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Evaluation of Engineering & Mathematics Majors' Riemann Integral Definition Knowledge by Using APOS Theory

Dr. Emre Tokgoz, Quinnipiac University

Emre Tokgoz is currently an Assistant Professor of Industrial Engineering at Quinnipiac University. He completed a Ph.D. in Mathematics and a Ph.D. in Industrial and Systems Engineering at the University of Oklahoma. His pedagogical research interest includes technology and calculus education of STEM majors. He worked on an IRB approved pedagogical study to observe undergraduate and graduate mathematics and engineering students' calculus and technology knowledge in 2011. His other research interests include nonlinear optimization, financial engineering, facility allocation problem, vehicle routing problem, solar energy systems, machine learning, system design, network analysis, inventory systems, and Riemannian geometry.

Evaluation of Engineering & Mathematics Majors' Riemann Integral Definition Knowledge by Using APOS Theory

In this study senior undergraduate and graduate mathematics and engineering students' conceptual knowledge of Riemann's definite integral definition is observed by using APOS (Action-Process-Object-Schema) theory. Seventeen participants of this study were either enrolled or recently completed (i.e. 1 week after the course completion) a Numerical Methods or Analysis course at a large Midwest university during a particular semester in the United States. Each participant was asked to complete a questionnaire consisting of calculus concept questions and interviewed for further investigation of the written responses to the questionnaire. The research question is designed to understand students' ability to apply Riemann's limit-sum definition to calculate the definite integral of a specific function. Qualitative (participants' interview responses) and quantitative (statistics used after applying APOS theory) results are presented in this work by using the written questionnaire and video recorded interview responses. Participants are asked to calculate the definite integral of the function f(x) = x^2 on the interval [1, 2] by using the limit definition of Riemann integral. Missing conceptual knowledge of the participants in calculus are observed when they were incapable of determining the solution to the problem.

Key Words: Riemann integral, functions, derivative, triad classification, APOS theory.

Introduction

Riemann integral is an important concept in calculus that is often used by engineering and mathematics majors during their undergraduate and graduate studies. Given a continuous function f on an interval [a, b], the Riemann integral (for definite integral) of f on the given interval can be determined by using the limit of sums:

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(a+i\Delta x)\Delta x.$$

This definition will be called the limit definition of Riemann integral throughout this work. This definition of Riemann integral is taught at early stages of calculus education, therefore Riemann sum approximation needs to be known by the Numerical Methods/Analysis students to be able to solve a question related to the Riemann integral's limit definition.

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This definition involves knowledge of concepts such as algebra, functions, limit, and summation rules. Integral calculations by using limit concept can be challenging and a mistake throughout the calculations can result in a misleading path towards finding the solution.

In this work, the goal is to observe senior undergraduate and graduate mathematics and engineering students' ability to apply the limit definition of Riemann integral to calculate the integral of $f(x)=x^2$ on the interval [1, 2].

Methodology

Seventeen participants of this study are asked to complete a questionnaire with a follow up interview to explain their written questionnaire responses. The questionnaire questions covered concepts such as functions, limits of functions, function derivatives, Riemann integral, power series of functions, and programming preferences of the participants. The participants of this study are engineering and mathematics undergraduate and graduate students who were either enrolled or recently completed a numerical methods or analysis course in a particular semester at a large Midwest university in the United States. The participants completed a series of pre-requisite calculus courses in which the questionnaire concepts are covered. Post-interview results are designed to have a better understanding of the pre-interview (i.e. written) responses of the participants. The responses to the Riemann integral question are evaluated by considering the concepts that take place in the solution of the research question. The data collected in this work is expected to help understanding the missing conceptual knowledge of STEM majors during the application of the limit definition of Riemann integral. The collected written and interview response data is evaluated by using the Action-Process-Object-Schema (APOS) theory of Asiala, Brown, DeVries, Dubinsky, Mathews, and Thomas (1996) and the triad classification taking place in schema development.

Relevant Literature

In this section theories used for evaluation of the research question are explained. APOS theory and triad classification will be used to observe students' ability to apply Riemann integral definition to the research question. The pedagogical literature on determining the Riemann integral of functions by using paper-pencil solution is limited. Asiala et al. (1996) pointed out the difficulty of writing a code to find the integral of functions and asked the participating students to write a code to approximate the integral by sampling points. Thompson (1994) states

...We must think of integration as the culmination of a limiting process, but at the same time consider that process, applied over an interval of variable length, as producing a correspondence...

and invites to do research on determining the integral of functions:

...A curricular and instructional emphasis in algebra and pre-calculus on having students develop images of arithmetic operations in analytically-defined functions as operations on functions would seem to prepare them for a deeper understanding of this aspect of the calculus. At the same time, a conception of operations in expressions as operating on numbers and not on functions would seem to be an obstacle to understanding the derivative and integral as linear operators. These are empirically testable hypothesis; I would welcome research on them...

Thompson (1994) observed senior mathematics undergraduate and graduate students' weak rate of change concept knowledge resulted in weak understanding of the integration concept. The first derivative knowledge of the students appeared to be the major problem in answering the research question of Thompson (1994).

Schema Development

Clark, Cordero, Cottrill, Czarnocha, DeVries, St. John, Tolias, and Vidakovic (1997) used the stages of the triad classification; Intra, Inter and Trans to investigate how first year calculus students construct the concept of chain rule. Their attempt to use the APOS theory resulted in insufficiency by itself therefore they included the schema development idea of Piaget et al. (1989). Clark et al. (1997) used triad classification after realization of not being able to apply the APOS theory. Similar to Clark et al. (1997) APOS theory appears to be inappropriate for evaluating the research question in this work because students' responses didn't reflect a proper setting to apply the APOS theory; therefore, participating students' responses are analyzed by using the schema development idea. The Triad classification in this setting is as follows:

- **Intra Stage:** Students classified in this category if they didn't know how to start solving the problem algebraically. This categorization includes
  a) Students who started solving the problem by writing the summation terms,
  b) Suggested to solve the problem by approximating it (i.e. Choose a particular value of n.)
- **Inter Stage:** Students in this stage knew how to solve the problem but made a mistake either during the summation or limit calculations.
- **Trans Stage:** Students were able to apply the definition and successfully found the correct answer.

The following terms are excluded from the triad classification because this information is provided to the students.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad and \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Action-Process-Object-Schema (APOS) Theory

By relying on Piaget`s study of functions in 1977 (Piaget et al. 1977), Action-Process-Object idea in mathematics education for the undergraduate curriculum was initiated by Breidenbach, Dubinsky, Hawks and Nichols in 1992 who studied students' conceptual view of the function in their research. In 1996, Asiala, et al. applied APOS theory to understand students' function knowledge and explained this theory as the combined knowledge of a student in a specific subject based on Piaget's philosophy. Dubinsky and McDonald (2001) explained the components of the APOS theory as follows:

An action is a transformation of objects perceived by the individual as essentially external and as requiring, either explicitly or from memory, step-by-step instructions on how to perform the operation...

When an action is repeated and the individual reflects upon it, he or she can make an internal mental construction called a process which the individual can think of as performing the same kind of action, but no longer with the need of external stimuli...

An object is constructed from a process when the individual becomes aware of the process as a totality and realizes that transformations can act on it...

A schema is an ... individuals collection of actions, processes, objects, and other schemas which are linked by some general principles to form a framework in individual's mind...

Baker, Cooley and Trigueros (2000) applied APOS theory to understand undergraduate students conceptual function knowledge by using the data set collected for a calculus graphing problem. Cooley, Trigueros and Baker (2007) continued in the line of their previous work from 2000 (Baker et al. 2000) by focusing on the schema thematization with the intent to expose those possible structures acquired at the most sophisticated stages of schema development. For a detailed review of the APOS theory see Dubinsky and McDonald (2002).

APOS theory is widely used in several educational research areas in the past decade: It is used by Parraguez and Oktac (2010) to lead the students towards constructing the vector space concept, Mathews and Clark (2007) to observe successful students' conceptual knowledge of mean, standard deviation, and the central limit theorem who completed an elementary statistics course with a grade of "A", by Trigueros and Martinez-Planell (2009), and Kashefi, Ismail, and Yusof (2010) to observe students' ability to construct and develop two variable functions. One of the most recent comprehensive APOS theory work is by Arnon, Cottrill, Dubinsky, Oktac, Fuentes, Trigueros, and Weller (2014). In their work APOS theory application to definite integral is explained. Due to this explanation, a participant is classified to have the Process classification after the Action level if he/she has the mental ability to progress the solution from Riemann partition to the continuous function interval (pg. 21) In this work we have a different classification that will be explain later throughout this work. Tokgöz and Gualpa (2015), and Tokgöz (2015) recently worked on understanding undergraduate and graduate students' ability to respond to a variety of calculus questions by using APOS theory. Evaluation of the results indicated a variety of APOS classification of the participants depending on the research question.

Research Question

Participants of this study are asked to answer the following Riemann integral question:

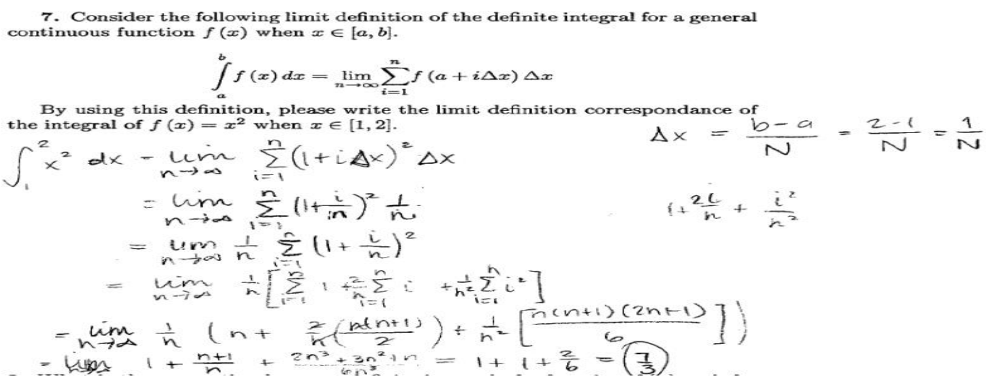
Question: Consider the following limit definition of the definite integral for a general continuous function f(x) when x ∈ [a, b].

$$\int_a^b f(x)dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(a+i\Delta x)\Delta x.$$

By using this definition, please write the limit definition correspondence of the integral of $f(x)=x^2$ when $x\in[1,2]$.

All seventeen participants' written and interview responses are evaluated. The researcher interviewed the participants and the video recorded interview responses are transcribed by the researcher. Post interview results indicated the following results:

- Only one student answered the research question correct before the interview without any mistakes.
- Only one student answered the research question correct during the interview without any mistakes.


7. Consider the following limit definition of the definite integral for a general continuous function $f(x)$ when $x \in [a,b]$.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(a+i\Delta x)\,\Delta x$$

By using this definition, please write the limit definition correspondance of the integral of $f(x)=x^2$ when $x\in[1,2]$.

$\Delta x = \frac{b-a}{N} = \frac{2-1}{N} = \frac{1}{N}$

$\int_1^2 x^2\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}(1+i\Delta x)^2\Delta x$

$= \lim_{n\to\infty}\sum_{i=1}^{n}(1+\frac{i}{n})^2\frac{1}{n}$ $\quad 1+\frac{2i}{n}+\frac{i^2}{n^2}$

$= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}(1+\frac{i}{n})^2$

$= \lim_{n\to\infty}\frac{1}{n}\left[\sum_{i=1}^{n}1+\frac{2}{n}\sum_{i=1}^{n}i+\frac{1}{n^2}\sum_{i=1}^{n}i^2\right]$

$= \lim_{n\to\infty}\frac{1}{n}\left(n+\frac{2}{n}\left(\frac{n(n+1)}{2}\right)+\frac{1}{n^2}\left[\frac{n(n+1)(2n+1)}{6}\right]\right)$

$= \lim\ 1+\frac{n+1}{n}+\frac{2n^3+3n^2+n}{6n^3} = 1+1+\frac{2}{6} = \frac{7}{3}$

Figure 1: Response of RP 1

- Only 3 out of 17 students found the right solution to the research question.

Interview Results

In this section qualitative and quantitative results for the concepts that take place in the solution to the research question are covered. Some of the students who could not recall the formula of Δx in the limit definition of Riemann integral either tried to give an example to explain Δx or tried to find an intuitive explanation:

RP 6: … This Δx is. Should just be a infinitesimal amount, just a small add on like we are looking at the definition of limit, they are basically the same Δx, just a really small, arbitrarily small slice of the graph.
Interviewer: … if that is the case, like, can you pick in particular what that is?
RP 6: … I guess you could pick. You can define your own Δx. But the smaller it is, the better accurate you have...

RP 8: Δx is the slightest change that you can do in x towards the (points on the interval [1, 2].) So basically if Δx is 0.1 then x is 1.1.
Interviewer: ...Do you remember the general definition?
RP 8: Nodding (to indicate that doesn't know)...

Interviewer: ...Do you recall what delta x is?
RP 12: Delta x supposed to… this is from basic Riemann sum... What I remember from calculus now. x^2 is something like this

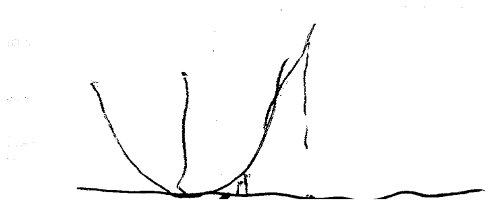


Figure 2: Graph of RP 12

We are going from one to two. So what we are basically trying to do is trying to break up this graph into smaller and smaller delta x step, okay (and explains on the graph of x^2) because when you blow up the graph, instead of, basically it makes rectangles (draws rectangles below the curve of x^2.)

Limit Knowledge

Limit knowledge is an important part of the schema development that is directly related to the definition of Riemann integral. Limit calculation is the last step to find a solution to the research question. During the interviews some of the participants miscalculated the limit:

RP 5: So this one is actually (pointing

$$\sum_{i=1}^{n}\left[\frac{1}{n}+\frac{2}{n^2}+\frac{i^2}{n^3}\right]$$

writes

$$\frac{i}{n}n+\frac{2}{n^2}\sum i+\frac{1}{n^3}\sum i^2$$

and underlines ∑i and $\sum i^2$.) If I know this (pointing ∑i) and this (pointing $\sum i^2$) from the table then I can actually calculate this (circles

$$\frac{i}{n}n+\frac{2}{n^2}\sum i+\frac{1}{n^3}\sum i^2$$

Interviewer: ... Do we have anything else from the beginning or is it just this we need to calculate? Is there anything else coming from the question's solution? …for example, you know that this is (pointing $\frac{n^2+n}{2}$)) n square plus n over two, you plug it in and is that it or is there anything else that we are looking at?

RP 5: n should go to infinity (writes $n\to\infty$)

Interviewer: Okay, n should go to infinity.

RP 5: This one should be zero and this one is one.

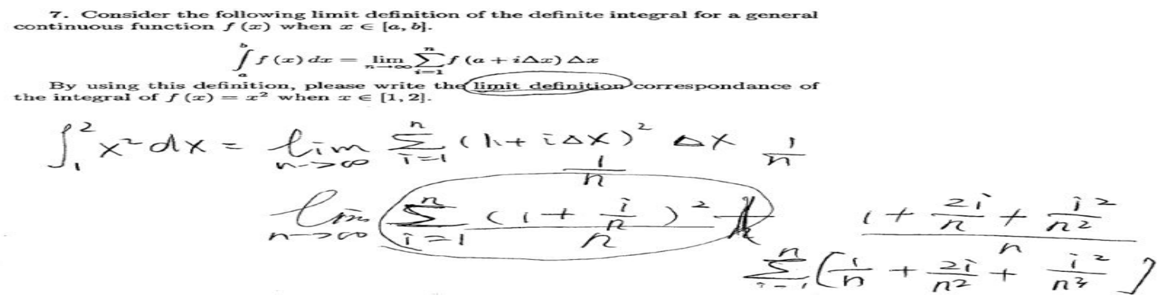


Figure 3: Response of RP 5

RP 8: Yeah. It would be (starts calculating

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left[\frac{1}{n}+\frac{2i}{n^2}+\frac{i^2}{n^3}\right]$$

as

$$1+(n+1)+\frac{(n+1)(2n+1)}{6}$$

Interviewer: So, if that is the case, what can be the answer here?

RP 8: Simplified version of that (pointing his last answer.)

Interviewer: And what would that be?

RP 8: (Rewrites

$$1+(n+1)+\frac{(n+1)(2n+1)}{6}$$

and also writes

$$=\frac{(6n^2+6n^3+6n^2+2n^2+n+2n+1)}{6}=\frac{14n^2+6n^3+3n+1)}{6n^2}$$

Interviewer: And if you take the limit of that what do you think that could be? What could that be?

$$=\lim_{n\to\infty}\frac{14n^2+6n^3+3n+1)}{6n^2}$$

RP 8: One.

Summation Term

Some of the participants ignored the summation term and made simple algebraic mistakes related to this term:

RP 6: … Okay, let's see. This is gonna be the (starts writing

$$=\lim_{n\to\infty}\left(\frac{1}{n}+2i\frac{1}{n^2}+\sum_{i=1}^{n}\frac{i^2}{n^3}\right)$$

These two are going to be equal (pointing

$$=\lim_{n\to\infty}\sum_{i=1}^{n}\left[\frac{1}{n}+\frac{2i}{n^2}+\frac{i^2}{n^3}\right]==\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n^3}+\sum_{i=1}^{n}2i\frac{1}{n^2}\right)$$

The original written response of RP 6 is displayed below.

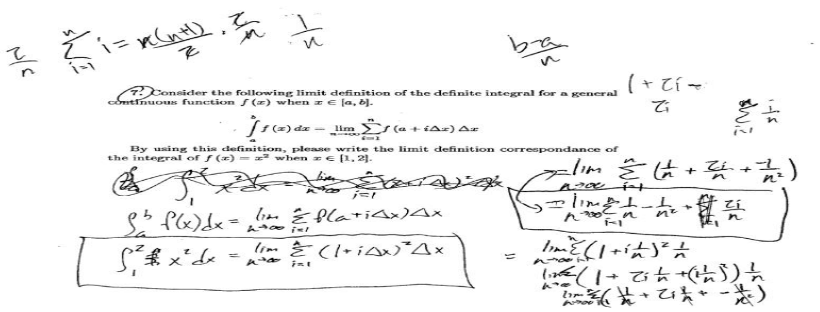


Figure 4: Response of RP 6

RP 2 mixed up the i^{th} and n^{th} terms and tried to calculate the summation by writing out the terms:

RP 2: …Maybe we can write it (continues writing

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{1}{n} + \frac{2i}{n^2} + \frac{i^2}{n^3} \right]$$

Interviewer: …can we improve this expression?
RP 2: … I don't know how to improve this.

Interviewer: What is $\sum_{i=1}^{n} i$? Can you write that?

RP 2: Yeah we can. (Starts writing

$$\left(\frac{1}{1} + \frac{2i}{1} + \frac{i^2}{1} \right) + \left(\frac{1}{2} + \frac{2i}{4} + \frac{i^2}{8} \right) + \ldots$$

Interviewer: So you would rather write it in its expanded form?
RP 2: Yeah…

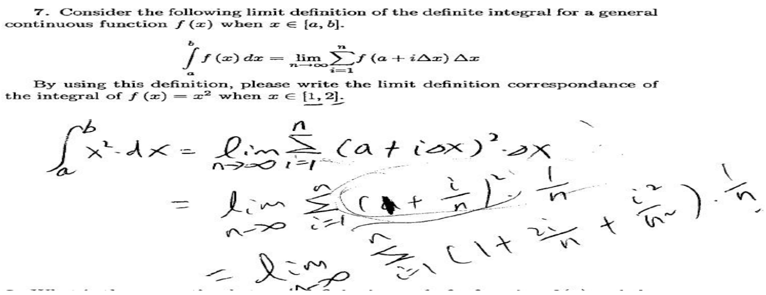


Figure 5: Response of RP 2

RP 7: So the sum of the limit loosens a little bit. (Starts writing

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + i^2 + \frac{1}{n} + \frac{(n+1)(2n+1)}{n^2} \right)$$

And just need to work out the (n+1)(2n+1).
Interviewer: And what is sum of i from 1 to n, one? Do you remember how you can find it? This term right here (Pointing $\sum_{i=1}^{n} 1$ written previously)
RP 7: It is going to be 1 because you don't have a, no n (laughs). Not really straight A′s. So that would cross that out. (Changes

$$= \lim_{n\to\infty} \left(\frac{1}{n} + n^2 i + \frac{1}{n} + \frac{(n+1)(2n+1)}{6n^2} \right)$$

RP 7: … yeah. That could get it straight. It would just be here, it wouldn't have anything to do with the other ones. So we have (starts writing

$$=\lim_{n\to\infty}\left(1+1+\frac{1}{3n}+\frac{1}{2n^2}+\frac{1}{n^3}\right)$$

what then just goes to (writes 2) as all the other terms cancel out…

RP 9: …Let's see. Do I want to simplify this first? Yeah… Okay, cool. (Writes

$$=\lim_{n\to\infty}\sum_{i=1}^{n}\left(1+i\frac{1}{n}\right)\frac{1}{n}\Rightarrow\left(1+i\frac{1}{n}\right)^2\frac{1}{n}+\left(1+\frac{i-1}{n}\right)\frac{1}{n}$$

RP 10: ... (Writes

$$=\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{1}{n}+2i\frac{1}{n^2}+\frac{1}{n^3}\right)$$

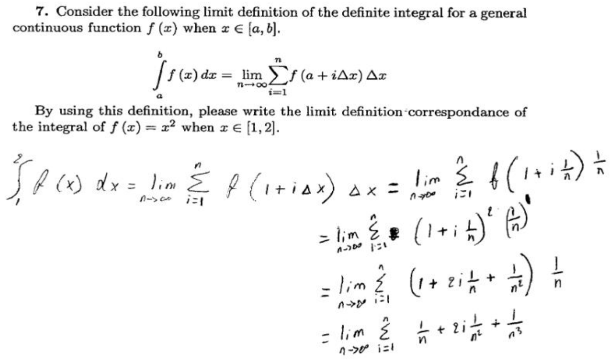


Figure 6: Response of RP 10

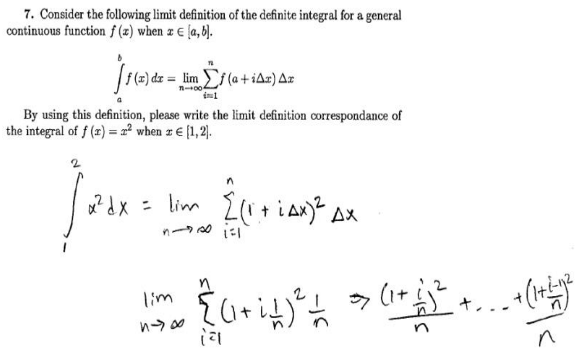


Figure 7: Response of RP 9

RP 12: ... Here we want i as one minus... Since we are calculating i and delta x is basically we have just one over n, squared all divided by one over n. (Writes

$$=\lim_{n\to\infty}\sum_{i=1}^{n}\left(1-i\left(\frac{1}{n}\right)^2\right)\frac{1}{n}$$

RP 14: … Here (pointing $\frac{1}{n}$ of $\left(1+\frac{i}{n}\right)^2\frac{1}{n}$) goes in (pointing $\left(1+\frac{i}{n}\right)^2$) of $\left(1+i\frac{1}{n}\right)^2\frac{1}{n}$).) (Writes

$$\left(1+i\frac{1}{n}\right)^2\frac{1}{n}=\left(1+\frac{2i}{n}+\frac{i^2}{n^2}\right)\frac{1}{n}=\left(\frac{1}{n}+\frac{2i}{n^2}+\frac{i^2}{n^3}\right)$$

Maybe we can use this (pointing the summation term $\sum_{i=1}^{n}$) with this (pointing $\left(\frac{1}{n}+\frac{2i}{n^2}+\frac{i^2}{n^3}\right)$ and calculate the limit…

RP 14′s response had an interesting response to this question:
Interviewer: And do you remember what summation of i from 1 to n (1/n) is?
RP 14: (Thinks)
Interviewer: Do you want me to remind you? What that is, i from 1 to n, summation of this (pointing summation of (1/n))?
RP 14: (Writes $\sqrt{2}$ and then scratches it. Then writes $\left(\frac{1}{1}+\ldots+\frac{1}{n}\right)$

Two (Writes 2)?

RP 16 made an algebraic mistake while calculating the summation terms and ignored both i and i^2 in the summation:

RP 16: (Writes

$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(1+\frac{2}{n}+\frac{1}{n^2}\right)\frac{1}{n}$$
$$\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{1}{n}+\frac{2}{n^2}+\frac{1}{n^3}\right)$$

And these summations, we know what these are right? (Looks at the previous page)

Interviewer: It is not given there, but the summation in terms of i

RP 16: Huh, huh. You can split it up into three…. (Writes

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{1}{n}+2\sum_{i=1}^{n}\frac{1}{n^2}+\sum_{i=1}^{n}\frac{1}{n^3}\right)$$

RP 17: …so do you want me to try keep going?

Interviewer: Yes please.

RP 17: (Previously had

$$=\lim_{n\to\infty}\sum_{i=1}^{n}(1+i\Delta x)^2\,\Delta x$$

and continues writing

$$\lim_{n\to\infty}\sum_{i=1}^{n}(1+i\Delta x)^2\Delta x=\lim_{n\to\infty}\sum_{i=1}^{n}\left(1+i\frac{1}{n}\right)^2\frac{1}{n}=\lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{1}{n}+\sum_{i=1}^{n}i\frac{1}{n^2}\right)$$

As it can be observed from some of the answers of the students given above, some of the participants had difficulty with the unknowns i and n.

Derivative Knowledge

Even though it is not evident that the derivative knowledge is required during the calculations of the limit definition of definite integral for the research question, the limit calculations can yield to L'Hospital's rule application. During the interview a participant applied L'Hospital's rule to a part of the question and applied quotient rule to the other part of the question:

RP 12: So I have 1 plus $\frac{n+1}{n}$. I can cancel out that n (pointing n of $\frac{n(n+1)(2n+1)}{6}$.) I have no other factor but I still have $\frac{(n+1)(2n+1)}{6n^2}$. (Wrote

$$1+\frac{n+1}{n}+\frac{(n+1)(2n+1)}{6n^2}$$

I would say as the limit approaches using L'Hospital's here is equal to 1. (Pointing to $\frac{n+1}{n}$). And then the derivative of this (pointing $\frac{(n+1)(2n+1)}{6n^2}$) with the product rule. So hang on. The derivative of that would be

$$\frac{1(n+1)+(2n+1)2}{6}$$

and we can take it again. Well hang on. This doesn't approach, oh, that's completely wrong... (Writes $\frac{2n^2+3n+1}{6n^2}$)

Interviewer: Okay.

RP 12: Now, I'll use my quotient rule. It's just basically... (Writes and explains

$$\frac{6n^2(4n+3)+(12n)(2n^2+3n+1)}{36n^4}$$

Interviewer: … Are you using L' Hospital's rule?

RP 12: No, I was just using. Oh, yeah, hang on. It's the derivative of the top and the bottom. Never mind (scratches

$$\frac{6n^2(4n+3)+(12n)(2n^2+3n+1)}{36n^4}$$

So this is what L' Hospital's. You take the derivative of the first (points out $\frac{2n^2+3n+1}{6n^2}$)

Writes $\frac{4n^2+3}{6n^2}$.) Since I still have a factor of n, I can take the derivative again. It is just $\frac{8n}{12}$. Therefore as this, I can't do it again. I wanna say we said that limit n approaches infinity, this is one plus one plus this (pointing $\frac{8n}{12}$), it looks like, not knowing any better, since I have an n on top, it approaches infinity...

L'Hospital's rule is not considered as a part of the APOS classification due to the fact that it doesn't have to be used for the solution of the question.

Need to Know n

Students learn how to approximate the Riemann integral of a function by using the sum of the function terms. Summation approximation of Riemann integral is covered in Numerical Methods/Analysis courses. In their responses to the research question, some of the students claimed that they either need to know n term or could pick n term to be able to calculate the definite integral. These students seem to ignore the limit term in the definition and appear to not have the full conceptual understanding of the definition.

Interviewer: … Can you …calculate more?

RP 4: I don't know. I would have to know what n is. What the step size gonna be. What the interval size is...

RP 9: Okay. Let's see. Do I want to simplify this first? ... Wait. We don't know what n is…

RP 14: Oh, but I don't know n...

RP 15: Oh, if I was going to actually solve it, I would pick, I wouldn't do the limit, I would just pick n equals like 5, and then 10 maybe...

APOS Theory Results & Triad Classification

A participant is qualified to be in the "Action" level of APOS if he/she was able to write the limit definition of Riemann integral correct for the function $f(x) = x^2$ in this work. Sixteen out of 17 participants (94%) are qualified to be in the "Action" level. RP 17 could not apply the definition correct:

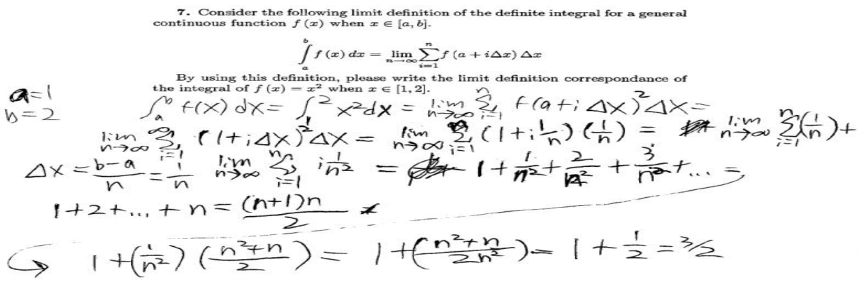


Figure 8: Response of RP 17

Participants of the action level are qualified to be in the “Process” level of APOS if they are qualified to have the right “Action” while writing the limit definition of Riemann integral (94%) and be able to progress the solution to the point of calculating algebra with the summation terms that have i and i^2. Only 7 out of 16 (43.75%) participants are qualified to be in the “Process” level. The obstacles that the participants faced appear to be needing to know n and algebraic mistakes.

Participants are qualified to be in the “Object” level if they are qualified to be in the “Process” level and be able to progress the calculations to the point that limit needs to be applied before finding the correct answer. Only 4 out of 9 (44.4%) participants are qualified to be in the “Object” level.

Participants are qualified to be in the “Schema” level if they are qualified to be in the “Object” level and be able to find the right answer by applying the limit right. Three out of 4 (75%) participants are qualified to be in the “Object” level. Only one participant could not calculate the final limit to find the right result.

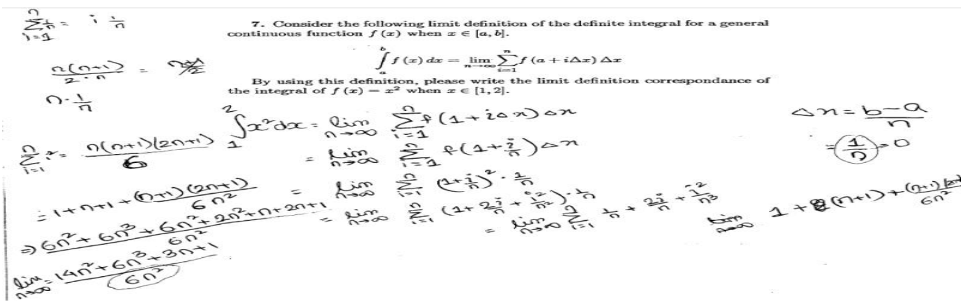


Figure 9: Response of RP 8

The Triad classification of the participants is ruled by their algebraic calculation ability. If the participants were able to write the limit definition of Riemann integral correct and did not make an algebraic mistake before starting to calculate the terms taking place in the summation then they are classified to be in the Intra stage. Ten out of 17 (58.8%) participants are classified to be in this stage. The Inter stage classification is determined by students’ ability to calculate the algebraic terms before calculating the limit. Four out of 17 (23.5%) participants are qualified to be in the Inter stage. The Trans stage classification of the participants depended on finding the right result with the right limit calculation. Three of 17 participants are classified to be in the Trans stage in this work.

Conclusion

The goal of the research question designed in this work is to investigate students' ability to implement calculus concepts on Riemann integral. The pedagogical goal of the collected data is to investigate graduate and advanced level undergraduate students' limitations while applying calculus concepts to a particular Riemann integral question by using APOS theory and Triad classification. APOS theory was particularly useful in measuring engineering and mathematics students' missing conceptual knowledge step-by-step and identifying research participants' thought process while responding to the research question of this work. Educators can investigate students' thinking process while solving a definite integral problem from the detailed description presented in this work. The concepts covered during the investigation of participants calculus knowledge included algebra, functions, limit, and power series. The limit definition of Riemann integral is defined to be the limit of the sums of a continuous function on the interval [a, b]. In particular, participants' responses to the limit definition of Riemann integral applied on $f(x) = x^2$ is investigated. Considering 17 participants who responded to the research question, there was only one student, an undergraduate mathematics student, successfully answered the research question prior to the interview. Only one student, an undergraduate engineering student, successfully answered the question during the interview. A math graduate student first made an algebraic mistake while responding to the research question but then corrected the mistake during the interview. The rest of the participating students made mistakes during the interviews and faced difficulties while responding to the research question. The main difficulty of the participants appeared to be the algebraic mistakes throughout the calculations. Other mistakes included limit and power series calculations.

APOS theory classification participants resulted in the following:

- 16 out of 17 participants at the Action level,
- 7 out of 16 participants at the Process level,
- 4 out of 9 at the Object level,
- 3 out of 4 at the Schema level.

Triad classification of the participants resulted in the following:

- 10 out of 17 participants at the Intra stage,
- 4 out of 17 participants at the Inter stage,
- 3 participants at the Trans stage.

Riemann integral has an important place in engineering and mathematics education therefore it is important to understand students' knowledge by using a measure. APOS theory with Triad classification is applied as a measure to scale student responses to the research question in the current work. One other importance of the research findings is to understand students' theoretical mathematics knowledge. The findings presented in this work can guide the educators in high schools and universities to pay attention to the missing concepts of the participants. This study suggests mathematics professors to break down the concepts in a concept (e.g. limit definition of function integral) into sub-concepts (e.g. functions, limits, summing terms, derivatives etc.) and re-teach these sub-concepts during the concept application with additional material.

The results of this study indicated graduate and senior undergraduate engineering and mathematics students' weak algebraic ability while solving a Riemann integral question. The results displayed in this work can be particularly useful for professors to acknowledge the challenges that students can face and at what stages mathematics learners face difficulties while solving an integral question. We invite other researchers to investigate undergraduate students' integral knowledge.

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