Integration of the Greatest Integer Function

David Fung and Steve Ligh

Department of Mathematics
Southeastern Louisiana University
Hammond LA 70402

There have been much discussions in recent years about the teaching of calculus. Among the various directions in the reform movement is the Consortium based at Harvard University. The text [2] for this project begins with a discussion of a library of functions. We introduced in our first semester calculus class several functions that are applicable to students' environment. The postage stamp function and the grading function are examples of step functions, such as the greatest integer function.

The greatest integer function, oft, denoted by [t], is defined as [t] = n for every $t \in [n, n+1)$ with n being an integer. Our study of the greatest integer function started with the use of the Computer Algebra System, *Derive version 2.0.* In order to study greatest integer function in *Derive,* one must first load the utility file MISC.MTH. The file contains the function FLOOR(a, b) which is defined as the greatest integer less than or equal to a/b. Therefore, the greatest integer function of t is given as FLOOR(t, 1). When we used the *Derive* command "Simplify" on this function, we obtained the following representation for $[t]$:

$$[t] = \frac{1}{\pi}\arctan(\cot(\pi t)) + t - \frac{1}{2}. \tag{1}$$

By means of elementary properties of trigonometric functions, we proved (1) for non-integral t. Representation (1) and its generalization have been studied by Fung and Ligh[1].

Let $F(x) = \int_0^x [t]dt$. In this paper, we will consider $F(x)$ and show how we can approximate it by two quadratic functions. In [1], by using (1) and the substitution $u = \frac{1}{\pi}\arctan(\cot(\pi t))$, we obtained the following integration formula:

$$F(x) \quad \frac{-[x]^2 + 2x[x] - [x]}{2} \tag{2}$$

The graph of F is given in Figure . Formula (2) was also found by Sy [3] using a different method.

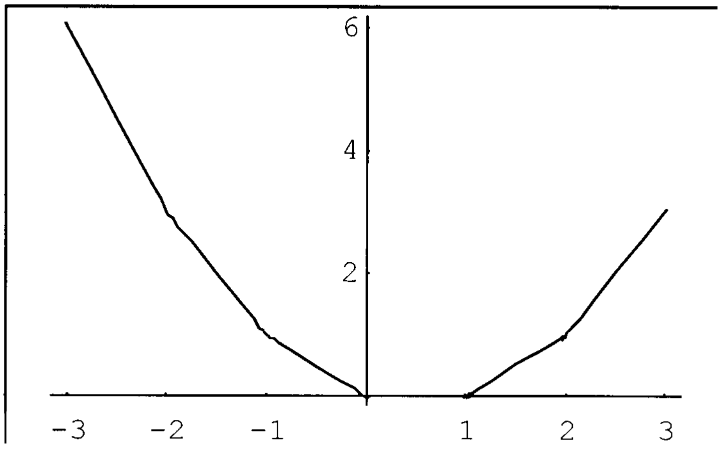

Figure 1. Graph of F.

Since formula (2) requires multiple uses of $[t]$, which in turn involves (l), approximations would be useful to avoid cumbersome calculations. As shown in Figure 1, the function F has a parabola-like graph. The fact that F mimics a parabola so much inspired us to find quadratic functions to approximately match it. We replace $[x]$ with x in (2) to obtain the quadratic function

$$q_1(x) \cdot \frac{x^2 - x}{2}$$

so that $q_1(x)$ agrees with $F(x)$ for integral x. We note that q_1 is a lower bound of F. In order to find an upper bound, we shift the graph of q_1 upward by $\frac{1}{8}$ unit to obtain the quadratic function

$$q_2(x) = \frac{x^2 - x}{2} + \frac{1}{8}.$$

We also note that the graph of F is tangent to the graph of q_2 at $n + \frac{1}{2}$ for every integer n.

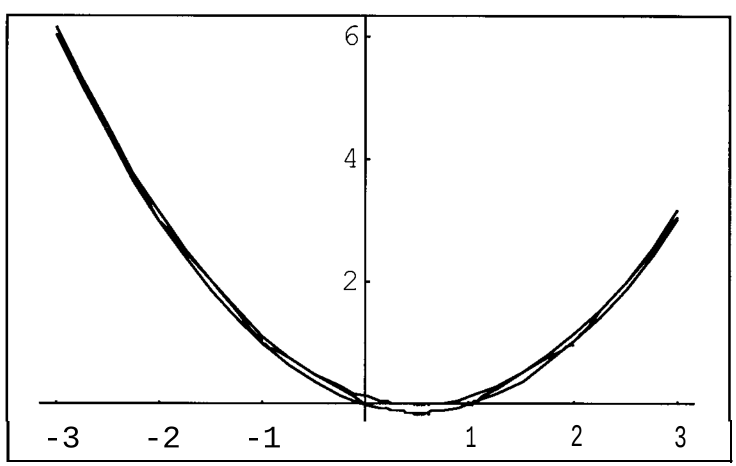

Figure 2. Graphs of q_1, F, and q_2.

“¯A-ssuggested in Figure2, the functions q_1 and q_2 are good approximations for F’. By using (1), the errors for $q_1(x)$ and $q_2(x)$ are obtained as

$$E_1(x) = \frac{1}{8} - \frac{1}{2}\left(\frac{1}{\pi}\arctan(\cot(\pi x))\right)^2 \text{ and } 132(z)= \frac{1}{2}\left(\frac{1}{\pi}\arctan(\cot(\pi x))\right)^2$$

respectively. The error here means that the absolute value of the difference between the approximation and the actual value of F. As one can see, the sum of the errors is always $\frac{1}{8}$, just as the difference of $q_2(x)$ and $q_1(x)$ is always $\frac{1}{8}$. It can also be proven that the errors are periodic of period 1 and are bounded by $\frac{1}{8}$. The graphs of the errors, E_1 and E_2, are displayed in Figures 3 and 4 respectively. The error bound $\frac{1}{8}$ may seem large, but as $|x|$ increases, the significance of $\frac{1}{8}$ becomes negligible. For example, the integral $\int_0^{123.4}[t]dt$ has an exact value of 7552.2, while the approximations $q_1(123.4)$ and $q_2(123.4)$ are 7552.08 and 7552.205 respectively.

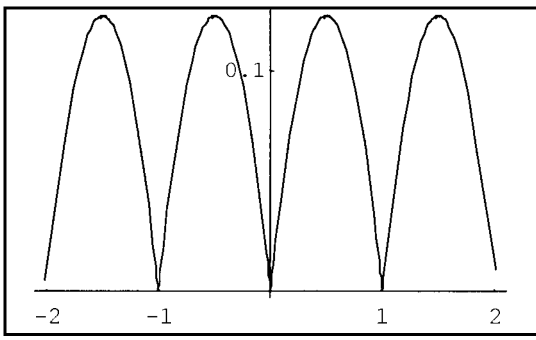

Figure 3. Graph of *El*.

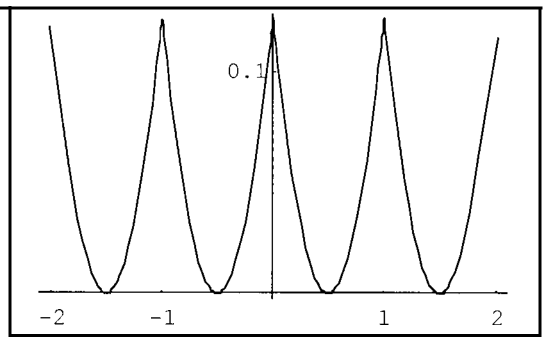

Figure 4. Graph of E_2.

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REFERENCES

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Biographical Information

DAVID FUNG, is an Assistant Professor of Mathematics at Southeastern Louisiana University. His research interest is in applied mathematics.

STEVE LIGH, is Professor and Head of the Department of Mathematics at Southeastern Louisiana University. His research areas include algebra, number theory and matrix theory.