead>
<tbody>
<tr>
<td>(T) (°C)</td>
<td>102.5</td>
<td>86.5</td>
<td>63.0</td>
<td>49.1</td>
<td>37.8</td>
<td>34.7</td>
<td>27.3</td>
<td>23.8</td>
<td>22.6</td>
</tr>
</tbody>
</table>

The local heat transfer coefficient can be calculated using Eqs. (15) and (18) and the experimental temperatures. A plot of the heat transfer coefficients along the fin is shown in Figure 3. The total heat transfer coefficient ranges from 15.0 W/m\(^2\)°C at the base to 6.80 W/m\(^2\)°C near the tip. At the base, radiation accounts for 20% of the heat loss, while at the tip radiation accounts for 30%. It is interesting to note that the heat transfer coefficients, \(h_c\) and \(h_r\), as predicted by Eqs. (15) and (18), do not approach zero at the tip of the fin; however the rates of heat transfer according to Eqs. (13) and (19) approach zero because the temperature differences approach zero.

![Figure 3: Variation of the heat transfer coefficients along the fin.](image-url)
Experimental Determination of $\tilde{h}$

Three approaches were used to estimate the average value of the heat transfer coefficient along the fin from the experimental data—nonlinear regression, linear least squares, and iteration with Eq. (11). An overview of these three methods is presented using the temperature data from Table 1.

First, a nonlinear regression analysis (Marquardt-Levenberg method) was performed on the data in Table 1 using the software package GraphPad Prism 4.0 [8]. This software package was used to fit the data to an equation of the form of Eq. (11). The nonlinear regression approach implemented allows for three degrees-of-freedom. To properly capture the exponential behavior in Eq. (11), this approach requires temperature measurements both near the base and near the tip. The data along with the best-fit curve and the 95% confidence intervals are shown in Fig. 4. This analysis, which made use of all the points from Table 1, resulted in the curve fit

$$ T = (105.4 - 20.95) \exp(-7.328x) + 20.95, $$  \hspace{1cm} (21)

where $T$ is in °C and $x$ is in m.

With the use of Eq. (9), this analysis yielded $\tilde{h} = 10.2$ W/m²°C. The value $\tilde{h}$ was found to lie between 6.37 W/m²°C and 15.0 W/m²°C according to the 95% confidence interval prediction. This wide range can be partially attributed to the fact that data point 2 appears to be an outlier with a residual of 5.3°C. With data point number 2 removed $\tilde{h} = 10.70$ W/m²°C, and the 95% confidence interval range on $\tilde{h}$ was reduced to 8.28 W/m²°C and 13.46 W/m²°C.
The second approach used to find $\bar{h}$ from the experimental temperature is linear least squares. The excess temperature distribution in Eq. (10) can be linearized with division by $\theta_0$ and application of the natural logarithm to yield

$$y = \ln(\theta/\theta_0) = -mx,$$

(22)

which is the equation of a line with the independent variable $\ln(\theta/\theta_0)$, the dependent variable $x$, a slope of $-m$, and an intercept of 0. The experimental data plotted in this form is shown in Fig. 5. Application of linear least squares to the experimental temperature data yields a slope of $m = 7.234$ and an intercept of $b = -0.0366$. From this analysis, the following exponential fit of the data is obtained

$$\theta(x) = 0.964 \theta_0 e^{-7.243x}.$$

(23)

Once $m$ from the least square analysis is determined, the heat transfer coefficient can be found from Eq. (9) to be $\bar{h} = 9.99 \text{ W/m}^2\text{C}$.

A sensitivity analysis [9] performed on the calculation of $y$ indicates that the temperature measurements near the tip (when the fin temperature is close to the ambient) should be avoided. The least squares approach, as described, allows for two degrees-of-freedom and thus, the line in Fig. 5 does not pass through the origin. This approach does not minimize the sum of the squares of the deviations of the non-dimensional temperature curve, but rather the deviations of the natural logarithm of the non-dimensional temperature curve. This amounts to minimizing the squares of the percentage errors [10], and thus gives greater influence to the temperatures near the tip of the fin.

The third approach to find $\bar{h}$ involves iteration and is based on the minimization of the error between the theoretically predicted temperature and the measured temperature at each location. In this approach, a value of $\bar{h}$ is assumed and a function indicative of the difference between the

Figure 5: Linear fit of $\ln(\theta/\theta_0)$ (left) and minimization of $E$ (right).
experimental and theoretical values is constructed, viz.

$$E(\bar{h}) = \sqrt{\frac{1}{n} \sum_{i=1}^{n} (T_i^{\text{exp}} - T_i^{\text{theory}})^2},$$  \hspace{1cm} (24)$$

where $T_i = T(x_i)$ and $n$ is the number of experimental data points. The theoretical temperature in Eq. (24) is found from Eq. (11) with the assumption that the base and ambient temperatures are known exactly. $\bar{h}$ is varied between 1 and 30 W/m$^2$K and a plot of $E(\bar{h})$ is shown in Fig. 5. The value of $\bar{h}$ that minimizes $E(\bar{h})$ is 10 W/m$^2$K. If data point number 2 is discarded, $\bar{h} = 11$ W/m$^2$K.

In comparison to the linear least squares approach, this iterative approach gives more influence to the higher temperatures where the exponentially varying curve is rapidly changing. Points near the tip of the fin have little effect. Also, note that with the base temperature assumed to be known exactly, this approach allows for only one degree-of-freedom.

As an example to illustrate the sensitivity of the three methods to the location of the temperature measurements, consider a point near the tip of the fin that is very near the ambient temperature, say $T_{10} = 22.0^\circ$C. With the nonlinear regression method, this value would help to confirm the plateau region shown in Fig. 4. With the least squares method, a temperature near the ambient would result in a value of $\theta$ near zero and after application of natural logarithm would result in a very large value that would greatly affect the calculation of the slope. Moreover, if the temperature at the tip of fin was equal to the ambient, the data point would have to be discarded. With the iteration method, a temperature measurement close to the ambient at the tip would have virtually no effect as the contribution to the function $E$ is minimal.

As expected, the experimentally determined average values found using the three methods in this section are in close agreement and fall somewhere between the value of the heat transfer coefficient at the base (maximum) and the tip (minimum) as shown in Fig. 3. Thus, the modeling approach described by Eqs. (15) and (18) appears to give a suitable bound on the average value of the heat transfer coefficient.

**Prediction of $\bar{h}$**

In the previous section, the average value of the total heat transfer coefficient $\bar{h}$ was found using the experimental temperatures. For design purposes, it would be useful to predict $\bar{h}$ *a priori*. In this section, two approaches to estimate the average value of the total heat transfer coefficient using the properties of the fin and the base and ambient temperatures are presented. The first method makes use of the average value of the temperature along the fin, while the second method is to simply average the maximum and minimum local heat transfer coefficients.

With the use of the analytical expression for the temperature distribution along the fin, the average excess temperature along the fin, i.e. $\bar{\theta} = \bar{T} - T_\infty$, can be found via integration, viz.

$$\bar{\theta} = \frac{1}{L} \int_0^L \theta(x) \, dx = \frac{1}{L} \int_0^L \theta_0 e^{-mx} \, dx = \frac{\theta_0}{mL} (1 - e^{-mL}),$$  \hspace{1cm} (25)$$

where $m$ is assumed to be constant at an average value. However, $m$ (actually $\bar{h}$) is unknown but can be found with iteration. The approach is to assume an average temperature and calculate $\bar{h}_c$...
from Eq. (15), $h_r$ from Eq. (18), and to let $\bar{h} = h_c + h_r$. Then, substitute $\bar{h}$ into Eq. (25), calculate a new value for the average temperature, and continue until convergence is achieved.

For the situation described in the previous sections, the average value of the heat transfer coefficient is found to be $\bar{h} = 10.8 \text{ W/m}^2\text{C}$ and the average temperature along the fin is found to be $37.6^\circ\text{C}$. Only three iterations are required using a starting value of the base temperature. This approach assumes that the average values of the heat transfer coefficients occur at the average temperature. With $\bar{h} = 10.8 \text{ W/m}^2\text{C}$, a temperature distribution can be constructed using Eq. (11). This temperature distribution is plotted in Fig. 6, along with temperature distributions that have been constructed with typical values of $\bar{h}$ as suggested in Ref. [3].

Equation (25) must be used carefully for the case of very long fins. When $mL \to \infty$, the average fin temperature approaches $T_\infty$ and according to Eq. (20) no heat is lost from the surface of the fin. However, the integration in Eq. (25) is over the entire fin length, i.e., even the portion from which no heat loss occurs. An important parameter that indicates when the heat transfer along the fin negligible is $m.x$ (see Eq. (12) and the discussion in Ref. [8]). For the situation analyzed in this paper $mL \approx 5$. If instead $mL$ is taken to be 3, (which implies an integration over the part of the fin closest to the base) the average temperature is found to be $48.7^\circ\text{C}$ and the average heat transfer coefficient is $\bar{h} = 11.9 \text{ W/m}^2\text{C}$.

A second approach to estimate the average heat transfer coefficient is to simply average the maximum local heat transfer coefficient calculated at the base temperature and the minimum local heat transfer coefficient calculated at the ambient temperature. With this approach, the average value of the heat transfer coefficient is found to be $10.9 \text{ W/m}^2\text{C}$. 
The predicted values in this section are similar and in close agreement with the experimentally determined values.

**Concluding Remarks**

A published correlation for natural convection and a simple model for radiative heat transfer have been used to estimate a total average heat transfer coefficient for a horizontal fin. This estimated value is in close agreement with values of the average heat transfer coefficient that have been found from the experimental temperatures.

Despite the relatively simple analytical model proposed, the actual physical situation is rather complicated. Effects such as (1) currents in the air which indicate forced (or mixed) convection, (2) an interaction between the flow around the mounting plate and the fin, and (3) radiative heat transfer between the mounting plate and the fin warrant additional consideration.

**References**


**Biography**

DON MUELLER is an Assistant Professor of Engineering at Indiana University–Purdue University Fort Wayne, in Fort Wayne, IN. He received his BS, MS, and PhD in Mechanical Engineering from the University of Missouri–Rolla. His teaching interests are in the areas of thermal-fluid sciences and numerical methods.

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