Introduction to MATLAB Using a
Theoretical Mechanics Study-Case

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Abstract
The paper describes a 5 weeks module of a 3 credits Advanced Mechanics of Materials class taught in the Fall 2002 semester at University of Bucharest. The module realized an introduction to the plotting, matrix computation and scientific algebra capabilities of the software MATLAB, while allowing the students to apply their newly acquired software knowledge to a meaningful and challenging physical problem, namely the behavior of a slender cylindrical bar in an uniaxial test. It is the conviction of the authors that this type of module it is well suited for the first part of an Advanced Mechanics of Materials class, or of any other class that starts by reviewing notions introduced in previous courses and that would benefit from the use by students of a symbolic calculation software.

Introduction

As the use of symbolic computation software (be it MATLAB, Mathcad, MAPLE or Mathematica) and numerical simulation becomes prevalent in the professional practice of mathematicians, theoretical mechanicians and engineers, so it becomes more important to teach the students the basics of these software programs. As many times the software used in a certain course is very specialized, it is not possible to have a course at freshman or sophomore level introducing all software do be used during the course of study.

The authors advocate for integrating the software learning into the teaching of the subject(s) where the software would be used. The introductory part of the course, usually dedicated to review of notions taught in prerequisite courses, is the natural time for introducing the software. In this way, all the tools necessary for the successful completion of the course are presented to all students, and the mix of old and new material keeps students interested in learning. The authors consider the approach superior to the one in which a review session of all theoretical notions is followed by labs/ home assignments in which students have to read through software tutorial. As common experience and pedagogical experience repeatedly prove, one learns better when what s/he learns has a visible application and is reinforced by use.

The paper describes a 5 weeks, three periods/week module of a first year graduate level Mechanics of Materials class taught in the Fall 2002 semester at University of Bucharest. The goal of the module was dual: (1) to introduce the students to the software MATLAB v.6.5 [1] and its use for scientific computation and (2) to provide a challenging problem for immediate application of the MATLAB knowledge. The challenges of this endeavor were multiple, as the students of the Mechanics of Materials class had extensive experience in theoretical mechanics but very variable English and computer proficiency, as the language of instruction was Romanian while most MATLAB documentation was in English, and as the traditional teaching format is based on lecture rather than on hands-on experiences.

The first author provided her expertise on MATLAB, while the second author, the principal instructor of the Mechanics of Materials class introduced the mechanical problem and stressed the physical significance of the results obtained from MATLAB computations. The students learning performance was consistent with their GPA, however all students were more open to asking questions and genuinely more active in the class. As a result, the module will be taught again for the next offering of the course.

Physical Problem Statement:

The behavior of a slender cylindrical bar made of hyperelastic material in a uniaxial test is a reasonably simple yet challenging problem. Variations in the constitutive law provide very interesting bifurcations in the solution of the boundary problem, as illustrated in the papers of Owen [2] and Spector [3]. On short, the in-depth study of this problem has illustrative potential for the whole syllabus of the Mechanics of the Material class.
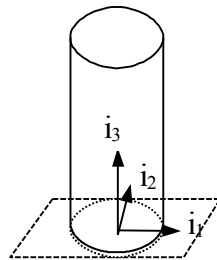

Figure 1 Cylindrical specimen in uniaxial test

A cylindrical specimen in a uniaxial test (see figure 1) supports an axial deformation of l_3 and a radial deformation of l_1. Assume that the material is hyperelastic. Denote by **F** the gradient of deformation, by $B = F\,F^T$ and by J_1, J_2, J_3 the invariants of **B**.

$$J_1 = \frac{trace(B)}{2}; J_2 = \frac{trace(B^2)}{4}; J_3 = \sqrt{\det(B)}$$.

Remember that for hyperelastic materials, the Piola-Kirchkoff stress of first type **S** can be obtained from a potential F depending only on the invariants J_1, J_2, J_3 using the formula

$$S = 2\frac{\partial \Phi}{\partial B} \, \mathrm{o} \, F$$
, where by definition the differential of the potential is $\left(\dfrac{\partial \Phi}{\partial B}\right)_{ij} := \dfrac{\partial \Phi}{\partial B_{ij}}$

Assume for the potential the form proposed by Blausius-Ko

$$\Phi_{BC}(J_1, J_2, J_3) = \left(\frac{J_1}{2} - J_2\right)\frac{1}{J_3^{\,2}} + J_3$$

1) Obtain a stress-deformation curve for axial component of the stress.

2) Repeat the procedure and obtain the stress-deformation curve for the rubber-type potential

$$\Phi_R(J_1, J_2, J_3) = J_1 + \frac{1}{mJ_3^{\,m}}$$
, where m=13.3

We dedicate two class periods in the first week to the almost complete analytical solution of the above problem. For the following class periods, up to the module completion, we intercalate steps in the solution with immediate MATLAB applications.

Lab and Assignment Plan

In the following section, we present the detailed plan for introducing different MATLAB functions and the assignments used to verify the learning. While some assignments are specific to the problem we have studied, most of them can be used to review fundamental notions of Mechanics of Materials while introducing the most common MATLAB functions. It is the authors' hope that others will be able to use these plans in their class, with minimal modifications.

Lab 1: Introduce the MATLAB software and the search procedures (in-line `help` and `lookfor`, help menu search). Introduction to matrix operations (read and work examples in MATLAB Help –>Getting Started-> Manipulating Matrices). Introduce the notion of symbolic and numerical variable, as well as the notion of script.

Assignment 1: Define in at least 3 different ways the 3x3 matrix F having the elements [1 1 5] on the diagonal (5 minutes, in class). Compare the results obtained in class and decide which definition is the simplest.
Objective: Checks the understanding of the notion of variable, and the use of element-by-element matrix definition, `zeros` and `diag` built in function.

Assignment 2: Compute the transpose of matrix F, F'. Compute the matrix B=F*F' and its invariants (5 minutes, in class). What does it mean that F=F'?
Objective: Check the use of matrix arithmetical operations and of the built-in MATLAB functions `transpose`, `trace` and `det`.

Assignment 3: For the matrix R having ones everywhere but in the upper left corner where it has 2, compute S=R*R'. Compute T=R*.R'(3 minutes, in class). Justify whether R is symmetrical or not. Why are S and T different?

<u>Objective</u>: Check the use of matrix arithmetical operations and of predefined matrix `ones.` Instructor, please discuss about the difference between matrix operations and element-by-element operations (exemplify with *times(*)* and *mtimes(.*)* operations).

Activity 1: From Help menu, read and do the exercises from the 'Symbolic Math Toolbox''Getting started'
<u>Objective</u>: Covers the definition of symbolic variables and the conversion between symbolic and numeric variables (functions `sym, syms, double`). Introduces symbolic matrices, symbolic functions, as well as the function `findsym`.
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Activity 2:
- Verify what variable you have in the work space and what is their type (who/`whos`).
- Define the symbolic variables lambda_1, lambda_2, lambda_3 and redefine matrix F having lambda_1, lambda_2, lambda_3 on the diagonal.
- Verify again what variables you have in the workspace.
- Copy the definition for lambdas and F in a new M-file, save it in a directory as `L1_YourInitials.m.` Congratulations, you had created a script.
- Clear the workspace. Check that you don't have any variables. (`clear, whos`)
- Run the script. Check that you now have variables.(`run`)

Assignment 4: For F defined as above, compute the matrix B=F*F' and its invariants (last 5 minutes of class/ home assignment).

Lab 2: Review of last lab (run, matrix definition and multiplication, definition of symbolic variables, operations with symbolic variables). Calculus with symbolic variables. Simplifications of symbolic expressions, substitution.

Activity 3: Check the results of the home assignment. You should have obtained

$$F = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad B = FF^T = \begin{pmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{pmatrix},$$

$$J_1 = \frac{tr(B)}{2} = \frac{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}{2}, \; J_2 = \frac{tr(B^4)}{2} = \frac{\lambda_1^4 + \lambda_2^4 + \lambda_3^4}{4}, \; J_3 = \sqrt{\det(B)} = \lambda_1 \lambda_2 \lambda_3$$

Activity 4: From Help menu, read and do the exercises from the 'Symbolic Math Toolbox'->'Using the Symbolic Math Toolbox' screens on 'Calculus'.

Assignment 5: Open a new M-file, define a symbolic matrix B, B symmetrical, and its invariants J1=1/2*trace(B); J2=1/4*trace(B*B); J3=sqrt(det(B)); you can use part of your previous file `L1_Yourinitials.m.` Add a comment to it. Save your new script as

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`L2_Yourinitials.m` and close it (5 minutes, in class).
Objective: Review the matrix introduction and the definition of symmetrical matrix; check the use of `copy` and `paste` or `insert` and comment lines (%)

Assignment 6: Open your file `L2_YourInitials.m`. Define the potential

$$\Phi_{BC}(J_1,J_2,J_3) = \left(\frac{J_1}{2} - J_2\right)\frac{1}{J_3^{\,2}} + J_3$$

and compute the matrix of partial derivatives

$$\left(\frac{\partial \Phi}{\partial B}\right)_{ij} := \frac{\partial \Phi}{\partial B_{ij}}$$

(Attention, greek letters are not used in MATLAB commands, try phi_BC for the name of the potential and Dphi_DB for the name of the matrix). Run and debug the script. Compare (in the command window) the results with what you would obtain by running diff(F*). Justify your result. (5 minutes, in class).
Objective: Check the use of functions `diff` and `findsym`.

Activity 5: From Help menu, read and do the exercises from the 'Symbolic Math Toolbox'->'Using the Symbolic Math Toolbox' screens on 'Symplifications and Substitutions'

Assignment7: At the end of `L2_Yourinitials.m,` add the definition of F=diag([lambda_1, lambda_2, lambda_3]), and of B=FF^T. Compute

$$S = 2\frac{\partial \Phi}{\partial B} \circ F$$

by substituting in the matrix Dphi_DB the values of b_i with the new values found in B (now they are functions of lambda_i)
Objective: Use the function `subs`.

Assignment 8: Define S11_0 as S_{11} in which we substitute lambda_2 with lambda_1. Simplify its form and pretty type S11_0. Repeat the procedure for S33_0. See what is the difference between simple(S11_0) and simplify(S11_0).
Objective: Use of functions `subs`, `simple`, `pretty` and `simplify`.

Activity 6:a) In a new file `L3_Yourinitials_HW1.m`, define the potential F as a function of the symbolic variables J1, J2, J3. Compute the partial derivatives of F *w.r.t.* J1, J2, J3. Define the new variables S33 and S11 as

$$S_{33} := I_3\left[\frac{\partial \Phi}{\partial J_1} + I_3^{\,2}\frac{\partial \Phi}{\partial J_2}\right] + I_1^{\,2}\frac{\partial \Phi}{\partial J_3}, \quad S_{11} := \frac{\partial \Phi}{\partial J_1} + I_1^{\,2}\frac{\partial \Phi}{\partial J_2} + I_3\ \frac{\partial \Phi}{\partial J_3}$$

Introduce F=diag([lambda_1, lambda_1, lambda_3]), B=FF^T and the definition of the invariants. Substitute in the expression of S11 and S33 the expressions of the invariants as functions of lambdas and rename the simplified expressions S11_00 and S33_00. What is the difference between S11_0 and S11_00? What about S33_0 and S33_00 ? What theoretical result have you obtained?
b)Save `L3_Yourinitials_HW1.m` as `L3_Yourinitials_HW2.m` and move the potential definition AFTER the definition of F, B and J. Run the new file. Why do you think you get an error message? (last 15 minutes of class/home)

Lab 3: Review of home assignment (diff, substitution in and simplification of symbolic expressions). Solving equations. 2D graphics

Activity 7: Review of home assignment. You should have obtained that S11_0= S11_00 and S33_0= S33_00 which shows that

$$S_{33} := \left(2\frac{\partial\Phi}{\partial B}\circ F\right)_{33} \equiv \lambda_3\left[\frac{\partial\Phi}{\partial J_1} + \lambda_3^2\frac{\partial\Phi}{\partial J_2}\right] + \lambda_1^2\frac{\partial\Phi}{\partial J_3},$$

$$S_{11} := \left(2\frac{\partial\Phi}{\partial B}\circ F\right)_{11} \equiv \frac{\partial\Phi}{\partial J_1} + \lambda_1^2\frac{\partial\Phi}{\partial J_2} + \lambda_3\frac{\partial\Phi}{\partial J_3}$$

For $\lambda_2 = \lambda_1$, the above expressions with respect to λ_1, λ_3 become

$$S_{33\_0} = \frac{1}{2\lambda_1^4} - \frac{1}{\lambda_1^2\lambda_3^4} + \frac{1}{\lambda_3^4} + \frac{\lambda_1^2}{\lambda_3} = \frac{-\lambda_3^4 - 2\lambda_1^2 + 2\lambda_1^4 + 2\lambda_1^6\lambda_3^5}{2\lambda_1^4\lambda_3^4}$$

$$S_{11\_0} = -\frac{1}{2\lambda_1^4\lambda_3^2} - \frac{1}{2\lambda_1^6} + \frac{\lambda_3^2}{2\lambda_1^6} + \lambda_3 = \frac{-\lambda_1^2 - \lambda_3^2 + \lambda_3^4 + 2\lambda_1^6\lambda_3^3}{2\lambda_1^6\lambda_3^2}$$

The condition $S_{11\_0}=0$ is equivalent to the physical condition that the specimen is free of stresses on the lateral surfaces. We will use this condition in a simplified form to obtain an implicit equation in λ_1, λ_3, which solved for λ_1 gives $\lambda_1 = \lambda_1(\lambda_3)$.

We will further substitute $\lambda_1 = \lambda_1(\lambda_3)$ in the expression of S33_0 to obtain a representation of the axial stress as function only of the axial elongation.

We will shortly show how MATLAB solves the implicit equation in λ_1, λ_3. This equation can be simplified to

$$-\lambda_1^2 - \lambda_3^2 + \lambda_3^4 + 2\lambda_1^6\lambda_3^4 = 0$$

and after the substitution $z = \lambda_1^2$ becomes

$$2z^3\lambda_3^3 - z - \lambda_3^2(1 - \lambda_3^2) = 0$$

From the 3 roots of the equation in z we are interested only in the positive one(s). We can make a parametric study of the number of solution of the equation in z w.r.t. λ_3 and we obtain that for $\lambda_3 < 1$ (compression) there is at least a positive root in z.

Activity 8: From Help menu, read and do the exercises from the 'Symbolic Math Toolbox'->'Using the Symbolic Math Toolbox', screen on 'Solving Equations'..

Assignment 9: Identify the numerator and the denominator of S_{11_0}(hint: search for functions working on numerator). Denote by *n* the numerator. Substitute in *n* λ_1=z, denote the new expression by *eq1* and simplify (hint: search for functions working on numerator). Solve the equation *eq1=0* and stock the roots in the vector V. Choose what component of V is the positive solution using the fact that in the neighborhood of λ_3=1 there is for sure a positive solution (and

for $\lambda_3=1$ on this solution we obtain $\lambda_1=1$). Denote this component by *z*.
<u>Objective</u>: Use functions `lookfor` (or search after full text in 'Help'), `numden`, `solve,` and `subs`.

Activity 9: At this point, we observe that for the initial form of the potential, the condition $S_{11}=S_{22}=0$ for $\lambda_1=\lambda_2=\lambda_3=1$ it is not satisfied. We ask the class what does it mean and where do they think there is a mistake (2 minutes individual thinking time, 5 minutes collective discussions). We decide that the mathematical model (the initial potential form) is not compatible with the physical interpretation (the lateral surface is stress-free in any configuration, including the initial configuration i.e. $S_{11}=S_{22}=0$ for $\lambda_1=\lambda_2=\lambda_3=1$). We decide to try a new form of the potential, for example

$$\Phi_{BC}(J_1,J_2,J_3) = \left(J_1^2 - J_2\right)\frac{1}{J_3^2} + J_3 .$$

Assignment 10: Redo assignments 6-9 for the new form of the potential. Save the results obtained by editing `L3_YourInitials.m` in `L4_YourInitials.m` (10 minutes in class). <u>Objective</u>: Exercise editing and saving a file.

Activity 10: From Help menu, read and do the exercises from the 'Getting Started'->'Graphics', screens on 'Basic Plots', 'Editing Plots'. Read about the function ezplot.

Assignment 11: Plot λ_1 for λ_3 between 0 and 5, using `ezplot`. Label properly the axis and add a useful title. Save the figures in .fig and .tif formats. Plot *z* for λ_3 between 0 and 5 using the function plot (last 10 min in class/ homework). <u>Objective</u>: Use of functions `ezplot` and `plot.`

Lab 4: Review of home assignment (2D graphics, `ezplot`). 3D graphics. Development environment

Activity 11: From Help menu, read and do the exercises from the 'Getting Started'->'Graphics', screens on 'Basic Plots', 'Editing Plots'.

Assignment 12 : Plot S_{11} as a general function of λ_1, λ_3 varying between 0.5 and 2 (out of these limits you are probably out of linear elasticity). Mark on this graph the contour line $S_{11}=0$. <u>Objective</u>: Use of function `ezmesh` and `contour3`.

Assignment 13: Substitute your choice of Z in the expression for S_{33}. you have obtained a function only of λ_3. In the same figure, do: (a) the plot of Assignment 12; (b) the 2D plot of Assignment 11(the one using ezplot); (c)a 3D surface plot of S_{33} as a function of λ_1, λ_3 varying between 0.5 and 2 (d) a 2D plot of S_{33} as function of λ_1. Save the obtained graphics in .fig and .tif formats.

Objective: Check 2 D and 3D graphics (functions `subplot, ezplot, ezmesh, surf` and operations on graphics).

Assignment 14: In a report, present the physical problem and its solution (with interpretation and graphics). Describe what facilities of MATLAB can you use now and how do you plan to learn more about MATLAB (homework).

Conclusions

The paper described a 5 weeks module of a 3 credits Advanced Mechanics of Materials class taught in the Fall 2002 semester at University of Bucharest. The module realized an introduction to the plotting, matrix computation and scientific algebra capabilities of the software MATLAB, integrated with review of notations and mechanics notions introduced in prerequisite courses and illustrated with the reasonably simple yet challenging problem of describing a slender cylindrical bar made of hyperelastic material in a uniaxial test.

The students learning performance during this module was consistent with their previous GPA, however all students were more open to asking questions and genuinely more active in the class. In an informal survey, all students expressed enthusiasm in the format, with few students complaining about the pace of assignments. As a result, the module will be taught again for the next offering of the course.

While some assignments are specific to the problem we have studied, most of them can be used to review fundamental notions of Mechanics of Materials while introducing the most common MATLAB functions and symbolic manipulation capabilities. It is the authors' hope that others will be able to use these plans in their class, with minimal modifications.

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