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Evaluation of Mathematical Building Blocks Impacting STEM Majors' Ability to Solve Conceptual Power Series Questions

Dr. Emre Tokgoz, SUNY - Farmingdale State College

Emre Tokgoz is a faculty of Department of Computer Security at SUNY - Farmingdale. His research interests in STEM education include understanding and proposing improvement ideas for advanding undergraduate and graduate students' conceptual mathematics, engineering, computing, and cybersecurity knowledge.

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[1]Emre Tokgöz

tokgoze@farmingdale.edu

State University of New York, Farmingdale, New York, 11779, USA

Abstract. Success at university-level calculus and more advanced concepts require extensive time and effort due to the building blocks of the associated sub-concepts. Critical thinking is an essential part of demonstrating calculus knowledge in STEM fields to make a connection between the theory and practice. Establishing such a connection requires training learner's mind over time to develop a well-established theoretical background. In this work, Conceptual Learning Distribution (CLD) is introduced for the first time that can be used as a pedagogical methodology to determine a distribution for students' conceptual progressive understanding and it can also be used by instructors to measure K20 students' accumulating mathematics knowledge during the conceptual knowledge progression over time. CLD is applied to measure 18 STEM students' calculus sub-conceptual knowledge that relate to power series expansion of functions. These results attained for CLD will be compared with the results attained by the application of Action-Process-Object-Schema (APOS) theory.

Key Words. Calculus Education; Conceptual Learning Distribution; APOS Theory; STEM education

1. Introduction

Solving power series questions in calculus requires demonstration of the associated sub-conceptual knowledge and the ability to progress from one content to another for deriving the desired outcome. In this building blocks of calculus concepts, furthering algebraic question solutions is a process of advancing from one sub-concept to another. For instance, a student who is trying to solve a question that requires to determine the derivative of a function's power series term-by-term would need to know how to apply derivative formulas term-by-term, be able to simplify and calculate algebraic expressions, and demonstrate cognitive ability to design a solution that integrates sub-concepts simultaneously. Failing to advance the solution in one of the sub-concepts as a part of a traditional paper-pencil solution would be a failure in determination of the correct result. The goal of this research is to investigate the impact of sub-concepts that influences STEM students' responses to a power series related research question. This investigation requires to focus on the building blocks of several concepts to determine a solution to the power series research question displayed in Figure 1 by using a function graphing research question. Participating students' understanding of concepts such as algebraic and derivative calculations, power series, infinity, factorials, and functions' equality are investigated through these two research questions.

Q. Is it a true statement to say

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty}\frac{x^n}{n!}\right)=\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

Please explain your answer.

Figure 1. The power series research question.

Written responses of 18 senior undergraduate and graduate STEM students' responses to this research question are initially collected after attaining Institutional Review Board (IRB) approval. A follow-up video recorded interview with each participant is conducted upon the written response collection and all participants are compensated money for their participation to both interviews. The data is analyzed by introducing a new evaluation method called *Conceptual Learning Distribution* (CLD) along with the application of APOS theory by the author on the same data set. Qualitative (participants' interview responses) and quantitative (statistics used after applying APOS theory) results are evaluated in this work by using the written questionnaire and video recorded interview responses. STEM educators and researchers can benefit from the results attained from this work by adapting the CLD as a method of student performance evaluation and apply it as a research methodology for determining their comprehension levels of mathematical concepts.

1.2 Conceptual Learning Distribution Framework

The typical approach in pedagogical research of mathematics is the introduction of a statistical distribution fitting to a set of empirical data to be able to quantitatively measure outcomes [1]. In such an approach, a researcher can extract meaningful outcomes from the results attained while the real meaning of the sub-content and content knowledge of the students may not be tracked due to the transition from the data collected into the mathematical distribution. The purpose of CLD is to recognize pedagogical differences in sub-conceptual dimensions of a concept that need to be observed to be able to have a broad understanding of learners' in-depth content knowledge. This approach would allow the educators and researchers to determine and mind the real sub-conceptual gaps that students have while learning a concept to be able to act upon them. For instance, an educator may recognize that CLD is indicating the need for improving the limit knowledge of a group of learners during the education of limit definition of derivative therefore the educator can quickly attempt to recover this gap.

A similar statistical distribution approach to the development of CLD is followed in [2] with the exception that a statistical distribution was fit to the responses of mathematics secondary teachers to a questionnaire. While such a model can help with determining the "opposites" conceptualized as a "teacher-centered" versus a "student-centered" orientation towards teaching mathematics, it doesn't help with the identification of depth in the missing sub-conceptualization.

CLD is a distribution that can demonstrate the success of a group of students on two or more dimensions. The input of a CLD in two dimensions can be considered as the participants' conceptual comprehension progression to determine a solution while the output can be viewed as either the number of students or the percentage of students with success on the sub-concepts. For instance, in the case when CLD would be applied to evaluate student responses to find the derivative of a function by using the limit definition of derivative, the input axis can consist of the following sub-concepts that take place in the definition:

- ✓ Basic addition, subtraction, division, and multiplication of numbers
- ✓ Function factorization
- ✓ Functional translation
- ✓ Subtraction and division of functions
- ✓ Limit of functions
- ✓ Function simplification

Three-dimensional (3D) version of CLD can be viewed as the two-dimensional CLD with the third dimension being the level of student in the K20 categorization. The third dimension is still a continuous variable due to time being considered within this categorization. For instance, students at K16, K17, and K18 line up back-to-back on the axis of the third dimension. This axis has finite nature due to the limitation of schooling. Figure 2 below demonstrates this 3D view.

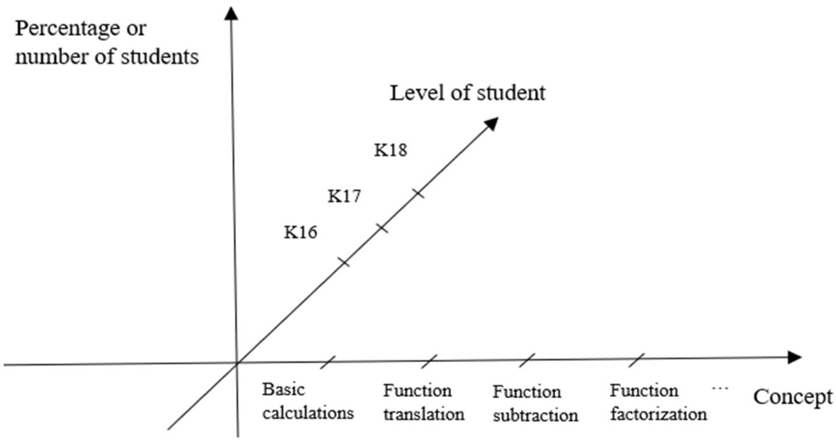

Figure 2. Example of a 3D CLD attained for algebraic derivative calculations.

1.3 Action, Process, Object, and Schema (APOS) Theory

APOS theory is applied in several studies to understand sub-conceptual knowledge of STEM students based on a variety of calculus sub-concepts through suggestions to the educators [3]-[18], [29]-[32]. Applications of this theory to evaluate sub-conceptual understanding of STEM learners included functions, derivatives, limits, Riemann sums, and the use of technology for advancing calculus education. APOS theory application in the pedagogical literature for understanding students' mental construction of power series concept is limited [13]. Several STEM majors' advanced level calculus conceptual comprehension (such as the concept of power series expansion of functions) that also relates to numerical methods is critical in advanced level of mental ability to solve more advanced problems. Designing the curriculum right with the right instructional techniques that covers missing sub-concepts is the key to success for helping students in learning calculus. This approach may help students to develop a better conceptual understanding with an improved ability to mentally construct concepts and their sub-concepts in calculus, and CLD can help in accomplishing these tasks.

Action, Object and Process (APO) idea is used in mathematics education as a part of the undergraduate curriculum in [19] for the first time during a study on students' conceptual view of the function concept. APO is extended to Action, Process, Object and Schema theory (called APOS theory) in [21] to understand students' function knowledge. APOS theory is explained as the combined knowledge of a student in a specific subject based on Piaget`s philosophy. APOS theory was designed in [22] as follows:

- *An action is a transformation of objects perceived by the individual as essentially external and as requiring, either explicitly or from memory, step-by-step instructions on how to perform the operation...*
- *When an action is repeated and the individual reflects upon it, he or she can make an internal mental construction called a process which the individual can think of as performing the same kind of action, but no longer with the need of external stimuli...*
- *An object is constructed from a process when the individual becomes aware of the process as a totality and realizes that transformations can act on it...*
- *A schema is an ... individuals' collection of actions, processes, objects, and other schemas which are linked by some general principles to form a framework in individual's mind...*

APOS theory on area calculations are explained in [23] as follows:

In Calculus: *Actions are needed to construct an estimate of the definite integral as the area under a curve: for example, in dividing an interval into specific subintervals of a given size, constructing a rectangle under the curve for each subinterval, calculating the area of each rectangle, and calculating the sum of the areas of the rectangles.*

…The area under the curve for a function on a closed interval is the limit of Riemann sums—an Action applied to the Riemann sum Process. In order to determine the existence of this limit and/or to calculate its value, the student needs to encapsulate the Riemann sum Process into an Object.

The development of the individual schemas can also be accomplished by using the triad classification in APOS theory; a progression of three stages proposed in [28]. Triad stages Intra, Inter, and Trans are used in [5] to investigate how STEM students' ability to relate integral to area under the curve. The APOS theory classification is determined to be insufficient therefore they included the schema development idea. Intra-Inter-Trans level categorization is one way of analysis of the post interview student responses based on a three-level triad classification [30]:

- **Intra-level:** Responses reflected only elementary level of sub-conceptual knowledge with mistakes made in two or more analytical properties on two or more intervals. This level of students couldn't demonstrate correlated calculus conceptual knowledge indicating that they cannot apply two or more calculus sub-concepts simultaneously.
- **Inter-level:** Participants were able to apply one or two calculus sub-concepts correctly on several places but not at all places. The responses in this category indicate application mistakes or not ability to respond to the question due to the lack of conceptual knowledge, possibly for one or more analytical properties on a certain interval.
- **Trans-level:** The participants in this category made no mistakes in the application of the analytical properties throughout the entire domain of the question.

Evaluation of the results attained in [3]-[18] indicated a variety of APOS classification of the participants depending on the research question for analysis of integral, series, function, limit, derivative, and asymptote knowledge. Application of APOS theory in other areas of interest included vector space concept-related observations in [24], mean, standard deviation, and the central limit theorem related conceptual understanding in [25], and observing students' ability to construct and develop two-variable functions in [26] and [27].

2. CLD and APOS Theory Applications on Empirical Data

In this section we will demonstrate the applications of CLD and APOS theory together on the empirical data collected from the participants. STEM students' comprehension of several sub-concepts taking place in a concept can be easily scaled by the design of a CLD. We initially observed participants' responses to the following sub-concepts by using the research question displayed in Figure 3 below prior to the power series related question displayed in Figure 1.

- Limit application on horizontal axis
- Limit application on vertical axis
- First derivative
- Second derivative

- Vertical asymptote
- Horizontal asymptote

The questions chosen in this research aimed to understand the participants' sub-conceptual knowledge that relates to power series.

Please draw a graph of a function that verifies all of the given information below. Write the necessary values on the coordinate axis and explain the details if you think they are necessary.

$$\begin{aligned}
\lim_{x\to-\infty} f(x) &= 0,\ \lim_{x\to\infty} f(x) = 0, \\
\lim_{x\to-3^-} f(x) &= -\infty,\ \lim_{x\to 2^+} f(x) = \infty, \\
\textit{Vertical Asymptotes at } x &= -3 \textit{ and } x = 2, \\
\textit{Horizontal Asymptote at } x &= 0 \\
f'(-2) &< 0, f'(1) < 0, \\
f''(x) &< 0 \text{ when } x < -3, \\
f''(x) &> 0 \text{ when } x > 2, \\
f''(c) &= 0 \text{ for a } x = c \text{ such that } -1 < c < 1
\end{aligned}$$

Figure 3. A research question used for sub-concept knowledge investigation of limit, derivative, and asymptote.

In this research, the CLD is initially hypothesized to have some form of a linear or exponential distribution; however, as Figure 4 indicates, first derivative (1st Diff) responses of the participants to this question was the key factor that made a difference from the hypothesized CLD structure. The following can be seen on Figure 4:

- ✓ Vertical asymptote (VA) and second derivative (2nd Diff) had the same number of correct responses from 18 participants.
- ✓ Horizontal axis knowledge (H. axis knowledge) had 17 correct responses.
- ✓ Vertical axis knowledge (V. axis knowledge) and horizontal asymptote knowledge had 16 correct responses.
- ✓ 1st Diff had 3 correct responses.

The line displayed in Figure 4 is the linear CLD that would be expected to be attained from the participants for a proper distribution of the participants' knowledge of sub-concepts; In this case, a CLD with a maximum R-square value of 92.57% is attained in the case of a total of 13 participants correctly responding to 1st Diff related questions. The linear model displayed is a very strong fit to the data.

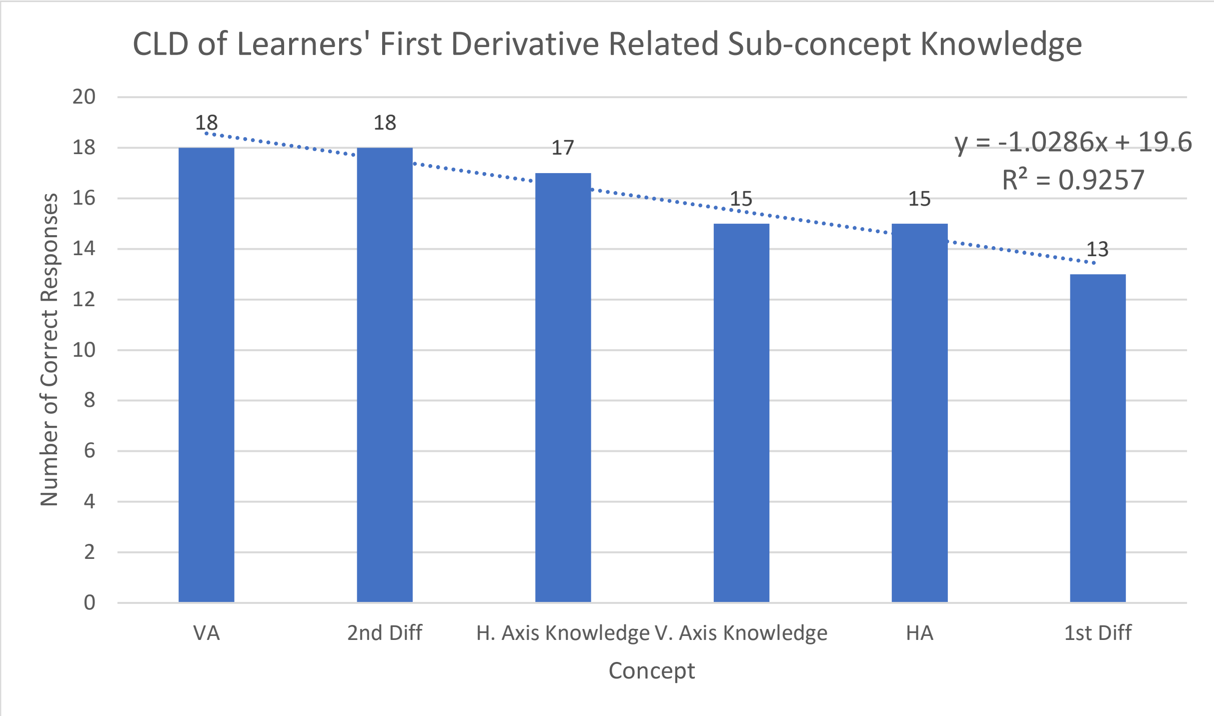

Figure 4. Expected CLD distribution of the participants for the research question in Figure 2.

The APOS classification's distribution displayed in Figure 5 is based on the Intra-Inter-Trans classification outlined in Section 3 [30]. As pointed out in [30] for this classification, a student is placed in the intra-level if the second derivative and the asymptote information are not applied correctly on two or more intervals. This is a result of participant's confusion by the union or intersection of other intervals and the failure to interpret every analytical property independently one at a time. If there is only one analytical property application mistake, such as the first derivative information on a certain interval that cannot consist of the union of independent intervals, then the response is categorized as inter-level. Students' trans-level triad classification is based on their ability to answer the question correctly in the entire domain. APOS distribution displayed in the figure solely depends on the mixed sub-conceptual knowledge of the participants that does not allow to back-track each sub-content knowledge of the students. The benefit of such an approach is to determine the correlated cognitive comprehension of students' and their content knowledge without individually inspecting sub-concepts. The APOS classification of the data has an exponential distribution nature with an R-square value of 99.38%, therefore we can conclude that a strong variation in the correlated sub-concept knowledge of the participants exists due to gaps exist between Intra, Inter, and Trans levels.

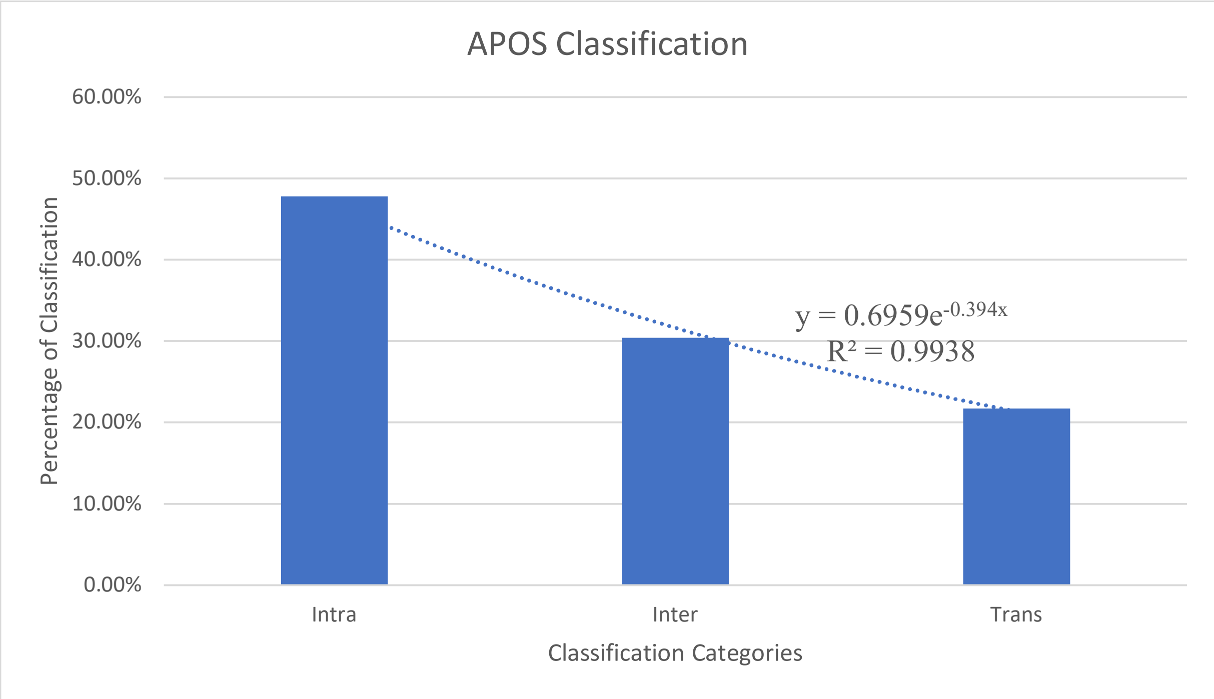

Figure 5. APOS classification's linear distribution.

3. Participants' Power Series Expansion of Function Understanding

Participants' qualitative analysis of the power series research question displayed in Figure 1 is covered in this section in conjunction with the qualitative and quantitative analysis of participants' responses to the research question displayed in Figure 3 in prior sections. The qualitative analysis of student responses to the power series question during the interview phase helped to understand students' step-by-step sub-conceptual knowledge construction. Each participant was initially asked to write out several terms of the power series and calculate the derivative of several terms of the power series to demonstrate their derivative sub-conceptual knowledge. In alignment with the results attained for the research question in Figure 1, participants had hard time to initiate basic polynomial derivative calculations. This result by itself indicated that the primary weakness of the participants is the first derivative knowledge as they had hard time to calculate the first derivative of each consecutive term in the power series.

4. Conclusions & Future Work

Critical thinking is an essential part of demonstrating calculus knowledge in STEM fields to make a connection between the theory and practice. Understanding learners' ability to construct the building blocks of calculus sub-concepts for success in university-level calculus and more advanced concepts is essential for educators to develop better curriculums. To understand detailed cognitive sub-conceptual construction of STEM students better, research results are derived in this work based on the STEM students' responses to two research questions. Qualitative and quantitative data is collected from the participants after attaining IRB approval, and each participant is compensated money for their participation. CLD is introduced and used for the first time in this work for empirical data analysis along with the use of APOS theory for deriving the results. The analysis of the first research question that was related to derivative, asymptote, and limit understanding of STEM students indicated weak first derivative knowledge of the students, and this outcome was also supported by the results attained for the second research question that was related to power series expansion of functions. Table 1 below summarizes the results attained in this research:

Table 1. Triangulation and APOS classification summaries.

	Triangulation Classification						APOS Classification		
Sub-classifica	VA	2nd Diff	H. Axis	V. Axis	HA	1st Diff	Intra	Inter	Trans
Percentage	20.93%	20.93%	19.77%	17.44%	17.44%	3.49%	47.83%	30.43%	21.74%

The Principal Investigator of this research determined the CLD to be a useful tool to determine a distribution for students' conceptual progressive understanding, and it can also be used by instructors to measure K20 students' accumulating mathematics knowledge during the conceptual knowledge progression over time.

Educators and researchers can adopt CLD and benefit from this distribution in several ways:

- Structuring simple quizzes (such as multiple choice) that target different aspects of a concept; Such quizzes can be given to STEM students for determining their detailed understanding of each calculus sub-concept as they are completed. For instance, if the chain rule concept is covered then the educator can prepare a quiz that covers the associated sub-concepts to understand learners' ability of answering relevant concepts. The educator can use the results attained above to support missing sub-concept knowledge of the students with additional materials through flipped classroom approach.
- Researchers can use different calculus concepts to apply CLD to scale all three dimensions of the distribution shown in Figure 2.
- Additional dimensions can be added to the 3D CLD for more in-depth analysis of the STEM students by both educators and researchers.

The outcomes of this research indicated the importance of the design of the research or quiz questions to derive the associated research and educational outcomes therefore a variety of question types may have to be asked to students. CLD is found to be useful for understanding in-depth sub-conceptual calculus comprehension of students while APOS theory is helpful for learners' collective understanding of concepts based on the correlated sub-concepts. Educators and researchers are invited to further test the significance of CLD and investigate how much recovery of the sub-concepts is possible through the use of additional materials (such as quizzes mentioned above) by further analyzing STEM majors' progressing calculus comprehension.

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