Integrating Various Mathematical Tools with a Senior Mechanical Engineering Laboratory Experiment

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Abstract

A senior mechanical engineering laboratory utilizes a simple experiment to provide application of several mathematical tools, including: fitting of experimental data using multi-variable linear regression, integration of non-linear, ordinary differential equations, solution of the heat diffusion equation by finite difference methods, and probability and statistics. All of the required mathematical tools are available in Excel® as functions, or can be solved by using a spreadsheet. The experiment involves heating a fine, resistive wire (nichrome) by DC current from room temperature up through break (due to melting). In the first lab meeting, heat transfer analysis of the problem is applied to predict break time, considering the expected modes of heat transfer. These results are submitted to the laboratory instructor. In the second meeting, students make break-time measurements for 30, presumably identical experiments. The computed results are then compared to the experimental results, and the model is refined, if warranted, to rationalize the comparison.

I. Introduction

The senior Mechanical Engineering laboratory "Experimental Methods II" is focused on demonstration of the principles of the thermal and fluid sciences and includes group interactions with both written and oral presentation of results. Companion lectures which correspond to each exercise, reviews both the technical principles and mathematical tools to be used in that exercise. Of the six required exercises, the first four begin with a group analysis of a physical problem by application of theoretical principles and the prediction of outcome, followed by experimental measurement to verify the model which was applied. Of these, one deals with the prediction of break-time for a resistive wire that is heated by passage of sufficient current to result in electrical continuity break (melting followed by liquid beading, due to surface tension).

Metallic electrical conductors passing current may be encountered in various applications, including but not limited to fuses, hot wire ignition of pyrotechnics and explosives, illumination, and heating elements, and trip wires for strand burner timing. Models based on the conservation of energy, coupled to heat transfer and thermodynamic principles, can be analyzed by application of various mathematical tools.

The following discussion will deal with various aspects of the analysis and experiment, but will focus on how various mathematical tools can be used in the predictive process and in the presentation of results. Findings from two student groups will also be given and discussed.

II. Theoretical Analysis

The problem being addressed is depicted in Figure 1, where the axial dimension is assumed to be infinite.

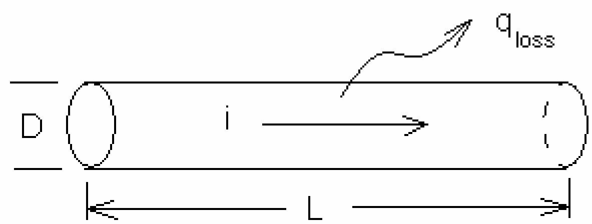

Figure 1 - Schematic of Problem

In order to adequately describe the problem, the appropriate heat transfer principles for the problem of interest will be discussed. An energy balance is written for a horizontal wire of resistance $\Re$ that is electrically heated and loses heat to the environment by both radiation to the environment and interaction with surrounding air. With constant thermal properties and a "lumped mass" analysis, an initial value equation of the following form can be written:

$$mc\frac{dT}{dt} = i^2\Re - \sum \dot{Q}_{Loss} \qquad \text{Eq. (1)}$$

where mc is the product of the mass and heat capacity of the wire per unit length, $i^2\Re$ is Joule heating, T is temperature which is the dependent variable, and t is time which is the independent variable. The quantities shown are typical symbols and can be found in popular heat transfer textbooks, e.g., Incropera and DeWitt [1]. The heat loss term will reflect the expected modes of thermal coupling between the wire and the environment. This formulation is used at temperatures below the wire melt temperature, and for subsequent time, the term cdT is replaced by the specific enthalpy of fusion multiplied by the melt fraction, i.e., $h_{melt}\, dz$, and the wire temperature remains constant during the melt period. Hence, the analysis includes two different regimes and the total time to wire break will be the sum of the time for heating from room temperature to melt plus the time required to complete melting. In other words, once full melting has occurred, then the surface tension of the melted wire will cause transition from the wire cylindrical shape to a spherical shape, thus breaking electrical continuity.

Four cases are considered for the heat loss term: 1) the adiabatic case , 2) radiation to room, 3) radiation to room plus convection to surrounding air, and 4) radiation to room plus conduction to surrounding air. Implications of the choice of heat loss models will now be discussed. The adiabatic case provides the simplest analysis and Eq. (1) can be easily integrated for the regimes before and after initiation of melt. This will result in a "lower bound" for total time to melt. Radiation will obviously be the dominant mode of heat transfer at temperatures approaching and during melt (approximately 1672 K) and is expected to be a significant heat loss factor for the latter three cases. The case that includes convection to surrounding air is expected to be appropriate for "slow" processes, where there is sufficient time for convective air movement to develop. For shorter times, the heat loss to surrounding air is expected to be governed by conduction. These four cases are listed in order of increasing difficulty with respect to heat transfer analysis. A comparison of results of the predictions will ultimately be made to experimental results to provide insight as to the most appropriate choice for the modes of heat transfer, and illustrate comparison for the four cases treated.

A major presumption in this approach is that the convective coefficient, the heat capacity, the emissivity and the resistance of the wire do not change substantially during the experiment. The former of these presumptions bears further examination on the part of the students; the latter was studied in an elective group project and found to be approximately valid, i.e., about 10% variation in electrical resistivity throughout the experiment was found. Consequence of variation in wire emissivity and heat capacity will be discussed later when analytical predictions are compared to experimental measurements.

In order to complete the integration of Eq. (1) for the latter three case, values for both the convective heat transfer coefficient and the emissivity of the wire must be obtained. Tabular values for the wire emissivity are found to be varied and highly dependent on material, surface condition, and temperature[1]. In order to obtain appropriate estimates for the emissivity and convective coefficient, the manufacturer of the wire (Omega ® [2] in this case) publishes for each wire size, the electric current versus steady-state wire temperature. Hence, by utilization of a multi-variable linear regression, e.g., the LINEST function in Excel®, it is possible to obtain values for the needed heat transfer coefficients. If the convective coefficient is presumed to be a constant over the range of experimental temperatures, LINEST returns an emissivity which is greater than unity; a violation of thermodynamic principles. The implication of this result is that LINEST, in order to obtain a "best fit", attempts to allocate too much of the heat loss into the term which has the highest temperature dependence, i.e, the radiative component. Examination of the dimensionless correlations for the convective coefficient, e.g., Morgan correlation [1], indicates that because the Nusselt number contains the thermal conductivity of air, which is highly temperature dependent over this temperature range, the convective coefficient may also vary significantly. And, in fact, from this correlation, the convective coefficient is found to be approximately linear with temperature (or temperature difference from room temperature). Assuming that the convective coefficient can be reasonably expected to be represented by a linear function of temperature of the form $h = C_1 + C_2\,(T-T_A)$ and emissivity is presumed to remain constant throughout the wire heating, then a very good fit to the Omega data can be obtained. The problem can be formulated by starting with the linear expression:

$$y = b + m_1 x_1 + m_2 x_2 + m_3 x_3 \qquad \text{Eq. 2}$$

and identifying the following equivalents: $y = i^2\Re$, $b = 0$, $m_1 = C_1$, $x_1 = A_S(T-T_A)$, $m_2 = C_2$, $x_2 = A_S(T-T_A)^2$, $m_3 = \varepsilon$, $x_3 = \sigma A_S(T^4 - T_A^{\,4})$

Figure 2 indicates the correspondence of the left hand side of this equation to the right hand side. From this fit, the values returned by LINEST were $\varepsilon = 0.74$, $C_1 = 64$ and $C_2 = 0.14$ where units

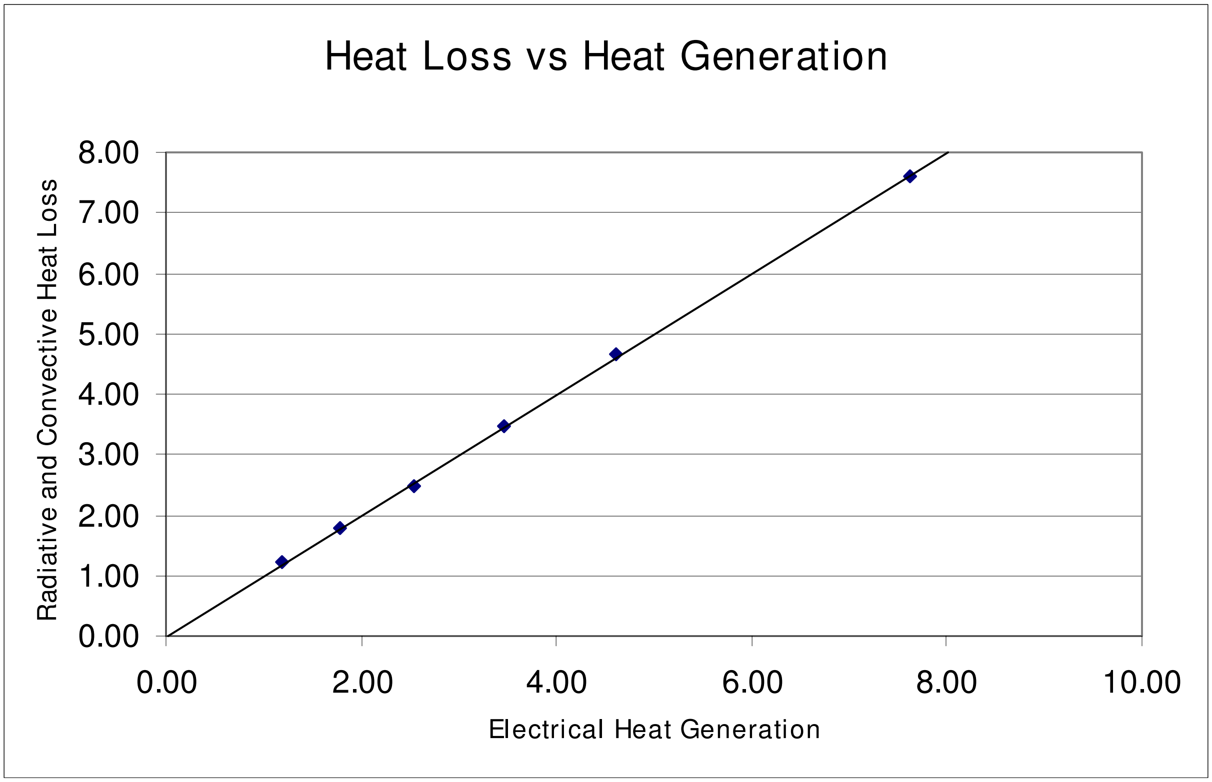

Figure 2 - Heat Loss versus Heat Generation according to Eq. (2)

are SI. The resulting convective coefficient is roughly twice that predicted by the Morgan correlation (which represents data for a wide range of measurements from horizontal cylinders of various diameters and in various environments). The emissivity value from the fit of Omega data is not inconsistent with tabulated values for stably oxidized metallic surfaces.

With a value for wire emissivity, the second case (heat loss by radiation only) can be integrated to obtain both the time for the wire to heat to incipient melt and the time for the wire to melt. This is accomplished by simply separating the variables in Eq. (1) and integrating for the two time regimes, i. e., before melt and during melt. Many students use hand calculators which can perform this integration, or they may have access to mathematical software, such as *Mathematica*, which can also perform the integration. Alternately, a simple trapezoidal rule integration can be easily performed on a spreadsheet.

Similarly, the case including both radiation and convection can also be integrated utilizing the coefficients which were found from the multivariable linear regression.

For the case that considers radiation from the wire to the room in conjunction with conduction to air at the wire-air interface, the mathematics becomes much more difficult because there is not a known "closed-form" solution to this problem. Thus, the non-linear boundary condition representing the electrically heated wire must be coupled to the thermal diffusion equation representing thermal response of the surrounding air via conduction. The problem can be formulated as:

$$\nabla^2 T = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$
$$B.C.1\ldots\frac{\partial T(R,t)}{\partial r} = \frac{1}{kA_S}\left(i^2\Re - \varepsilon A_S (T^4(R,t) - T_A^4\right) - mc\frac{\partial T(R,t)}{\partial t}) \qquad \text{Eq. 3}$$
$$B.C.2\ldots\frac{\partial T(\infty,t)}{\partial r} = 0$$
$$I.C.\ldots\ldots T(r,0) = T_A$$

where T_A is ambient temperature, R is the wire radius, ε is wire emissivity, α and k are the air thermal diffusivity and thermal conductivity respectively $\Re$ is wire resistance and A_S is the wire surface area per unit length. While textbooks such as Carslaw and Jaeger [3] offer short time and long time approximations, the boundary condition has been linearized, which is not appropriate over large temperature variations. Therefore, the students can solve the field equation by finite-difference (or finite element) for the period before melt and the period corresponding to melting. For finite element solution, the nodal configuration is illustrated in Figure 3:

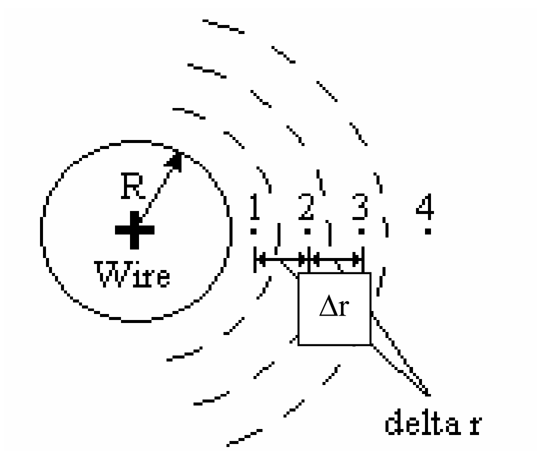

Figure 3 - Illustration of Radial Conduction into Air surrounding Wire of Radius R

The nodal equations are derived by considering a heat balance with boundary heat transfer and internal energy storage where the area and volume of each node increases with increasing radius. The first node represents the wire with electrical heat generation, internal energy storage and radiative and conductive boundary heat losses. The first air node will be located at the wire radius plus half of the selected radius increment, and subsequent nodes can be represented by a recursive relation which can be easily applied for the second and higher air nodes.

The first node is subscripted 0 and represents the electrically heated wire. The energy balance can be written:

$$T_o' = T_o + \frac{\Delta t}{\rho \cdot \pi R^2 Lc}\left[i^2 \Re' \frac{L}{\pi R^2} - \varepsilon\sigma\, 2\pi RL\left(T_o^4 - T_s^4\right) - k\, 2\pi RL\left(\frac{T_o - T_1}{\Delta r / 2}\right)\right]$$

Eq. 4

where $\Re'$ is the wire resistivity and the prime temperature represents values at future time and unprimed temperatures represent present time values. For the first air node, the equation can be written:

$$m_1 c_p \frac{dT_1}{dt} = k\left[A_s\left(\frac{T_o - T_1}{\Delta r / 2}\right) - A_1\left(\frac{T_1 - T_2}{\Delta r}\right)\right]$$

$$m_1 = \rho V_1 = \rho\left[\pi (R + \Delta r)^2 L - \pi R^2 L\right]$$

$$A_s = 2\pi RL$$

$$A_1 = 2\pi (R + \Delta r) L$$

$$T_1' = T_1 + \alpha \frac{\Delta t}{V_1}\left[A_s\left(\frac{T_o - T_1}{\Delta r / 2}\right) - A_1\left(\frac{T_1 - T_2}{\Delta r}\right)\right]$$

Eq. 5

and for higher nodes, a recursive relation can be written:

$$m_n c_p \frac{dT_n}{dt} = k\left[A_{n-1}\left(\frac{T_{n-1} - T_n}{\Delta r}\right) - A_n\left(\frac{T_n - T_{n+1}}{\Delta r}\right)\right]$$

$$m_n = \rho V_n = \rho\left[\pi (R + n\Delta r)^2 L - \pi \left[R + (n-1)\Delta r\right]^2 L\right]$$

$$A_{n-1} = 2\pi \left[R + (n-1)\Delta r\right] L$$

$$A_n = 2\pi (R + n\Delta r) L$$

$$T_n' = T_n + \alpha \frac{\Delta t}{V_n}\left[A_{n-1}\left(\frac{T_{n-1} - T_n}{\Delta r}\right) - A_n\left(\frac{T_n - T_{n+1}}{\Delta r}\right)\right]$$

Eq. 6

A spreadsheet approach, where time advances down the column according to the time increment, and radius advances along the cell rows according to the radius increment, can be easily set-up. For this case, the thermal conductivity of the air is presumed constant. The results of the calculation can be used to justify this assumption, in that the wire is observed to reach melt temperature when the first air node temperature has increased by less than 100 ^{O}C.

The finite difference analysis requires the students to impose a stability criteria for the step sizes, i.e., the Fourier modulus based on air diffusivity multiplied by the time increment and divided by the square of the radius increment must be less than 0.5. Since conduction to air is presumed to be geometrically infinite with radius, the students must impose a condition for truncation of the analysis, i.e., imposition of zero gradient at an imaginary outer boundary, provided that the temperature at this node does not respond during the time of interest. A typical plot generated by the above procedure is shown in Figure 4.

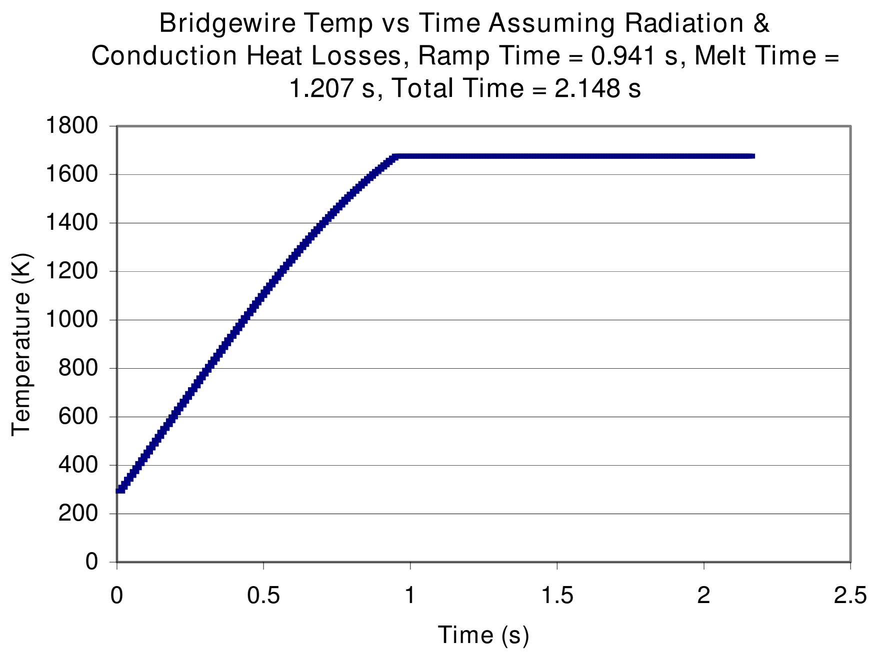

Figure 4 - Temperature versus Time for Radiation and Air Conduction Heat Loss from Wire

Results of these four cases were reported by two student groups and their answers are shown in Table 1.

Group	Wire AWG	Current, amps	Adiabatic case	Radiation only	Radiation + Conduction	Radiation + Convection
1	28	6.4	1.131	2.348	3.659	3.839
2	28	6.0	1.218	1.66	2.148	4.311

Table 1 - Illustration of Predictive Results for Four Cases

where times are in seconds and the wire is Nichrome®. These results show the expected order. That is, the adiabatic case provides the shortest time to melt and the radiation only is the second shortest. The fact that the radiation + conduction case is faster than the radiation + convection case can be rationalized by recognizing that convection heat loss has conduction as its lower limit, i.e., where there is no buoyant air movement. Therefore, the magnitude of convective heat transfer exceeds conduction to air, and the time for melting is consequently longer.

III. Experiment

These analytic results are presented to the laboratory instructor during the first lab meeting, and in the second lab meeting, the students perform a series of 30 experiments, all with the same gage wire and subject to the same current. The test rig is shown in Figure 5, where a 1" length of heating wire is clamped between two posts.

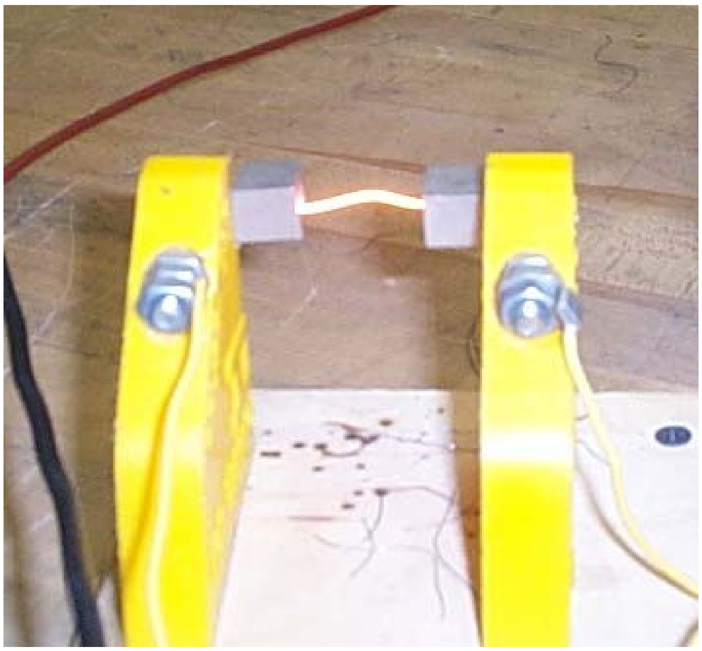

Figure 5 - Heated Wire before Break

A constant current power supply is first adjusted to deliver the current input specified by the laboratory instructor. Two different wire sizes could be accommodated: 28 AWG and 30 AWG, so that some variation in results between student groups could be expected. And, various currents could be specified. If the current is low, then melting of the wire would not be achieved. If the current is too high, then heating would be so fast that the results would closely match the adiabatic case. Therefore, an intermediate current should be specified to provide for variability of results. An oscilloscope is used to measure voltage across a known resistor in the circuit so that time from closure of the switch until the circuit opened could be accurately measured by use of cursors internal to the oscilloscope. Time to break was measured for 30 trials and the data recorded. Figure 6 illustrated an image of the test assembly. The oscilloscope is to the right of this image.

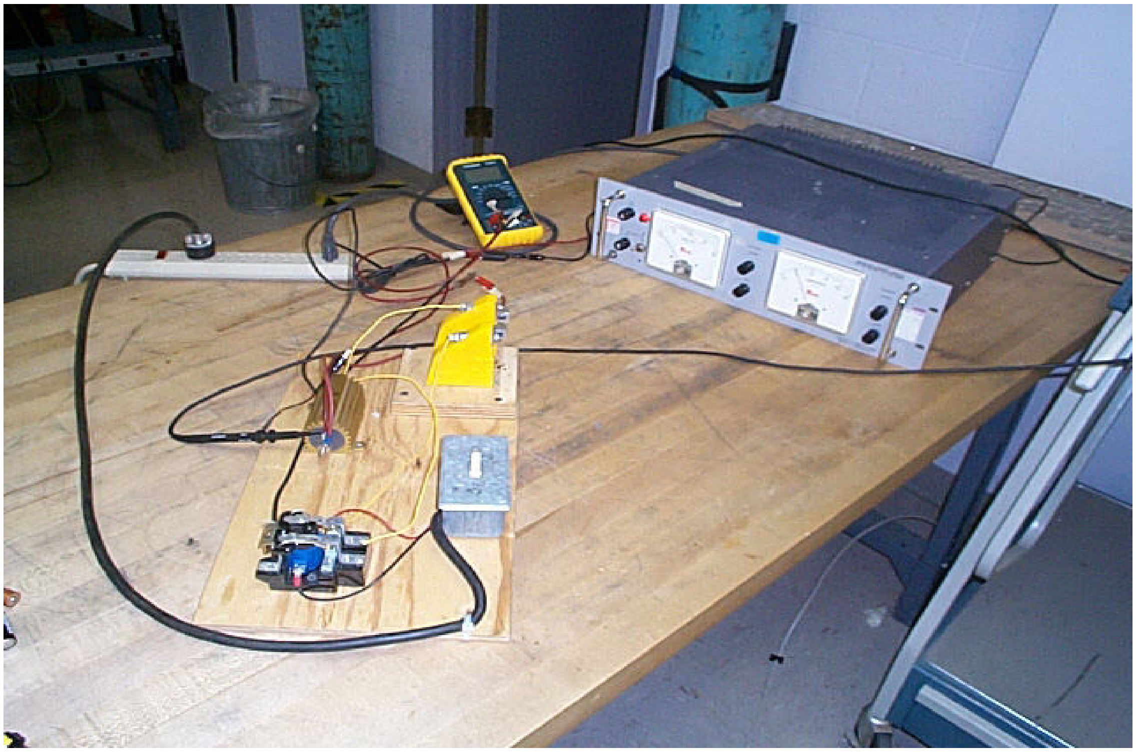

Figure 6 - Layout of Major Components of Experiment (Oscilloscope is off-image to the right)

V. Data Treatment

The break time data is analyzed for both the mean and standard deviation. Next, the data set is examined to determine if outliers can be discarded according to Chauvenet's criterion, i.e., maximum deviation of each datum from mean divided by the standard deviation should be less than 2.33 (for a set of 25 data points). If points are discarded, then the mean and standard deviation are recalculated and compared to the values predicted previously. And finally, from the data, the 90% and 95% confidence intervals are determined. For the two groups whose calculated results were shown previously, the experimental results are given below.

Group	Mean, (sec.)	Standard Dev.	Number of Points Discarded	90% Confidence Interval, (sec.)	95% Confidence Interval, (sec.)
1	4.132	0.313	0	+/- 0.093	+/- 0.11
2	4.633	0.331	1	+/- 0.104	+/- 0.126

Table 2 - Summary of Statistics for Thirty Trials

VI. Rationalization of Calculated and Experimental Results

In comparison of the calculations to the experimental results for both groups, the experimental break time is higher than any of the calculated results. The students are now asked to consider the most plausible explanation for this discrepancy. Some of the more probably explanations for an experimental break time which is longer than any of the predicted break times are.:

- Emissivity of wire increases due to surface oxidation while the wire is at melt conditions. Actual increase in heat loss will delay break time.
- Heat capacity of the wire is not constant but is instead, directly related to temperature. Hence, as wire temperature increases, then the ability of the wire to store energy and thus delay break is increased.
- Air movement in the room may not reflect quiescent conditions. Air move due to room fan, student movement, etc., would increase convection, if convection is a significant mode and delay break time.

Alternately, if the experimental break time is shorter than the predicted break time, there are other plausible explanations, e.g.:

- Resistivity of wire increases with increasing temperature and increases at melting conditions. That this is the case, at least for the solid phase, can be verified by examining the temperature dependence of the wire resistivity. Hence, if the power supply holds current constant, then power delivered to the wire will increase and shorten wire break time.
- The emissivity of the wire is lower than values given by Incropera and DeWitt, or obtained from a fit of the Omega data. This could be argued based on the degree of surface oxidation between the wire used in the experiment and the tabulated values from references.
- Students will also frequently list geometric wire anomalies such as nicks as causing discrepancy between the predictions and experimental results. While nicks would present a preferred location for concentrated heating, this would be expected to shorten break time. However, a severe nick probably could cause the data/datum to be classified as an outlier, and discarded according to Chauvenet's criterion.

Students will typically list heat conduction to the posts as being a cause for delayed break. However, without proof, it is unlikely that this consideration influences the experiment because the L/D ratio for the wire is around 100, and melting will obviously occur at an axial position away from the posts leading to a break time which would not likely be influenced by posts.

VII. Conclusion

A simple and inexpensive laboratory experiment has been described which can be used to exercise student's analytic abilities, as well as demonstrate the application of a number of mathematical tools which they are expected to have mastered. Additionally, it presents the opportunity for the student to evaluate the limitations of analytic models in the prediction of an outcome.

Biographic Information

1. <u>Fundamentals of Heat and Mass Transfer</u>, 2nd Ed., Frank P. Incropera and David P. DeWittJohn Wiley & Sons, New York, 1985
2. <u>The Electric Heaters Handbook</u>, Vol. 29, Omega Engineering, 1995
3. <u>Conduction of Heat in Solids</u>, 2nd Ed., H.S. Carslaw and J.C. Jaeger, Oxford University Press, Clarendon, England, 1959

Biographical Information

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