Graphical Visualization for commutative Sequential Rotations

Alamgir A. Choudhury, Mitchel J. Keil, Jorge Rodriguez

Western Michigan University

Abstract

The analysis of rotational motion in articulated mechanisms, and the subsequent design of a system involving sequential rotations is often a tedious task. Thus, simply comprehending the rotational motion required to orient a rigid link or associated reference frame can be quite challenging for students as well as practicing designers.

The introduction of a special chain rule allows users to treat finite rotations as vectors i.e. the option to apply the commutative law to the sequence of a specified set of rotations. This chain rule consists of a lemma for forward and backward propagation of the prescribed rotations. The proposed methodology has been demonstrated by using a commercial graphics package to visualize the initial and final orientations of a rigid body following three identical finite rotations composed in three different sequences. The unique final orientation of the link in each of the cases confirms the commutative nature of the constrained rotations.

Using this method, the consequence of sequential finite rotations becomes easy to comprehend. Therefore, it can be used as a tool for learning the concepts associated with finite rotations as well as in kinematic analysis involving sequential rotations.

1. Introduction

A mathematical model for design and analysis of interconnected mechanical bodies must deal with both translational and rotational motion. In kinematics, vectors are often used to analyze motion when commutativity is applicable to the motion sequence. Sequential finite rotations are encountered in the motion of many bodies such as robotic links. Finite rotations are not commutative [1]. Therefore, in treatment of such motion, one has to ensure that the method does not contradict this characteristic of rotational motion. In design and analysis involving complex rotational motion of interconnected mechanical links, students and practicing professionals in the field need to have a clear comprehension of this problem.

Due to the lack of commutativity of finite rotations, in order to achieve a desired position and orientation as a mechanical link undergoes a sequence of rotations, the order of rotation [2] is apriori. The noncommutativity of finite rotations has long been proven

[1]. Therefore, a method dealing with the sequential rotational motion needs to ensure that the commutativity of the rotations exists prior to achieving a predetermined motion so that the final outcomes will be independent of the sequence of rotations.

In this paper we address the problem with illustrations of rotational motion of a body for graphical visualization. A specially developed chain rule is utilized to constrain the rotational motion so that the commutativity of rotations can be ensured. Based on this constraint a series of finite rotations is shown to be independent of rotation order. Therefore, the constraint condition allows non-commutative rotations to behave like commutative rotations. Finally, for visual comprehension, we apply this condition on a graphical problem and establish commutativity of a series of rotations subject to the constraint.

2. Classic rotation problem

Here we present two sets of rotational problems to show that the finite rotations are not commutative; therefore, generally different sequence of rotations of a rigid body will produce different end results.

2.a Rigid body rotations

Considering the classic rigid body rotation problem [2], if the rotational matrices are multiplied in a different order, it will produce different rotation matrices representing the final orientations. Given a set of Eulerian angles and defining the orientation of a rigid body reference frame as shown in Figure 1, the matrices for the rotations α_1 and α_2, about x and y axes are given by

$$\mathbf{R}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_1 & s_1 \\ 0 & -s_1 & c_1 \end{bmatrix} \quad \text{and} \quad \mathbf{R}_2 = \begin{bmatrix} c_2 & 0 & -s_2 \\ 0 & 1 & 0 \\ s_2 & 0 & c_2 \end{bmatrix}$$

Where, c_1 = Cos α_1 and s_2 = Sin α_2 etc.

It has been shown that [3] by performing the two rotations α_1 first and then α_2, the resulting rotation matrix $\mathbf{R}_{1\to 2}$ is given by

$$\mathbf{R}_{1\to 2} = \mathbf{R}_1\,\mathbf{R}_2 = \begin{bmatrix} c_2 & s_1 s_2 & -c_1 s_2 \\ 0 & c_1 & s_1 \\ s_2 & -s_1 c_2 & c_1 c_2 \end{bmatrix} \quad \ldots \quad (1)$$

Reversing the sequence of rotations, i.e. α_2 first and then α_1, yields the transformation matrix $\mathbf{R}_{2\text{-}1}$, where

$$\mathbf{R}_{2\rightarrow 1} = \mathbf{R}_2\,\mathbf{R}_1 = \begin{bmatrix} c_2 & 0 & -s_2 \\ s_1 s_2 & c_1 & s_1 c_2 \\ c_1 s_2 & -s_1 & c_1 c_2 \end{bmatrix} \qquad \dots \qquad (2)$$

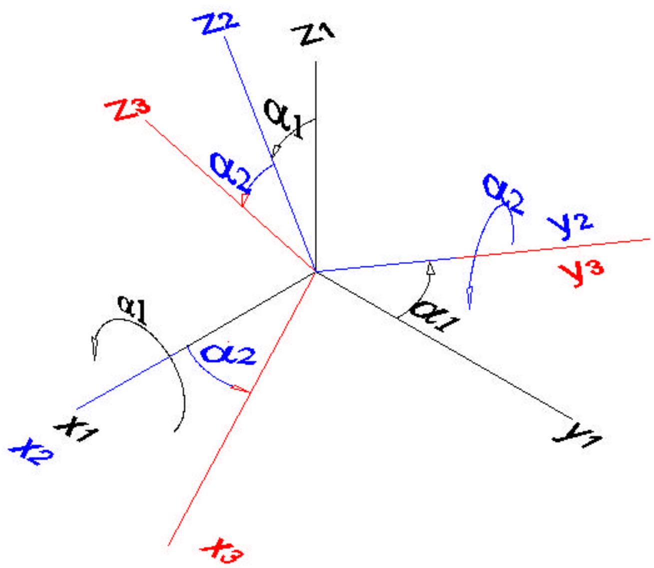

Figure 1. Rotations of reference frames

In general, $\mathbf{R}_{1\rightarrow 2} \neq \mathbf{R}_{2\rightarrow 1}$. Here, the same rotations about the rigid-body axes in reverse sequence produce a different result. Therefore, we may conclude that if the rotation angles are defined by a set of the Eulerian angles based on the reference frame fixed to a rotating rigid-body, finite rotations are not commutative in nature.

2.b Graphical visualization of rigid body rotation

The non-commutative nature of finite rotation exists whenever the rotations are defined based on a reference frame fixed to the rotating rigid-body. Comprehension of this phenomenon is more difficult as the rigid-body rotation of a complex system is studied. For graphical visualization of the problem, in Figure 2, we are presenting a series of three finite rotations of a rigid body in three different orders. The rotations are about the x, y, and z axes of a reference frame fixed to the rigid-body. Starting from the initial orientation a, the rigid body undergoes three 90^0 rotations about current x, y, and z axes respectively. The subsequent orientations are represented in a1, a2, and a3. Next the same body undergoes the 90^0 rotations about the axes z, x, and y respectively producing a different final orientation shown in b3. Finally, we apply the 90^0 rotations in a reverse sequence, i.e. about the axes z, y, and x respectively. This order of rotations produces the final orientation shown in c3. These rotations in different sequences and the consequent final orientations of the body again yield the conclusion that to achieve a specific final result, the order of rotations of the body cannot be changed. Therefore, this series of 90^0 rotations about the axes fixed with the body are not commutative.

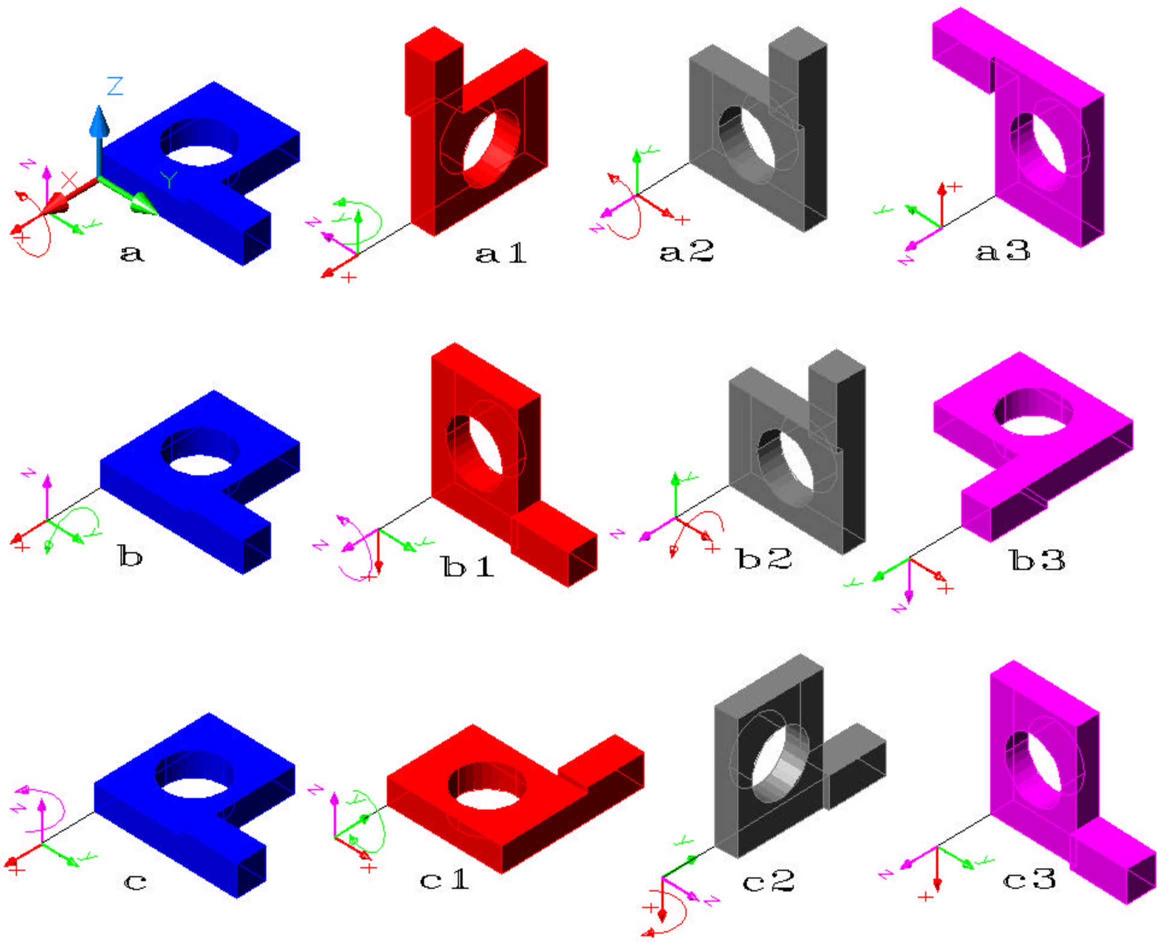

Figure 2. Unconstrained rotations of a rigid body in different sequence

3. Constraint for commutativity of finite rotations

To circumvent the problem of commutativity of finite rotational motions, the following constraint is introduced as an integral part of the modified Eulerian angle based kinematical notation [4, 5].

Forward constraint condition: Consider a system of reference frames forming a chain from the *0*th to *n*th frame, each of which defines an Eulerian angle of rotation. To affect the commutativity of a series of finite rotations, the rotation about an axis of the *i*th frame affects all other frames from *(i+1)*th to *n*th reference frames in a forward progression of rigid body rotation. Conversely, a rotation does not affect the orientation of the reference frames from *(i-1)*th to the *0*th (fixed) frame in the backward progression.

4. Commutativity of sequence of rotations

Utilizing the above constraint condition, next we re-examine the cases of rigid body rotation problem presented in section 2a and 2b.

4.a Commutativity of modified Eulerian rotations

In order to show the commutativity of the Eulerian angles, the proposed constraint condition is applied to the classic rotation problem presented in section 2.a. We use the following reference frame and corresponding set of modified Eulerian angles [3] as shown in Figure 3. According to this notation, the reference frame $x_i y_i z_i$ rotates in a rigid-body fashion with respect to its earlier *(i-1)* orientation. The nutation angle θ_i, is measured from z_{i-1} to z_i axis about the $\mathbf{B}_i \mathbf{B}_i^{/}$ axis which is a perpendicular to the projection of z_i axis on $x_{i-1}y_{i-1}$ plane. The precession angle ϕ_i is measured form the x_{i-1} axis to the same projection and ψ_i represent the spin angle about the z_i axis.

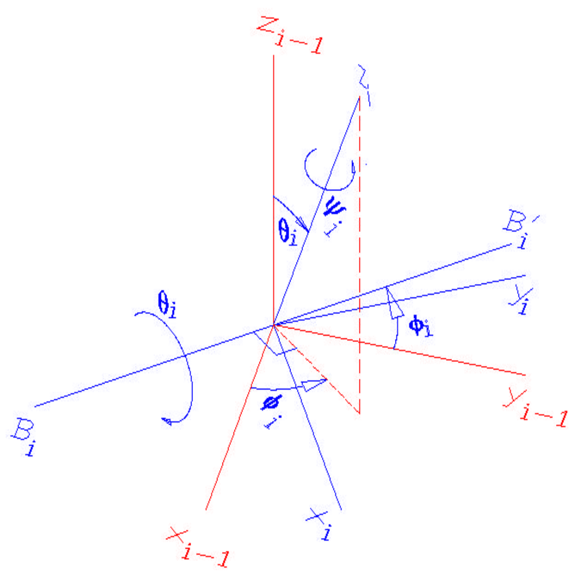

Figure 3. Modified Eulerian angles

Based on this definition of the modified Eulerian angle and the above constraint condition, a vector transformation from the *i*th to the *(i-1)*th reference frame is represented by the *i*th transformation matrix [3] shown below:

$$\mathbf{T}_i = \begin{bmatrix} c_1 c_2 c_{2-3} + s_2 s_{2-3} & c_1 s_2 c_{2-3} - c_2 s_{2-3} & -s_2 c_{2-3} \\ c_1 c_2 s_{2-3} - s_2 c_{2-3} & c_1 s_2 s_{2-3} + c_2 c_{2-3} & -s_1 s_{2-3} \\ s_1 c_2 & s_1 s_2 & c_1 \end{bmatrix} \quad \ldots(3)$$

where, $c_1 = \text{Cos}\ \theta_i$, $s_2 = \text{Sin}\ \phi_i$, $c_{2-3} = \text{Cos}\ (\phi_i - \psi_i)$, etc.

This transformation matrix can now be used to formulate the classic rotation problem stated in section 2.a. For this purpose, let $\theta_1 = -\alpha_1$, $\phi_1 = \pi/2$, $\psi_1 = 0$, $\theta_2 = \alpha_2$, $\phi_2 = 0$ and $\psi_2 = 0$ in the transformation matrix $\mathbf{T}_i$. This yields same rotation matrices as in section 2.a.

$$\mathbf{R}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_1 & s_1 \\ 0 & -s_1 & c_1 \end{bmatrix} \quad \text{and} \quad \mathbf{R}_2 = \begin{bmatrix} c_2 & 0 & -s_2 \\ 0 & 1 & 0 \\ s_2 & 0 & c_2 \end{bmatrix} \quad \ldots \quad (4)$$

Due to orthogonal nature of the transformation matrix $\mathbf{T}_i$, the constraint condition is an integral part of the system when the matrices are multiplied in the order of $\mathbf{T}_i\mathbf{T}_{i-1}\mathbf{T}_{i-2}\ldots$ etc. Therefore, we multiply the rotational matrices in the order of $\mathbf{R}_2$ and $\mathbf{R}_1$ to produce the final rotational matrix

$$\mathbf{T}_{2\to 1} = \mathbf{R}_2\,\mathbf{R}_1 = \begin{bmatrix} c_2 & 0 & -s_2 \\ s_1 s_2 & c_1 & s_1 c_2 \\ c_1 s_2 & -s_1 & c_1 c_2 \end{bmatrix}$$

Now let us apply the constraint condition in the classic problem stated in section 2.a. In the chain of reference frames $x_1y_1z_1 \to x_2y_2z_2$, rotation α_2 does not affect the earlier axis of rotation x_1 but α_1 affects the axis of rotation y_2. The corresponding constrained rotational matrices become

$$\mathbf{R}_{1C} = \begin{bmatrix} c_2^2 + c_1 s_2^2 & s_1 s_1 & c_2 s_2 - c_1 c_2 s_2 \\ -s_1 s_2 & c_1 & s_1 c_2 \\ c_2 s_2 - c_1 c_2 s_2 & -s_1 c_2 & c_1 c_2^2 + s_2^2 \end{bmatrix} \quad \text{and}$$

$$\mathbf{R}_{2C} = \begin{bmatrix} c_2 & 0 & -s_2 \\ 0 & 1 & 0 \\ s_2 & 0 & c_2 \end{bmatrix}$$

Multiplication of these newly developed matrices in forward sequence produces the final transformation matrix

$$\mathbf{T}_{1\to 2} = \mathbf{R}_{1C}\,\mathbf{R}_{2C} = \begin{bmatrix} c_2 & 0 & -s_2 \\ s_1 s_2 & c_1 & s_1 c_2 \\ c_1 s_2 & -s_1 & c_1 c_2 \end{bmatrix} \quad \ldots \quad (5)$$

Comparing equation (4) and (5), $\mathbf{T}_{1\to 2} = \mathbf{T}_{2\to 1}$. Therefore the commutativity exists for the two sequences of rotations. The constraint is applicable for any number of rotations and will produce unique final outcomes irrespective of the rotation sequence.

4.b Graphical visualization of constraint finite rotations

In this section we apply the constraint condition on the rigid body rotation problem presented in section 2.b. The resulting orientations of the body are shown in Figure 4.

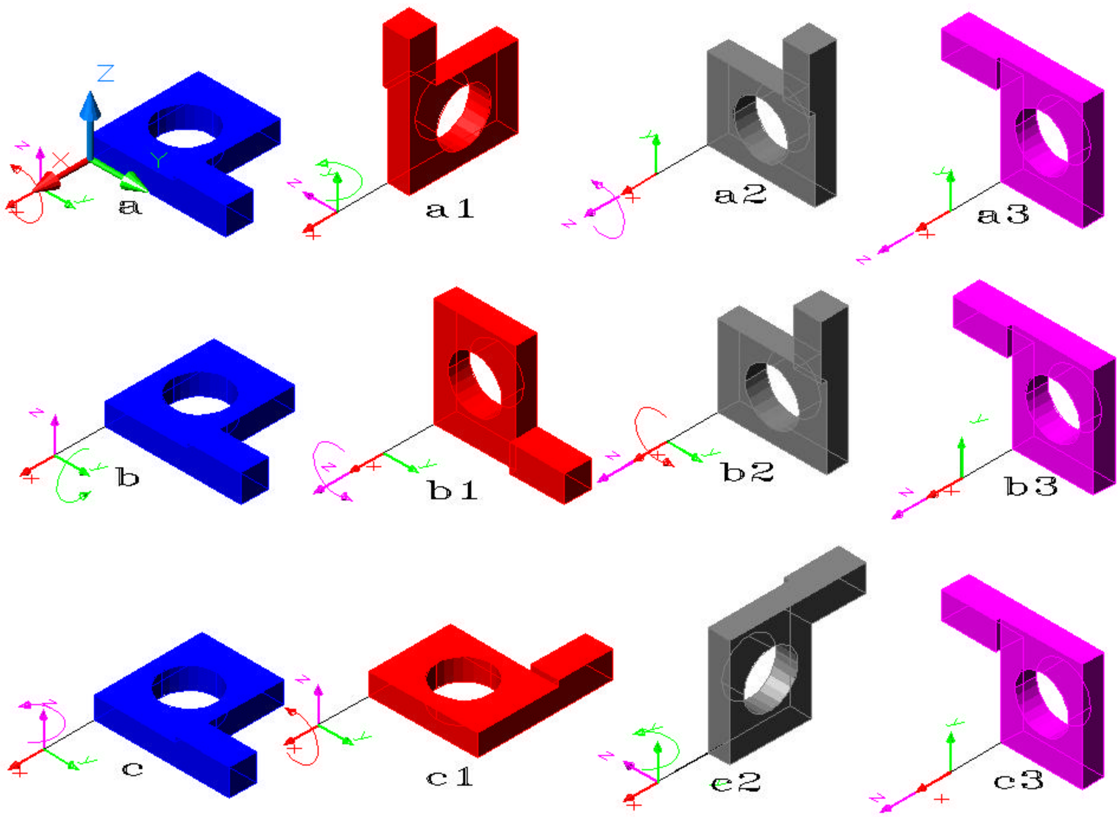

Figure 4. Constrained rotations of a rigid body in different sequence

According to the constraint condition, first a chain of reference frames is established in the progression of x $\rightarrow$ y $\rightarrow$ z. As the rotation about x axes is applied, it is transmitted to the current y and z axis as a rigid-body rotation. Similarly, the rotation about y axis is transmitted to z axis but not the x axis and rotation about z axis does not affect x or y axis. The 90^0 rotations of the block about current x, y, and z axis subject to the constraint condition are shown in case a1, a2, and a3. Next we apply the rotations about the current y, z, and x axis respectively subject to the same constraint. The resulting orientations are shown in case b1, b2, and b3. Similarly the rotations about the z, x, and y axes result orientations c1, c2, and c3. Notice that irrespective of the sequence of orientations, in all three cases, final orientations of the block shown by a3, b3, and c3 are identical. Though the principle is applicable for any number of rotations, such simple examples can be useful for visual presentation of the concept of finite rotation and its general non-commutative nature to the undergraduate kinematics students.

5. Conclusion

The visualization and comprehension of sequential finite rotations can be challenging for graphics and mechanics students as well as practitioners. The methodology is formulated as a chain rule that results in a transformation matrix for forward and backward propagation of a series of rotations. The tool presented here augments the learning process through the application of the commutative law to a sequence of specified rotations. It has the potential to assist in visualizing and comprehending the concepts related to finite sequential rotations.

Bibliography

1. Rosenberg, R. M., **Analytical dynamics of discrete systems**, Plenum Press, New York and London, 1977.
2. Ginsberg, J. and Genin, J., **Dynamics**, John Wiley & Sons, Inc., New York, 1984.
3. Choudhury, A. and Genin, J., "Kinematics of an n-degree-of-freedom multi-link robotic system", **Int. J. of Robotics Research**, 8(6), 1989.
4. Choudhury, A., "Modified Eulerian angle principle for the kinematics and dynamics of an n degree of freedom open chain mechanical system", doctoral diss., New Mexico State University, Las Cruces, NM, 1990.
5. Hong, J. and Genin, J., "A discussion on commutativity of finite rotations in robotic design", **Int. J. of Robotics and Automation**, Vol. 15, No. 3, 2000.

ALAMGIR CHOUDHURY
Alamgir A. Choudhury is an Assistant Professor of industrial and manufacturing engineering at Western Michigan University, Kalamazoo, Michigan. He earned BS in mechanical engineering from BUET (Dhaka), MS and PhD from NMSU (Las Cruces). His interest includes computer applications in curriculum, mechanics, MCAE, instrumentation & control and fluid power. He is also a Registered Professional Engineer in the State of Ohio and affiliated with **ASME, ASEE, SME** and **TAP**.

MITCHEL J. KEIL
Mitchel J. Keil is an Assistant Professor in the Industrial and Manufacturing Department at Western Michigan University. He received his Ph.D. from Virginia Polytechnic Institute (VPI&SU), his MS from Florida Atlantic University, and his BSME from VPI&SU. His areas of expertise are mechanisms, computer graphics, CAD/CAM/CAE, and vibrations. He is a registered professional engineer in the State of Florida. He has had industrial work experience at Litton Poly-Scientific, Motorola, and Bethlehem Steel Corp.

JORGE RODRIGUEZ
Jorge Rodriguez is an Assistant Professor in the Department of Industrial and Manufacturing Engineering and Research Associate of the Human Performance Institute at Western Michigan University. He received his Ph.D. in Mechanical Engineering from University of Wisconsin-Madison and received an M.B.A. from Rutgers University in Piscataway, NJ. Dr. Rodriguez teaches courses in Computer-Aided Design and Manufacturing, Mechanical Design, Biomechanics and Finite Element Analysis. His research is in the field of computers in engineering, with particular emphasis to machine design, modeling and biomechanics.