Integrating Integration

Lynn Kiaer
Rose-Hulman Institute of Technology

Abstract

Rose-Hulman's Integrated First Year Curriculum in Science, Engineering and Mathematics consists of a sequence of three one-quarter twelve-credit courses, which incorporate all the traditional technical courses of the first year: differential, integral and multivariate calculus, mechanics, electricity and magnetism, two quarters of general chemistry, engineering statics, graphical communication, computer science and engineering design. It has served approximately one fourth of the entering class of first year students for the last six years. From the inception of the program, many of the synchronicities between mechanics and differential calculus have been well-exploited. In the past two years, additional opportunities to coordinate the treatment of chemical kinetics, electricity and magnetism, and integral calculus have been identified, and a number of classroom, laboratory and homework experiences have been designed to assist students in understanding the relationships between these topics. A number of these activities are described.

Introduction

Students are often presented with similar problems in different surroundings, and it is not uncommon to find a student who can solve a problem easily in a physics class but is completely stymied when presented with the same problem in calculus class. Most students eventually make the connection between calculus and mechanics, but other connections are harder. Making those connections is vitally important to the student's ability to learn rather than merely be taught. Rose-Hulman's Integrated First Year Curriculum in Science, Engineering and Mathematics' was developed in part to assist students in learning by making those connections explicit.

Integration can be a difficult subject to integrate. Techniques of integration appear to be essentially unrelated to the traditional applications of integration, and the introduction of computer algebra systems can make them seem essentially unmotivated. Over the last two years motivating examples have been collected and explicit connections have been made to other disciplines in the areas of

- rates (orders) of chemical reactions and separable differential equations,
- work and other traditional applications of integration and Riemann sums, and
- electric field and trigonometric substitution.

Connections With Chemistry

As the mathematicians in the integrated curriculum begin the study of integral calculus, the physicists in the program are still using the differential calculus developed in the previous quarter, but the chemists are beginning to look at the rates of chemical reactions. Consequently, the initial mathematical thrust is antidifferentiation.

Chemists use the (differential) rate law for a reactant A: $Rate = -d[A] / dt = k[A]^n$, where [A] is the concentration, in moles per liter, of substance A. From the point of view of the mathematician, this is a straightforward, separable differential equation, with $A(t)$ being the concentration of substance A at time t. There are two general forms for the solution, one in the case where $n = 1$, and the other for **all** other values of n.

When $n = 1$, the steps of the solution are:

1. Setup the differential equation: $-dA(t) / dt = k A(t)$.
2. Separate: $(1 / A(t))(dA(t) / dt) = -k$
3. Integrate both sides with respect to t: $\ln(A(t)) = -k t + C$.
4. When $t = 0$, the value of C *can be* determined: $\ln(A(t)) = -k t + \ln(A(0))$.

The mathematician then wants to proceed to the final step and solve for $A(t)$, but the chemist values the information in its present form. It is apparent that the graph of the natural log of the concentration of substance A at time t versus time should be a straight line. Rearranging the terms of this equation yields $\ln(A(0)/A(t)) = k t$. *In* this form, we can easily obtain information about the half-life of the reaction, that is, the time when the concentration of substance A is half of $A(0)$. Since chemists speak of the half-life of a reaction even when the half-life is not constant, it is useful to use this form of the solution to demonstrate that, if $t_{1/2}$ is the time at which $A(t) = 1/2A(0)$, then $t_{1/2} = \text{in}(2) / k$, and thus is constant for first-order chemical reactions. In addition, students discover that no amount of calculation will allow them to obtain a value for k. It can only be found through experimentation.

For **all** $n \neq 1$, the steps of the solution are:

1. Setup the differential equation: $-dA(t) / dt = k [A(t)]^n$.
2. Separate: $[A(t)]^{-n} (dA(t) / dt) = -k$.
3. Integrate both sides with respect to t: $[A(t)]^{-n+1} / (-n + 1) = -k t + C$.
4. Simplify: $1 / [A(t)]^{n-1} = k t (n - 1) + C (1 - n)$.
5. When $t = 0$, the value of C (1 -n) *can* be determined: $1 / [A(t)]^{n-1} = k t (n - 1) + 1 / [A(0)]^{n-1}$.

Once again, the mathematician wants to solve for $A(t)$, but the chemist wants to examine the current equation to determine the half-life of an *nth* order chemical reaction. In this case, we find that $t_{1/2} = (2^{n-1} - 1) / (k [A(0)]^{n-1} (n - 1)$, so the half-life is not constant, but depends on the initial concentration $A(0)$. The special case of $n = 2$ also yields a quick test for second order reactions, based on the plot of $1 / A(t)$ versus time.

The extensive use of computer algebra systems often causes students - and faculty -to question the importance of by-hand skills. In making this connection with chemistry, the usefulness of by-hand skills is reinforced.

Applications With Riemann Sums

Soon after the start of the second quarter, the physicists begin to discuss work. Work is one of the classic applications of integration, but in a typical course in integral calculus, the applications are covered after the techniques have been developed. In the integrated first year curriculum, we take advantage of the computer algebra system to introduce and extensively use Riemann sums to solve applications problems. We find that the early emphasis on Riemann sums, which is made significantly less tedious with the computer algebra system, pays off later when the students are very comfortable with the idea of slicing. Thus they find modeling complex integration problems to be a routine task rather than a challenging one.

The *Mathematica* code

```
rightsum[f_,a_,b_,n_] := Sum[(b-a)/n * f[a+i(b-a)/n],{i,1,n}]
```

computes a Riemann sum. Once the students have had a little experience computing area under the curve, they can begin to solve applications problems using this Riemann sum.

Some illustrative examples:

Problem: Find the work done hauling 40 feet of chain that is hanging from a window whose sill is 50 feet above the street, if the chain weighs half a pound per foot.

Solution: Since work is force times distance, and the distance is different for each point along the chain, we can approximate the work done with a Riemann sum. For each slice, we can approximate its distance to the windowsill by using the furthest point on the slice - thus using the Riemann rightsum. The force is the weight per foot times the length of the slice, or $0.5\,(b-a)/n$, and the distance is $a + i\,(b - a)/n$. In this case, we want to make our slices between $a = O$ (at the windowsill) and $b = 40$ (at the end of the chain). Thus, we want our function f to be $f(x) = x/2$. For $n = 40$, we obtain 405 foot pounds of work. For $n = 80$, we obtain 404. For
n = 1000, the sum is 400.4. Since rightsum is an over-estimate in this case, and leftsum gives au underestimate, a little checking gives an answer of about 400 foot pounds.

Problem: Find the work done pumping water from a full hemispherical tank, having radius of 4 feet, to a truck whose intake is located 10 feet above the top of the tank. Assume that the water weighs 62.4 pounds per cubic foot.

Solution: Once again, we will make horizontal slices, and use rightsum. This time, the distance each slice will be lifted is the easy part: $10 + a + i\,(b - a)/n$. The weight of each slice is the weight per unit volume times the volume of the slice. The volume of the slice is given by $\pi r^2 (b - a)/n$, where r is $(16 - i\,(b - a)/n)^{1/2}$. So we want to sum $(10+ a + i\,(b - a)/n)\,(62.4@\,(16 - i\,(b - a)/n)\,(b - a)/n$, and we want our slices between $a = O$ (at the top of the tank) and $b = 4$ (at the bottom of the tank). Our function is therefore $f(x) = (10+x)\,(62.4\,\pi)\,(16 - x)$, and we obtain sums of 130842, and 130721 for $n = 20$ and $n = 100$, respectively. A little checking with leftsum suggests that the answer is about 130700 foot pounds.

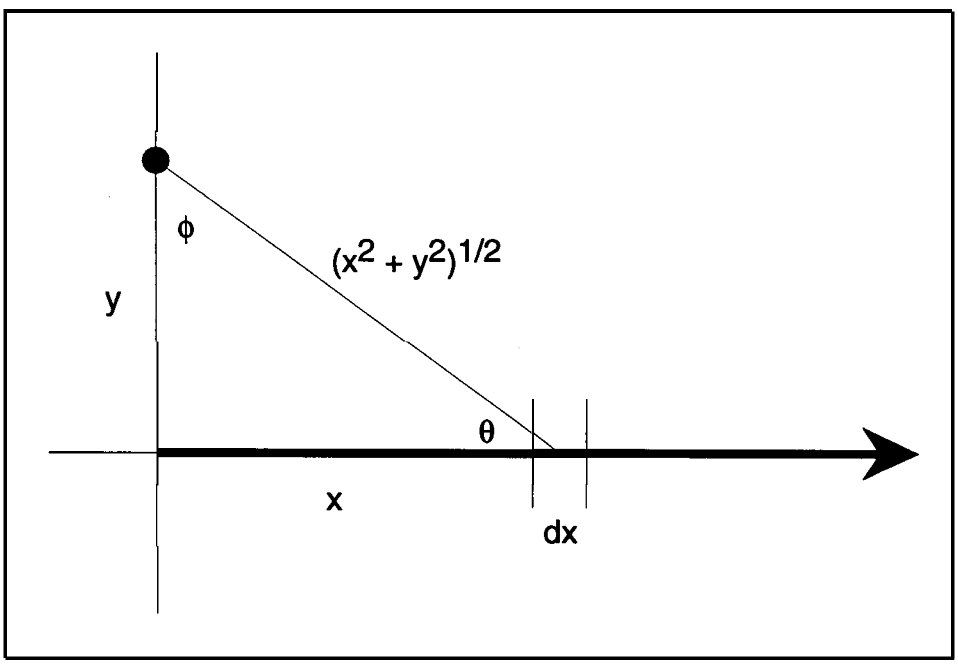
Solving practical problems with Riemann sums not only enhances students' problem formulation skills, it also provides some motivation as the course progresses through various techniques of integration. A set of well-chosen examples can be used to introduce all the techniques of integration.

Motivating Trigonometric Substitution

Trigonometric substitution is one of the most difficult techniques for students to master, in part because the technique consists largely of rules for substitution, and students memorize the rules with little sense of why they work. Introducing trigonometric substitution by way of electric fields has the double benefit of making trigonometric substitution seem perfectly reasonable and natural, and also of reinforcing the usefulness of understanding the technique.

A simple problem in electricity and magnetism is to determine the electric field at a point charge, given a line of charge. The magnitude of the electric field at a point charge q_1, due to another point charge q_2, is given by $1/(4\pi\varepsilon_0)\, q_1 q_2 / r^2$, where ε_0 is the permittivity of free space (a constant), and r is the distance between the charges. The direction of the field is toward q_2 if the signs of the charges are different, and away from q_2 if the charges have the same sign. The principle of superposition allows us to add the effects of many point charges, and thus to integrate. However, when we add or integrate charges, we must be careful about the direction of the resulting field; we cannot simply add magnitudes. Thus, we separate the field into its x and y components, and then the magnitudes can be added.

Consider the problem% Charge is distributed uniformly along the positive x axis with charge density λ. Find the electric field at a point (O, y). To find the magnitude of the field in the x direction, we note that the contribution of a slice of the line of charge is given by $1/(4\pi\varepsilon_0)\,\lambda / (x^2 + y^2)\; x / (x^2 + y^2)^{1/2}\, dx$, since, by convention, the charge at (O, y) is +1. The integral from O to ∞ can be found easily by a standard u-substitution.



However, the magnitude of the electric field in the y direction is not so easily found. The contribution of a slice is given by $1/(4\pi\varepsilon_0)\,\lambda / (x^2 + y^2)\; y / (x^2 + y^2)^{1/2}\, dx$, and no standard u-substitution will work. We seek to evaluate $\lambda\, y / (4\pi\varepsilon_0) \int_0^\infty dx / (x^2 + y^2)^{3/2}$. If we change our variable of integration from x to θ, and bring y back inside the integral, we need only to evaluate $\lambda / (4\pi\varepsilon_0) \int_{\pi/2}^{\pi} \sin(\theta)\, d\theta$. Equivalently, we could integrate with respect to φ, and work with $\cos(\varphi)$ as φ ranges from-n/2 to O.

Once students are comfortable making the transition between variables of integration, it is time to examine exactly what trigonometric substitution was made: What did we substitute for x? In this case, we substituted x = y tan(q). Teaching trigonometric substitutions to students who have already successfully done a trigonometric substitution is a much easier task than introducing trigonometric substitutions as a technique.

Trigonometric substitutions can also be motivated strictly through the use of a computer algebra system.
In introducing the technique of partial fractions, students can be asked to determine how the computer algebra system found the antiderivative of $2/[(x - 1)(x+ 1)]$ to be $\ln(x - 1) - \ln(x +1)$. Once students have puzzled it out, they generally have little trouble establishing the rules for simple partial fractions. However, once they encounter a rational function whose partial fraction decomposition includes a term with an irreducible quadratic denominator, they often want to know how that antiderivative was obtained.

Summary

Rose-Hulman's Integrated First Year Curriculum in Science, Engineering, and Mathematics provides fertile ground for the development of cross-disciplinary approaches to teaching calculus. Although some of these approaches depend upon the student's taking certain other courses at the same time as calculus, many require little or no background in other disciplines.

Bibliography

1. Rogers, Gloria J., and Brian J. Winkel, "Integrated First-Year Curriculum in Science, Engineering, and Mathematics at Rose-Hulman Institute of Technology: Nature, Evolution, and Evaluation," Proceedings of the 1993 ASEE Conference, Champaign, Illinois, June 1993.

2. Crummett, William P. and Arthur B. Western, *University Physics: Models and Applications*, William C. Brown Publishers, 1994.

LYNN KIAER received a B.A. in French and Political Science and a B.S. in Mathematics from Norwich University, and an M.S. in Operations Research and a Ph.D. in Applied Mathematics from Florida Institute of Technology, and is now an Assistant Professor of Mathematics at Rose-Hulman Institute of Technology. She has taught in the Integrated First Year Curriculum in Science, Engineering and Mathematics for two years.