Integral Methods in Solving Governing PDEs in the Undergraduate Heat Transfer Course

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Integral methods in solving governing partial differential equations in the undergraduate heat transfer course

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Although undergraduate students in mechanical engineering are required to take a course in ordinary differential equations (ODE), they are not obligated to acquire formal instruction in partial differential equations (PDE) before graduation. This may be why very popular and widely used undergraduate heat transfer textbooks such as those by Incropera\textsuperscript{1}, Kreith\textsuperscript{2} use finite difference method as their main solution approach to solve heat conduction problems in transient and steady 2-dimensional cases. With broad range of topics covered in a single undergraduate heat transfer course, the current authors are of the same belief that there would be no room and time to teach them separation of variable and transform techniques for solving the governing PDE equations. Nonetheless, another fresh approach besides finite difference formulation of steady 2-dimensional and unsteady heat conduction problems is the use of integral methods. The integral method of solving 2-D and transient heat conduction, though is extensively covered in advanced conduction textbooks such as the ones by Arpaci\textsuperscript{3} and Özışık\textsuperscript{4}, yet this powerful, promising approach seems to be almost totally missing from undergraduate engineering education, except for a treatment of forced and natural heat convection problems.

To illustrate the effectiveness and usefulness of the integral method, the authors will use the method to solve two heat conduction problems. One problem involves a 2-D steady conduction case, and the other one deals with transient heat conduction. The first problem is taken from Incropera's textbook, which uses only separation of variables and finite difference methods to solve it. By contrast, it will be seen in this paper that the integral method offers a more effective solution process for this problem, with much less mathematical manipulation and more accurate numerical results in comparison with the solution to this problem obtained by finite difference methods. For the sake of brevity, the transient problem considered in this article is chosen from the volumetric rise type; but the method is applicable to penetration kind problems as well. See, for instance, Arpaci's text.

Before discussing the two heat conduction problems in question, it is worthwhile to mention a few words regarding the academic value of the integral method approach. To implement this
method one has to arrive, first, at the partial differential equation of heat conduction from the first law of thermodynamics, then state the boundary and initial conditions associated with the problem at hand. Next the derived partial differential equation of heat conduction is integrated over space, and an approximate profile for the temperature composed of product functions, each of which depends on only one independent variable, is selected. One of these functions, known as parameter function, is left unspecified, whereas the second product function is taken as a polynomial or circular function of the space direction. This polynomial or circular function is forced to satisfy the temperature boundary conditions in that independent direction. The selected product profile is then substituted into the integrated partial differential equation, which renders an ODE subject to boundary conditions for the case of 2-D steady problems, or initial condition for the unsteady problems.

As it is evident from the above paragraph, the attempt made to implement this solution approach involves valuable educational objectives. First, students learn to derive the governing differential equation of conduction from the fundamental law of conservation of energy, and then they become well adapt at supplementing their governing equation with appropriate boundary and initial conditions. Moreover, to ensure uniqueness of solution to the problem, the ever-important role of boundary conditions is reinforced once again by getting students to satisfy the boundary conditions in the space direction of the polynomial/circular function. The last part of this undertaking also serves to refresh students' skills in solving ordinary differential equations.

At this point, the integral method will be presented in terms of the aforementioned two problems. The first problem is taken from Incropora's text; its statement is as follows:

A fireclay brick 1 m by 1 m on a side, is subject to a maintained temperature of 500 K on its three sides while the remaining surface is exposed to an airstream of 300 K and convective heat transfer coefficient of 10 $\frac{W}{m^2.K}$. Thermal conductivity of fireclay brick is $\frac{1}{m.K}$. Determine the (steady state) temperature distribution throughout the unit square.

![Figure 1: 2-D steady-state heat conduction with prescribed surface conditions](image-url)
Integral Solution

Selecting the coordinate system at the middle of the square, the governing partial differential equation after applying the first law of thermodynamics yields,

\[
\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \quad (0)
\]

\[
\frac{\partial T(0,y)}{\partial x} = 0 \quad T(L,y) = T_0 \quad (1)
\]

\[
T(x,L) = T_0 \quad k \frac{\partial T(x,-L)}{\partial y} = h[T(x,-L) - T_{air}] , \quad (2)
\]

where \( T(x,y) \) denotes the temperature field of the brick, and \( T_o = 500 \) K is the sides' temperature of the brick as illustrated in Figure 1. Assuming a new variable \( \theta(x,y) \) as

\[
\theta(x,y) = T(x,y) - T_{air} \quad (3)
\]

where \( T_{air} = 300 \) K. In terms of \( \theta \), the above differential equation together with its boundary conditions become:

\[
\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = 0 \quad (4)
\]

\[
\frac{\partial \theta(0,y)}{\partial x} = 0 \quad , \quad \theta(l,y) = T_0 - T_{air} = 200 \ K \quad (5)
\]

\[
\theta(x,L) = T_0 - T_{air} = 200 \ K \quad (6)
\]

\[
k \frac{\partial \theta(x,-L)}{\partial y} = h\theta(x,-L) \quad (7)
\]

Because of the linearity of the system, the principle of superposition is now employed, and the above problem is divided into two sub-problems as follow:

\[
\theta(x,y) = \theta_1(x,y) + \theta_2(x,y) \quad (8)
\]
where $\theta_1$ satisfies the following equations:

$$\frac{\partial^2 \theta_1}{\partial x^2} + \frac{\partial^2 \theta_1}{\partial y^2} = 0$$

(9)

$$\frac{\partial \theta_1(0,y)}{\partial x} = 0 \quad , \quad \theta_1(l,y) = 0$$

(10)

$$\theta_1(x,L) = T_0 - T_{air} = 200 \text{ K}$$

(11)

$$k \frac{\partial \theta_1(x,-L)}{\partial y} = h \theta_1(x,-L)$$

(12)

and $\theta_2$ satisfies this system of equations:

$$\frac{\partial^2 \theta_2}{\partial x^2} + \frac{\partial^2 \theta_2}{\partial y^2} = 0$$

(13)

$$\frac{\partial \theta_2(0,y)}{\partial x} = 0 \quad , \quad \theta_2(l,y) = 200 \text{ K}$$

(14)

$$\theta_2(x,L) = 0 \quad , \quad \frac{\partial \theta_2(x,-L)}{\partial y} = 0$$

(15)

Choosing $\theta$ relative to $T_{air}$ instead of $T_o$ and expressing it as in Eq. (8) give us one extra chance to illustrate how to use the integral method to solve the problem under consideration.

At this point, these two systems of differential equations are solved by integral methods. Starting with the system of Equations (9)-(12), Equation (9) is multiplied through by $dxdy$ and is integrated over the unit square to yield:

$$\int_0^l \int_{-L}^L \left( \frac{\partial^2 \theta_1}{\partial x^2} + \frac{\partial^2 \theta_1}{\partial y^2} \right) dxdy = 0$$

(16)

A product solution for $\theta_1$ is:

$$\theta_1(x,y) = X(x)Y(y)$$

(17)

Where function $X(.)$ is chosen as a circular function satisfying the homogeneous boundary conditions in the $x$-direction. Because of symmetry in the $x$ direction, $X(.)$ is an even function of $x$ and can be readily written as:

$$X(x) = A \cos \left( \frac{\pi x}{2l} \right)$$

(18)
Substituting (18) in (17), and the outcome in (16), and then carrying out the integration, one arrives at

\[ \int_{-L}^{L} \left( \frac{d^2 Y}{dy^2} - \left( \frac{\pi}{2l} \right)^2 Y \right) dy = 0, \]

which yields the following second order ordinary differential equation of

\[ \frac{d^2 Y}{dy^2} - \left( \frac{\pi}{2l} \right)^2 Y = 0, \quad (19) \]

subject to the boundary conditions in the y direction which now takes the form:

\[ A \cos\left( \frac{\pi x}{2l} \right) Y(L) = 200 \]

and

\[ k \frac{dY(-L)}{dy} = h(Y(-L)). \quad (21) \]

Solution to ODE (19) is readily obtained as:

\[ Y(y) = Ce^{\frac{\pi y}{2l}} + De^{-\frac{\pi y}{2l}} \quad (22) \]

Substituting Eq. (22) into Eq. (21), inserting the numerical values for \( k \) and \( h \) from the above, and setting \( l = L = 0.5 \text{ m} \) yield the following relation between constants \( C \) and \( D \) as;

\[ D = \frac{C(\pi - 10)}{(10 + \pi)e^{\pi}}. \quad (23) \]

Substituting Eq. (22) into Eq. (20), with \( l = L \), the following relation is obtained:

\[ A \cos\left( \frac{\pi x}{2l} \right) \left( Ce^{\frac{\pi}{2}} + De^{-\frac{\pi}{2}} \right) = 200 \quad (24) \]

Substituting \( D \) from Eq. (23) in Eq. (24) and setting \( AC = B \), one obtains:

\[ B \cos\left( \frac{\pi x}{2l} \right) \left( e^{\frac{\pi}{2}} + \frac{(\pi - 10)}{(10 + \pi)e^{\pi}} e^{-\frac{\pi}{2}} \right) = 200. \quad (25) \]

To find constant \( B \), the orthogonality condition of the circular functions is employed. Multiplying both sides of Eq. (25) by \( \cos\left( \frac{\pi x}{2l} \right) dx \) and integrating the result over the domain, \((0, l)\), yield:
\[ \int_0^l B \cos^2 \left( \frac{\pi x}{2l} \right) \left( e^{\frac{\pi}{2}} + \frac{(\pi - 10)}{(10 + \pi)e^{\pi}} e^{-\frac{\pi}{2}} \right) dx = \int_0^l 200 \cos \left( \frac{\pi x}{2l} \right) dx. \quad (26) \]

Now solving for the constant \( B \) in the above equation results in:

\[ B = 52.98. \]

With \( B \) found, then:

\[ \theta_1(x, y) = 52.98 \cos \left( \frac{\pi x}{2l} \right) \left( e^{\frac{\pi y}{2l}} + \frac{(\pi - 10)}{(10 + \pi)e^{\pi}} e^{-\frac{\pi y}{2l}} \right). \quad (27) \]

Proceeding to solve set of equations (13)-(15), circular functions are used again to satisfy the homogeneous boundary conditions, but this time, in the \( y \) direction as follows:

\[ Y(y) = E \cos(\lambda y) + F \sin(\lambda y) \quad (28) \]

Making Eq. (28) satisfy boundary conditions of Eq. (15) yields the \( y \) direction function as:

\[ Y(y) = E \left( \cos \left( \frac{\pi y}{4L} \right) - \sin \left( \frac{\pi y}{4L} \right) \right). \quad (29) \]

Then,

\[ \theta_2(x, y) = X(x)Y(y) = EX(x) \left( \cos \left( \frac{\pi y}{4L} \right) - \sin \left( \frac{\pi y}{4L} \right) \right). \quad (30) \]

Substituting for \( \theta_2(x, y) \) in Eq. (13) from Eq. (30), multiplying the result by \( dx \, dy \), and integrating the outcome over the unit square, one obtains:

\[ \int_0^l \left( \frac{d^2 X}{dx^2} - \left( \frac{\pi}{4l} \right)^2 X \right) dx = 0. \quad (31) \]

This, in turn, renders the following ODE problem:

\[ \frac{d^2 X}{dx^2} - \left( \frac{\pi}{4l} \right)^2 X = 0 \quad (32) \]

\[ \frac{dX(0)}{dx} = 0, \quad (33) \]

\[ \theta_2(l, y) = EX(l) \left( \cos \left( \frac{\pi l}{4L} y \right) - \sin \left( \frac{\pi l}{4L} y \right) \right) = 200 \quad (34) \]
The solution to Eq. (31) now is readily found as:

\[ X(x) = He^{-\frac{\pi}{4}x} + Ge^{\frac{\pi}{4}x} \]  

(35)

Equation (33) renders a relation between constants \( G \) and \( H \) as:

\[ G = H. \]  

(36)

Substituting Eq. (36) into Eq. (35), and the result in Eq. (34), the following is obtained:

\[ EG \left( e^{\frac{\pi}{4}x} + e^{-\frac{\pi}{4}x} \right) \left( \cos \left( \frac{\pi}{4L} y \right) - \sin \left( \frac{\pi}{4L} y \right) \right) = 200 \]  

(37)

Setting the product \( EG \) as constant \( M \), the orthogonality of circular functions is then used. Multiplying both sides of Eq. (37) by \( \cos \left( \frac{\pi}{4L} y \right) - \sin \left( \frac{\pi}{4L} y \right) \) \( dy \) and integrating it over \((-L, L)\), yield:

\[ \int_{-L}^{L} M \left( e^{\frac{\pi}{4}x} + e^{-\frac{\pi}{4}x} \right) \left( \cos \left( \frac{\pi}{4L} y \right) - \sin \left( \frac{\pi}{4L} y \right) \right)^2 \, dy = \int_{-L}^{L} 200 \left( \cos \left( \frac{\pi}{4L} y \right) - \sin \left( \frac{\pi}{4L} y \right) \right) \, dy \]  

(38)

The solution of the above gives:

\[ M = 67.96. \]

With \( M \) determined, then one can find:

\[ \theta_2(x, y) = X(x)Y(y) = 67.96 \left( e^{\frac{\pi}{4L}x} + e^{-\frac{\pi}{4L}x} \right) \left( \cos \left( \frac{\pi}{4L} y \right) - \sin \left( \frac{\pi}{4L} y \right) \right). \]  

(39)

Substituting Eqs. (27) and (39) into Eq. (8), and the outcome into Eq. (3), the temperature distribution in the unit square is found as:

\[ T(x, y) = 52.98 \cos \left(\frac{\pi x}{2L}\right) \left( e^{\frac{\pi}{2L}y} + \frac{(\pi-10)}{(10+\pi)e^\pi} e^{-\frac{\pi}{2L}y} \right) + 67.96 \left( e^{\frac{\pi}{4L}x} + e^{-\frac{\pi}{4L}x} \right) \left( \cos \left( \frac{\pi}{4L} y \right) - \sin \left( \frac{\pi}{4L} y \right) \right) + 300. \]  

(40)

Since the above temperature field is obtained by satisfying the boundary conditions, the largest error in this approximation method is going to be at the origin of the coordinate system, namely at (0,0).

A comparison will now be made with the temperature value for \( T(0, 0) \) obtained in Incropora's by finite difference methods:
Equation (40) yields the square's temperature at (0, 0) as:

\[ T(0,0) = 487.69 \, K \]

The finite difference solution, after several iterations and step size adjustments, gives:

\[ T_{finite-dif}(0,0) = 461.2 \, K \]

The discrepancy between the two results is

\[ \frac{T(0,0) - T_{finite-dif}(0,0)}{T_{finite-dif}(0,0)} \times 100\% = 5.7\% \]

One can wring more accurate results by adding more terms to the circular functions, which satisfy the homogeneous boundary conditions. In general, the accuracy of the integral solutions depends on the form of the assumed profile\(^4\).

We now discuss the second problem, which deals with transient conduction. Its statement is as follows:

Consider a long, wide and relatively thin slab of aluminum, of thickness \( L \), one side of which is immersed in a condensing fluid at \( T_\infty \), and the other face of it being adiabatic (see Figure 3). To begin, the aluminum plate is in thermal equilibrium with its surrounding. Suddenly an electric current passes through the plate, which produces a uniform volumetric energy generation \( \dot{u}' \) in the plate. What is the temperature of the adiabatic face of the aluminum plate 1 minute after the passage of the current? Assume \( \dot{u}' = 50000 \, \text{W/m}^3 \).

\[ \text{Figure 3: 1-D transient heat conduction with uniform volumetric heat generation} \]

Obviously, one can find the exact answer to this problem by means of separation of variables or Laplace transform techniques, which are not familiar concepts to most ME undergraduate students. This is one of the main reasons the integral method presented here, though utilized as an approximation technique, is a good choice to tackle the transient conduction problem. It is noteworthy that the error involved in this approximation is rather small, provided that suitable polynomial or circular function is selected for the homogeneous direction.
The governing differential equation to the problem, assuming very high convective heat transfer coefficient (condensing fluid), is:

\[ \frac{\partial \theta(x, t)}{\partial t} = \alpha \frac{\partial^2 \theta(x, t)}{\partial x^2} + \frac{u'''}{\rho c} \]  \hspace{1cm} (41)

\[ \theta(x, 0) = 0 \]  \hspace{1cm} (42)

\[ \frac{\partial \theta(0, t)}{\partial x} = 0 \]  \hspace{1cm} (43)

\[ \theta(L, t) = 0, \]  \hspace{1cm} (44)

where \( \alpha \) is the thermal diffusivity of the solid, \( \rho \) is the density of the solid, \( c \) is the specific heat of the solid, and

\[ \theta(x, t) = T(x, t) - T_\infty. \]  \hspace{1cm} (41a)

Here, \( T(x, t) \) is the temperature distribution in the plate. Integrating Eq. (41) in the space yields:

\[ \frac{d}{dt} \int_0^L \theta(x, t)dx = \alpha \int_0^L \frac{\partial^2 \theta(x, t)}{\partial x^2} dx + \int_0^L \frac{u'''}{\rho c} dx. \]  \hspace{1cm} (45)

Assuming that

\[ \theta(x, t) = X(x)\tau(t), \]  \hspace{1cm} (41b)

where \( X(.) \) is chosen as a polynomial that satisfies homogenous boundary conditions (43) and (44). The simplest polynomial that fulfills this condition is:

\[ X(x) = (L^2 - x^2). \]  \hspace{1cm} (46)

Substituting Eq. (46) into Eq. (45) and carrying out the integral yield:

\[ \frac{d\tau(t)}{dt} + \frac{3\alpha}{L^2} \tau(t) = \frac{3}{2 \rho c L^2} u'''. \]  \hspace{1cm} (47)

The solution of this differential equation subject to the initial condition \( \tau(0) = 0 \) is:

\[ \tau(t) = \frac{u'''}{2k} \left( 1 - e^{-\frac{3\alpha t}{L^2}} \right), \]  \hspace{1cm} (48)

where \( k \) is the thermal conductivity of the plate. Substituting Eqs. (48) and (46) into (41b), and the outcome into Eq. (41a) will render the temporal temperature field as:

\[ T(x, t) = \frac{u'''}{2k} \left( 1 - e^{-\frac{3\alpha t}{L^2}} \right) (L^2 - x^2) + T_\infty \]  \hspace{1cm} (49)
Equation (49) correctly predicts the temperature distribution in the plate once it reaches steady state ($t \to \infty$). To calculate the temperature on the adiabatic wall at the specified time, we set $k = \frac{w}{mK}$, $\alpha = 97.1 \times 10^{-6} \text{ m}^2/\text{s}$, and $T_\infty = 100 \, ^\circ\text{C}$ to get:

$$T(0,1 \text{ min}) = \frac{50000}{2 \times 237} \left(1 - \exp\left(-\frac{3 \times 97.1 \times 10^{-6} \times 60}{0.1^2}\right)\right)(0.01) + 100 = 100.87 \, ^\circ\text{C}$$

Once more the trend of temperature rise is correctly predicted.

Conclusion

All undergraduate heat transfer textbooks available today never discuss integral methods for solving the governing PDEs in heat transfer. In this paper, we presented our methodology in bringing integral methods to the undergraduate heat transfer classroom, with no prior student experience with PDEs, to train students, in under four hours, how to find approximate solutions to multidimensional steady and unsteady conduction problems with accuracy easily matching that found by finite difference methods under distinct temperature profiles. This is an important engineering education value that is not provided by alternative methods such as separation of variables for a number of objectives. For example, unlike the separation of variables or transform techniques, the integral method emphasizes the physical concepts by writing the first law of thermodynamics in integral form and choosing approximate temperature profiles satisfying boundary and initial conditions. In addition, the mathematical implications of using integral methods in this undergraduate course show students the value in reducing the order of the governing PDEs and/or the number of associated independent variables to yield ODEs. Moreover, if complications of nonlinearities, such as those associated with phase change and temperature-dependent thermal properties, are present, then the integral method presented here will still apply to solve the corresponding problem while the other methods will not. No knowledge of separation of variables or transform methods is needed to train students to obtain an equally reliable approximate solution to such multidimensional problems. In our case, we believe that this new approach and strategy in undergraduate engineering education represent an effective framework for teaching students the fundamentals of how to use integral methods to better understand and solve heat transfer problems.

Besides Heat Transfer, the integral method can be an effective tool to teach in a number of other undergraduate courses in Engineering and Applied Mathematics including introductory undergraduate courses on PDEs and Calculus of Variations. The problems presented here would make for an excellent interdisciplinary learning experience in such courses.

Bibliography