2006-823: LEARNING THE VIRTUAL WORK METHOD IN STATICS: WHAT IS A COMPATIBLE VIRTUAL DISPLACEMENT?

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Abstract

Statics is a course aimed at developing in students the concepts and skills related to the analysis and prediction of conditions of bodies under the action of balanced force systems. At a number of institutions, learning the traditional approach using force and moment equilibrium equations is followed by learning the energy approach using the virtual work method to enrich the learning of students. The transition from the traditional approach to the energy approach requires learning several related key concepts and strategy. Among others, **compatible virtual displacement** is a key concept, which is compatible with what is required in the virtual work method but is not commonly recognized and emphasized. The virtual work method is initially not easy to learn for many people. It is surmountable when one understands the following: (*a*) the proper steps and strategy in the method, (*b*) the displacement center, (*c*) some basic geometry, and (*d*) the radian measure formula to compute virtual displacements. For learning and pedagogical purposes, this paper includes seven examples with various levels of challenge.

I. Introduction

More often than not, it is manifested that the virtual work method is used to treat problems involving mainly *machines*. This manifestation comes about as a consequence of focusing on the determination of the equilibrium configuration of a series of pin-connected members by restricting virtual displacements to be *consistent* with constraints at the supports. In general, such a restriction is too strong and is an *over restriction*. It prevents the virtual work method from being effectively used to treat problems involving *beams* and *frames*, and it diminishes the usefulness of the virtual work method in Statics. As a result, some feel that the virtual work method lacks broad appeal in Statics. Nonetheless, the virtual work method is a standing topic contained in most textbooks of Statics. By and large, such a topic is covered in Statics at the discretion of the instructors to enrich the learning of students.

Both the traditional method and the virtual work method *equally* require and emphasize the drawing of free-body diagrams, although the former involves more algebra and the latter uses more geometry in solving problems. Most students find that learning the virtual work method is challenging, since they are generally better at algebra than geometry. It is *not* the intent of this paper to urge anyone to teach the virtual work method or to upstage the time-honored traditional method in Statics. Rather, this paper is mainly aimed at being an extension to previous efforts[1-9] of mechanics educators and textbook authors who included the virtual work method in Statics. In particular, this paper identifies key concepts, steps, and strategy that have been helpful to students in learning the virtual work method. Readers, who are familiar with this method, may **skip** the **refresher** on the rudiments included in the early part of this paper.

A *displacement* of a body is a change of position of the body. A *rigid-body displacement* of a body is a change of position of the body without inducing any strain in the body. A *virtual dis-*

placement of a body is an imaginary, first-order differential displacement, which is possible but does not actually take place. A **rigid-body virtual displacement** of a body is a rigid-body displacement as well as a virtual displacement of the body, where the body undergoes an imaginary, first-order differential deflection to a neighboring position *without* experiencing any *strain*.

Work is energy in transition to a system due to force or moment acting on the system during a displacement of the system. *Heat* is energy in transition to a system due to temperature difference between the system and its surroundings. Work, as well as heat, is dependent on the path of a process. Like heat, work crosses the system boundary when the system undergoes a process. Unlike kinetic energy and potential energy, work is not a property possessed by a system. In mechanics, a body receives work from a force or a moment that acts on it while it undergoes a displacement in the direction of the force or moment, respectively, during the action. It is the force or moment, rather than the body, which does work. A *virtual work* is the work done by force or moment during a virtual displacement of the body.

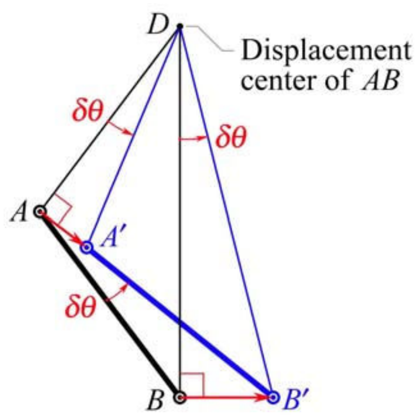

Fig. 1 Compatible virtual displacement of body AB to position $A'B'$

In virtual work method, *compatible virtual displacements* (besides rigid-body virtual displacements) are to be used, where second-order (*not* first-order) straining of members in a system is permitted in drawing *virtual displacement diagrams*. This may initially come across as being against the grain of the usual mentality of *rigid* bodies held for Statics. Notwithstanding, a working definition is in order. As shown in Fig. 1, a **compatible virtual displacement** of a body AB is an imaginary displacement resulting from a *first-order* differential angular displacement $\delta\theta$ of the body about a certain point D, called its **displacement center**,[6,7] during which the body *deflects* from position AB to another position $A'B'$ and the following conditions exist

$$\overline{AA'} \perp \overline{AD} \qquad \overline{BB'} \perp \overline{BD} \qquad \overline{A'B'} \geq \overline{AB}$$

The displacement vectors $\overrightarrow{AA'}$ and $\overrightarrow{BB'}$ in Fig. 1 are called the *compatible virtual displacements* of points A and B, respectively.

The concept of compatible virtual displacement is compatible with what is required in the virtual work method, and it is a critical one in teaching and learning the virtual work method. Note that this concept has a subtle shade of difference from the traditional concept of **rigid-body virtual displacement** defined earlier.

II. *Rigid-Body* versus *Compatible* Virtual Displacements

All bodies considered in this paper are rigid bodies or systems of pin-connected rigid bodies, where the pins are taken as frictionless and no resisting moments are developed in the joints. If a body AB undergoes a **rigid-body virtual displacement** by rotating about its end A, as its *displacement center*, through an angular displacement $\delta\theta$, it will experience no axial strain to reach the position AB'', not AB', as illustrated in Fig. 2. However, if the body AB undergoes a **compatible virtual displacement** by rotating about its end A, as its *displacement center*, through an angular displacement $\delta\theta$, it will experience some axial strain to reach the position AB', not AB'', as illustrated in Fig. 2.

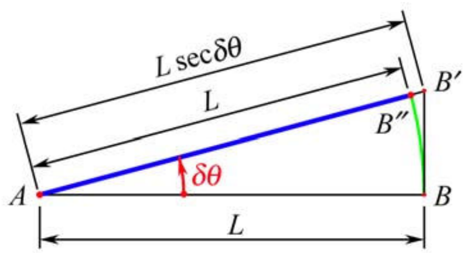

Fig. 2 ***Rigid-body*** versus ***compatible*** **virtual displacements** of body AB

Using series expansion in terms of the *first-order* differential angular displacement $\delta\theta$, which is *infinitesimal*, we find that the difference between AB' and AB'' is given by

$$\overline{B''B'} = L\sec\delta\theta - L = L\left[1+\tfrac{1}{2}(\delta\theta)^2+\tfrac{5}{24}(\delta\theta)^4+\tfrac{61}{720}(\delta\theta)^6+\cdots\right]-L$$

$$\therefore \quad \overline{B''B'} \approx \tfrac{L}{2}(\delta\theta)^2 + \text{higher order terms of } \delta\theta \tag{1}$$

Since $\delta\theta$ is infinitesimally small, the amount of *axial strain* experienced by body AB is

$$\varepsilon = \frac{\overline{B''B'}}{\overline{AB}} \approx \frac{\tfrac{L}{2}(\delta\theta)^2}{L} = \tfrac{1}{2}(\delta\theta)^2 \tag{2}$$

Thus, in undergoing compatible virtual displacement from position AB to position AB', the body AB may, *at most*, experience an axial strain of the second order of $\delta\theta$. Meanwhile, the *compatible virtual displacement* of point B in Fig. 2 is the displacement vector from B to B'. The magnitude of this displacement vector is

$$\overline{BB'} = L\tan\delta\theta = L\left[\delta\theta+\tfrac{1}{3}(\delta\theta)^3+\tfrac{2}{15}(\delta\theta)^5+\tfrac{17}{315}(\delta\theta)^7+\cdots\right] \approx L\,\delta\theta \tag{3}$$

In Fig. 2, the lengths of the chord $\overline{BB'}$ and the arc $\widehat{BB''}$ can be taken as equal as $\delta\theta \to 0$ in the limit. Equation (3) shows that the magnitude of the *compatible virtual displacement* of point B may indeed be computed by using the *radian measure formula* in calculus; i.e.,

$$s = r\theta \tag{4}$$

where s is the arc subtending an angle θ (in radians) included by two radii of length r. Note that Eq. (4) is *extremely* useful and important in solving problems by the virtual work method!

III. Relevant Fundamental Concepts

In teaching and learning the virtual work method, it is well to recall the following relevant fundamental concepts:

- **Work of a force**

If a force **F** acting on a body is *constant* and the displacement vector of the body from position A_1 to position A_2 during the action is **q**, then the *work* $U_{1\to 2}$ *of the force* **F** on the body is[2-6, 8,9]

$$U_{1\to 2} = \mathbf{F}\cdot\mathbf{q} = Fq_{\parallel} \tag{5}$$

where F is the magnitude of **F** and $q_{\parallel}$ is the scalar component of **q** parallel to **F**. If the force is not constant, then integration may be used to compute the work of the force.

- **Work of a moment**

If a moment **M** (or a couple of moment **M**) acting on a body is *constant* and the angular displacement of the body from angular position θ_1 to angular position θ_2 in the direction of **M** during the action is $\Delta\theta$, then the *work* $U_{1\to 2}$ *of the moment* **M** on the body is[2-6, 8,9]

$$U_{1\to 2} = M(\Delta\theta) \tag{6}$$

where M is the magnitude of **M**. If the moment is not constant, then integration may be used to compute the work of the moment.

- **Principle of virtual work**

Recall that bodies considered here are rigid bodies or systems of pin-connected rigid bodies. The term "force system" denotes a system of *forces* and *moments*, if any. The work done by a force system on a body during a *virtual displacement* of the body is the **virtual work** of the force system. By Newton's third law, internal forces in a body, or a system of pin-connected rigid bodies, must occur in pairs; they are equal in magnitude and opposite in directions in each pair. Clearly, the total virtual work done by the internal forces during a virtual displacement of a body, or a system of pin-connected rigid bodies, must be zero. When a body, or a system of pin-connected rigid bodies, is in equilibrium, the *resultant force* and the *resultant moment* acting *on* its free body must both be zero.

The total virtual work done by the force system acting *on* the free body of a body is, by the distributive property of *dot product of vectors*, equal to the total virtual work done by the resultant force and the resultant moment acting on the free body, which are both zero if the body is in equilibrium. Therefore, we have the **principle of virtual work** in Statics, which may be stated as follows: *If a body is in equilibrium, the total virtual work of the external force system acting on its free body during any compatible virtual displacement of its free body is equal to zero, and conversely.* [2-6, 8,9] Note that the body in this principle may be a particle, a set of connected particles, a rigid body, or a system of pin-connected rigid bodies (e.g., a frame or a machine). Using δU to denote the total virtual work done, we write the equation for this principle as

$$\delta U = 0 \tag{7}$$

- Conventional method **versus** virtual work method

With the **conventional method**, equilibrium problems are solved by applying two basic equilibrium equations: (*a*) *force equilibrium equation*, and (*b*) *moment equilibrium equation*; i.e.,

$$\Sigma \mathbf{F} = \mathbf{0} \tag{8}$$

$$\Sigma \mathbf{M}_P = \mathbf{0} \tag{9}$$

With the **virtual work method**, equilibrium problems are solved by applying the *virtual work equation*, which sets to zero the total virtual work δU done by the force system on the free body during a chosen *compatible virtual displacement* of the free body; i.e.,

$$\delta U = 0 \qquad \text{(Repeated)} \tag{7}$$

IV. Learning the Virtual Work Method: Examples

There are **three major steps** and **one strategy** in using the virtual work method. **Step 1:** *Draw the free-body diagram (FBD)*. **Step 2:** *Draw the virtual-displacement diagram* (*VDD*) with a strategy. **Step 3:** *Set to zero the total virtual work done*. The **strategy** in step 2 is to give the free body a compatible virtual displacement in such a way that *only the unknown to be determined*, besides the applied loads that are already known, will be involved in the total virtual work done. This strategy is the **key** to successful solutions of problems using the virtual work method. It is important to *understand and master this strategy* before attempting to solve any problem.

In a nutshell, the virtual work method simply consists of three major steps and one strategy! To be more helpful to those who wish to compare the features between the virtual work method and the traditional method, the following seven illustrative examples are chosen for pedagogical purposes and are arranged in increasing level of challenge. In particular, Example 1 presents a parallel comparison between the *traditional method* and the *virtual work method*. Naturally, teaching of the virtual work method in Statics is usually aimed at enriching the learning of students as circumstances warrant.

Example 1. Determine the vertical reaction force $\mathbf{B}_y$ at the roller support *B* of the simple beam loaded as shown in Fig. 3 by using (*a*) the *traditional method*, and (*b*) the *virtual work method.*

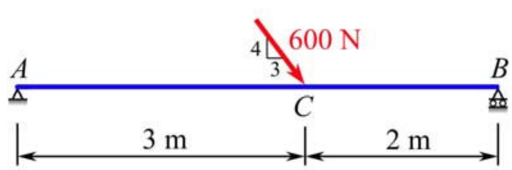

Fig. 3 A simple beam carrying an inclined concentrated load

Solution. Note that color codes are here employed in the solution to enhance head-to-head comparison of (*a*) the *traditional method*, and (*b*) the *virtual work method.*

(*a*) **Traditional method to solve for $\mathbf{B}_y$:** It is a usual procedure in the traditional method to first draw the *FBD* of the beam as shown in Fig. 4.

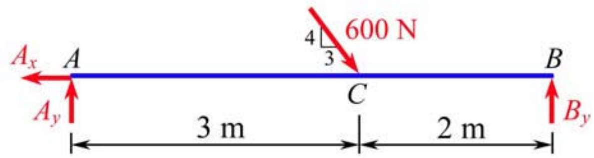

Fig. 4 Free-body diagram (*FBD*) for the simple beam

Next, we refer to the *FBD* in Fig. 4 and apply Eq. (9) to write

$$+\circlearrowleft \Sigma M_A = 0: \qquad -3(\tfrac{4}{5})(600) + 5B_y = 0$$

$$\therefore \quad B_y = 288 \qquad\qquad \mathbf{B}_y = 288 \text{ N} \uparrow$$

(*b*) **Virtual work method to solve for $\mathbf{B}_y$**: We shall follow the *steps* and *strategy* as outlined.

Step 1: We draw the *FBD* for the beam as shown in Fig. 4.

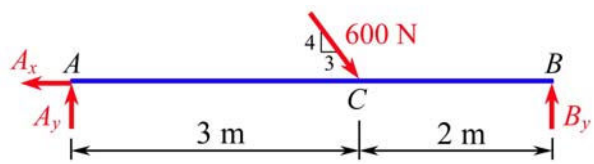

Fig. 4 *FBD* for the simple beam (Repeated)

Step 2: Keeping an eye on the *FBD* in Fig. 4, we draw a *VDD* for the beam with a **strategy** as shown in Fig. 5, where we let the beam rotate counterclockwise through an *imaginary* angular displacement $\delta\theta$ about point A as the *displacement center*, and we have applied Eq. (4), the *radian measure formula*, in labeling the magnitudes of the virtual displacements of points C and B as follows:

$$\overline{CC'} = 3\,\delta\theta \qquad\qquad \overline{BB'} = 5\,\delta\theta$$

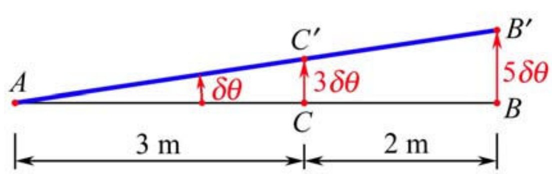

Fig. 5 Virtual-displacement diagram (*VDD*) for determining only B_y

The resulting *VDD* is well done because it will allow *no* unknowns *except* B_y to be involved in the total virtual work done.

Step 3: We refer to Figs. 4 and 5 and apply Eqs. (5) and (7) to write

$$\delta U = 0: \qquad \tfrac{4}{5}(600)(-3\,\delta\theta) + B_y(5\,\delta\theta) = 0$$

$$\therefore \quad B_y = 288 \qquad\qquad \mathbf{B}_y = 288 \text{ N} \uparrow$$

Remarks. We see in Example 1 that both the traditional method and the virtual work method can solve the same simple problem and arrive at the same solution. Although using the virtual work method to solve a simple problem may appear "unconventional," we shall see that this method has more advantage in solving more challenging problems, such as those shown below.

Example 2. A combined beam (a *Gerber beam*) is loaded as shown in Fig. 6. Using the *virtual work method*, determine *only* the reaction moment $\mathbf{M}_A$ at the fixed support A.

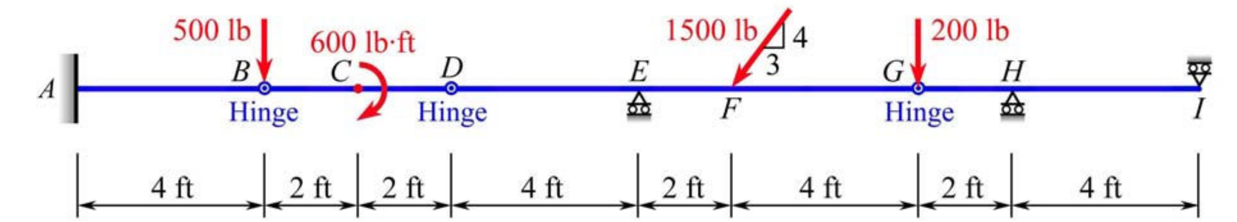

Fig. 6 A combined beam with hinge connections at B, D, and G

Solution. We shall follow the outlined *steps* and *strategy* in the method to *directly* solve for $\mathbf{M}_A$.
Step 1: We draw the *FBD* for the combined beam as shown in Fig. 7.

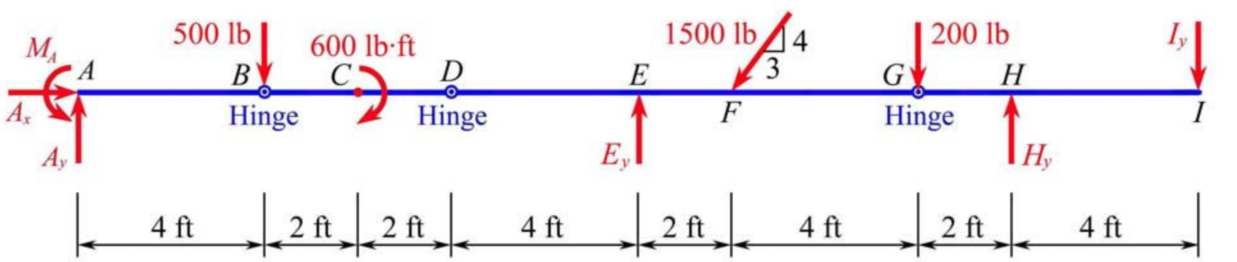

Fig. 7 *FBD* for the combined beam

Step 2: Keeping an eye on the *FBD* in Fig. 7 and applying Eq. (4), we draw a *VDD* for the beam with a **strategy** as shown in Fig. 8. Note that we let member *AB* rotate about A through an *imaginary* angle $\delta\theta \circlearrowleft$ to get a compatible virtual displacement of the beam as shown, where the members *DEFG* and *GHI* are unmoved according to the **strategy**. The resulting *VDD* is well done because it will allow *no* unknowns *except* M_A to be involved in the total virtual work done.

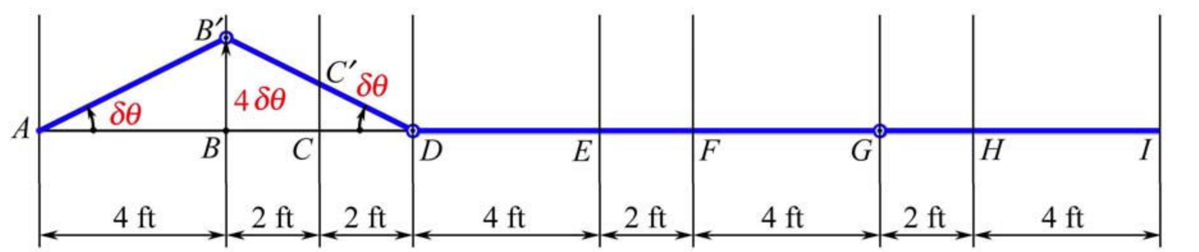

Fig. 8 *VDD* for determining only M_A

Step 3: We refer to Figs. 7 and 8 and apply Eqs. (5), (6), and (7) to write

$$\delta U = 0: \quad M_A(\delta\theta) + 500(-4\,\delta\theta) + 600(\delta\theta) = 0 \qquad M_A = 1400 \qquad \mathbf{M}_A = 1400 \text{ lb}\cdot\text{ft} \circlearrowleft$$

Remarks. If we are to use the **traditional method**, we refer to the *FBD* in Fig. 7 and write

- At hinge B, $M_B = 0$: $\quad M_A - 4A_y = 0 \qquad$ (i)
- At hinge D, $M_D = 0$: $\quad M_A - 8A_y + 4(500) - 600 = 0 \qquad$ (ii)

These *two* simultaneous equations yield: $A_y = 350$ and $M_A = 1400$.

Thus, the **traditional method** confirms the same solution: $\mathbf{M}_A = 1400 \text{ lb}\cdot\text{ft} \circlearrowleft$

Example 3. A combined beam is loaded as shown in Fig. 6. Using the *virtual work method*, determine *only* the vertical reaction force $\mathbf{A}_y$ at the fixed support A.

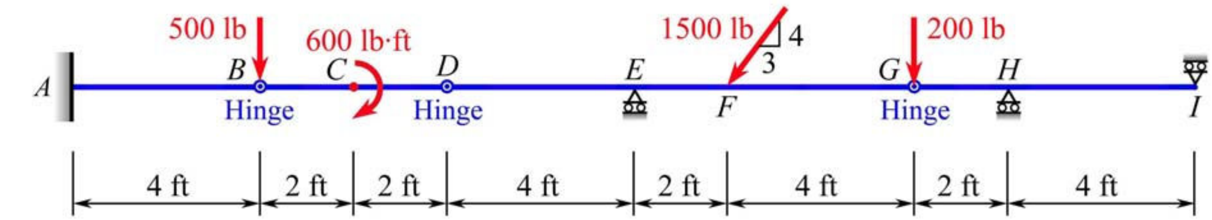

Fig. 6 A combined beam with hinge connections at B, D, and G (Repeated)

Solution. We shall follow the outlined *steps* and *strategy* in the method to *directly* solve for $\mathbf{A}_y$.
Step 1: We draw the *FBD* for the combined beam as shown in Fig. 7.

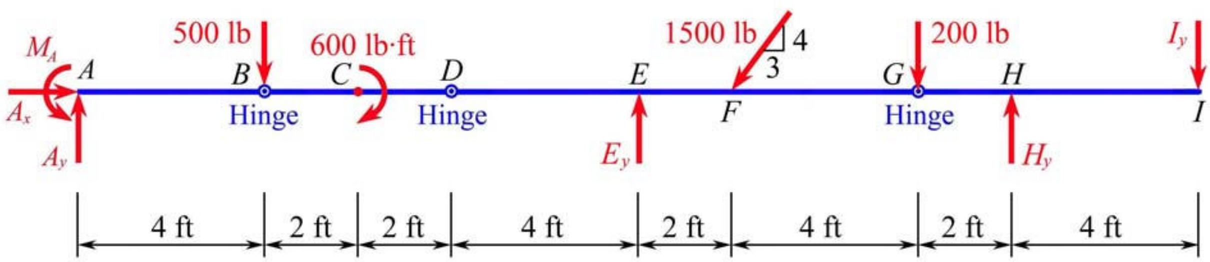

Fig. 7 *FBD* for the combined beam (Repeated)

Step 2: Keeping an eye on the *FBD* in Fig. 7 and applying Eq. (4), we draw a *VDD* for the beam with a **strategy** as shown in Fig. 9, where we let member *BCD* rotate about D through an *imaginary* angle $\delta\theta$ ↻ to get a compatible virtual displacement of the beam as shown, where the members *DEFG* and *GHI* are unmoved according to the **strategy**. The resulting *VDD* is well done because it will allow *no* unknowns *except* A_y to be involved in the total virtual work done.

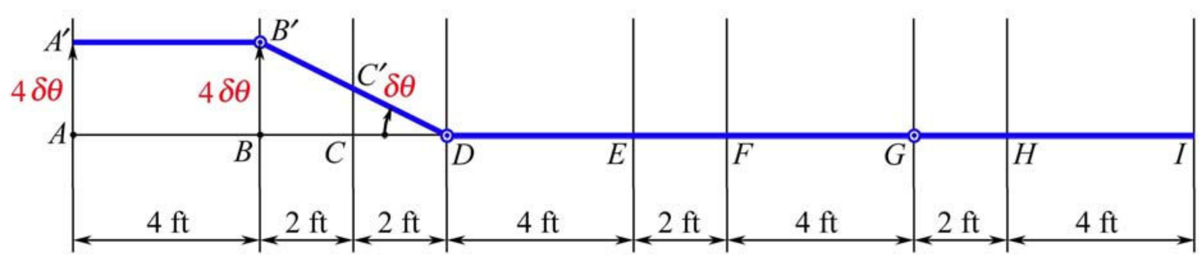

Fig. 9 *VDD* for determining only A_y

Step 3: We refer to Figs. 7 and 9 and apply Eqs. (5), (6), and (7) to write

$$\delta U = 0: \qquad A_y(4\,\delta\theta) + 500(-4\,\delta\theta) + 600(\delta\theta) = 0 \qquad A_y = 350 \qquad \mathbf{A}_y = 350\text{ lb } \uparrow$$

Remarks. If we are to use the **traditional method**, we refer to the *FBD* in Fig. 7 and write

- At hinge B, $M_B = 0$: $\quad M_A - 4A_y = 0 \quad$ (i)
- At hinge D, $M_D = 0$: $\quad M_A - 8A_y + 4(500) - 600 = 0 \quad$ (ii)

These *two* simultaneous equations yield: $A_y = 350$ and $M_A = 1400$.
Thus, the **traditional method** confirms the same solution: $\mathbf{A}_y = 350\text{ lb } \uparrow$

Example 4. A combined beam is loaded as shown in Fig. 6. Using the *virtual work method*, determine *only* the vertical reaction force $\mathbf{E}_y$ at the roller support *E*.

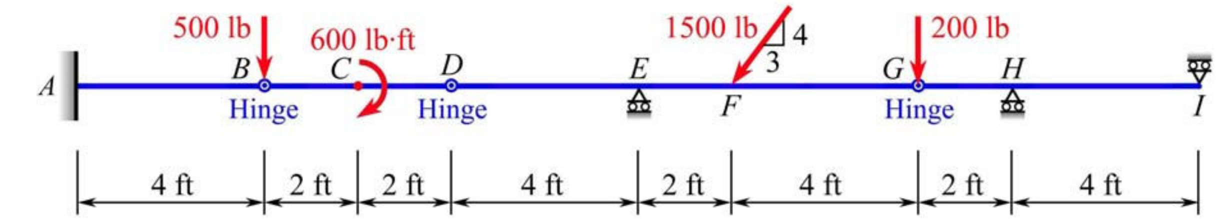

Fig. 6 A combined beam with hinge connections at *B*, *D*, and *G* (Repeated)

Solution. We shall follow the outlined *steps* and *strategy* in the method to *directly* solve for $\mathbf{E}_y$.

Step 1: We draw the *FBD* for the combined beam as shown in Fig. 7.

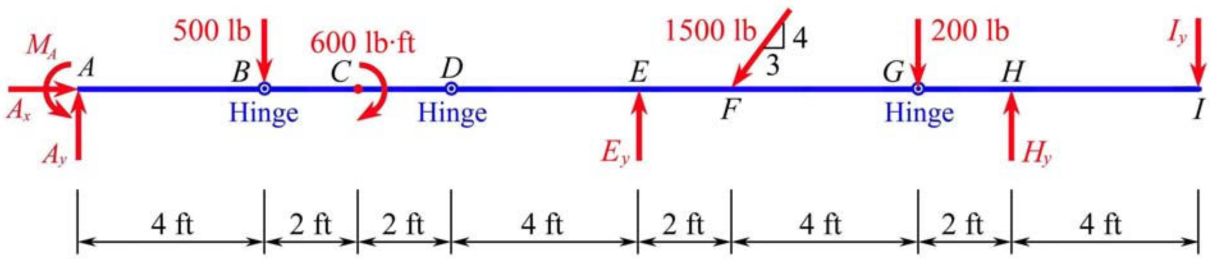

Fig. 7 *FBD* for the combined beam (Repeated)

Step 2: Keeping an eye on the *FBD* in Fig. 7 and applying Eq. (4), we draw a *VDD* for the beam with a **strategy** as shown in Fig. 10, where we let member *DEFG* rotate about *G* through an *imaginary* angle $\delta\theta$ ↻ to get a compatible virtual displacement of the beam as shown. Members *AB* and *GHI* are unmoved according to the **strategy**. The resulting *VDD* is well done because it will allow *no* unknowns *except* E_y to be involved in the total virtual work done.

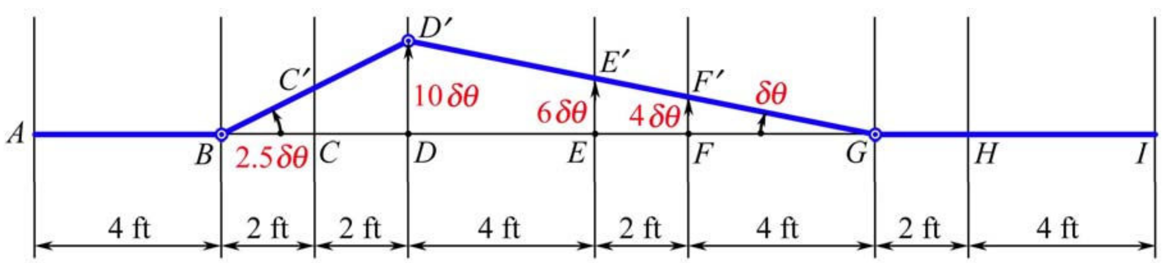

Fig. 10 *VDD* for determining only E_y

Step 3: We refer to Figs. 7 and 10 and apply Eqs. (5), (6), and (7) to write

$$\delta U = 0: \quad 600(-2.5\,\delta\theta) + E_y(6\,\delta\theta) + \tfrac{4}{5}(1500)(-4\,\delta\theta) = 0 \qquad E_y = 1050 \qquad \mathbf{E}_y = 1050 \text{ lb} \uparrow$$

Remarks. If we are to use the **traditional method**, we refer to the *FBD* in Fig. 7 and write

- At hinge *B*, $M_B = 0$: $\quad M_A - 4A_y = 0 \quad$ (i)
- At hinge *D*, $M_D = 0$: $\quad M_A - 8A_y + 4(500) - 600 = 0 \quad$ (ii)
- At hinge *G*, $M_G = 0$: $\quad M_A - 18A_y + 14(500) - 600 - 6E_y + 4(\tfrac{4}{5})(1500) = 0 \quad$ (iii)

These *three* simultaneous equations yield: $A_y = 350$, $M_A = 1400$, and $E_y = 1050$.
Thus, the **traditional method** confirms the same solution: $\mathbf{E}_y = 1050 \text{ lb} \uparrow$

Example 5. A combined beam is loaded as shown in Fig. 6. Using the *virtual work method*, determine *only* the vertical reaction force $\mathbf{H}_y$ at the roller support H.

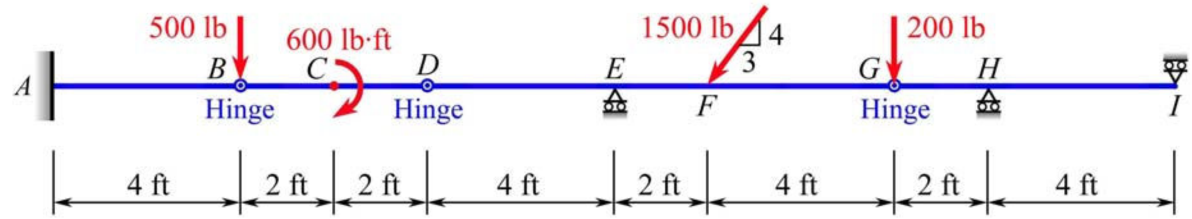

Fig. 6 A combined beam with hinge connections at B, D, and G (Repeated)

Solution. We shall follow the outlined *steps* and *strategy* in the method to *directly* solve for $\mathbf{H}_y$.

Step 1: We draw the *FBD* for the combined beam as shown in Fig. 7.

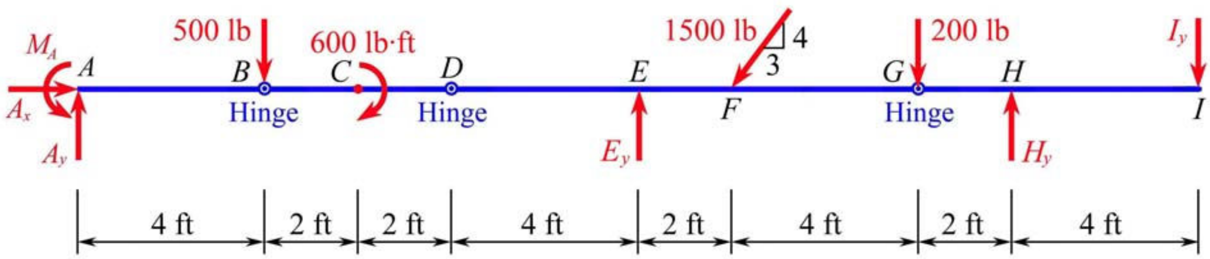

Fig. 7 *FBD* for the combined beam (Repeated)

Step 2: Keeping an eye on the *FBD* in Fig. 7 and applying Eq. (4), we draw a *VDD* for the beam with a **strategy** as shown in Fig. 11, where we let member *BCD* rotate about *B* through an *imaginary* angle $\delta\theta$ ↻ and unmoved points E and I to get a compatible virtual displacement of the beam as shown. Member *AB* is, by **strategy**, unmoved. The resulting *VDD* is well done because it will allow *no* unknowns *except* H_y to be involved in the total virtual work done.

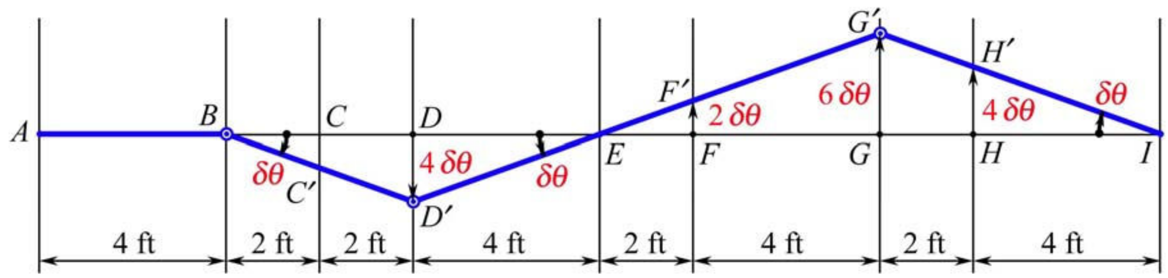

Fig. 11 *VDD* for determining only H_y

Step 3: We refer to Figs. 7 and 11 and apply Eqs. (5), (6), and (7) to write

$\delta U = 0$: $\quad 600(\delta\theta) + \frac{4}{5}(1500)(-2\,\delta\theta) + 200(-6\,\delta\theta) + H_y(4\,\delta\theta) = 0 \qquad H_y = 750 \qquad \mathbf{H}_y = 750 \text{ lb} \uparrow$

Remarks. If we are to use the **traditional method**, we refer to the *FBD* in Fig. 7 and write

- At hinge G, $M_G = 0$: $\quad -6I_y + 2H_y = 0 \qquad$ (i)
- At hinge D, $M_D = 0$: $\quad -16I_y + 12H_y - 10(200) - 6(\frac{4}{5})(1500) + 4E_y = 0 \qquad$ (ii)
- At hinge B, $M_B = 0$: $\quad -20I_y + 16H_y - 14(200) - 10(\frac{4}{5})(1500) + 8E_y - 600 = 0 \qquad$ (iii)

These *three* simultaneous equations yield: $E_y = 1050$, $H_y = 750$, and $I_y = 250$.

Thus, the **traditional method** confirms the same solution: $\mathbf{H}_y = 750 \text{ lb} \uparrow$

Example 6. A frame is loaded as shown in Fig. 12. Using the *virtual work method*, determine *only* the reaction moment $\mathbf{M}_A$ at the fixed support A.

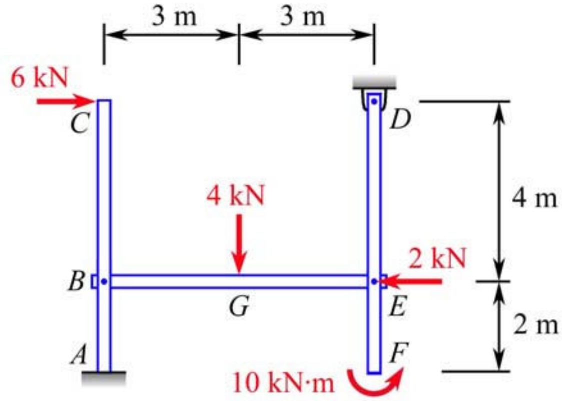

Fig. 12 A frame with a fixed support at A and a hinge support at D

Solution. We shall follow the outlined *steps* and *strategy* in the method to *directly* solve for $\mathbf{M}_A$.

Step 1: We draw the *FBD* for the frame as shown in Fig. 13.

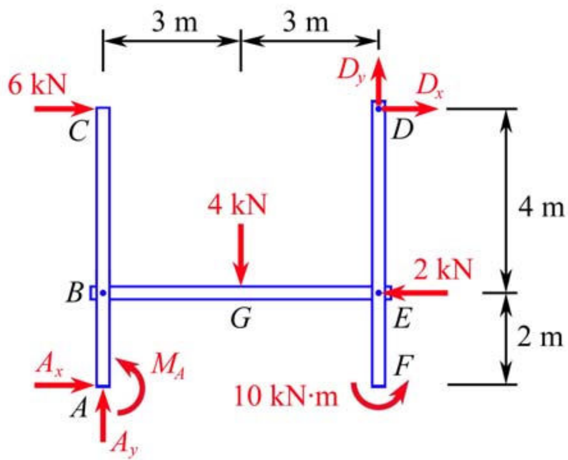

Fig. 13 *FBD* for the frame

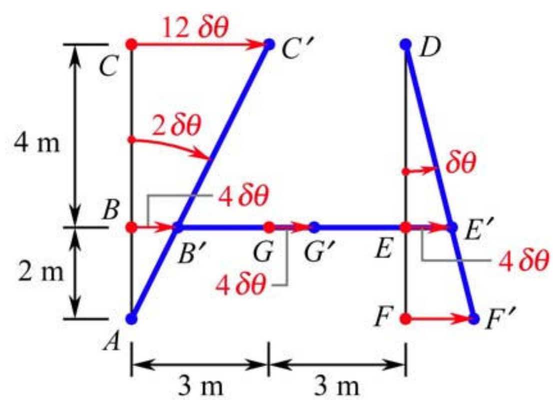

Fig. 14 *VDD* for determining only M_A

Step 2: Keeping an eye on the *FBD* in Fig. 13 and applying Eq. (4), we draw a *VDD* for the frame with a **strategy** as shown in Fig. 14, where we let member *DEF* rotate about D through an *imaginary* angle $\delta\theta$ ↺ and unmoved points A and D. The resulting *VDD* is well done because it will allow *no* unknowns *except* M_A to be involved in the total virtual work done.

Step 3: We refer to Figs. 13 and 14 and apply Eqs. (5), (6), and (7) to write

$\delta U = 0$: $\quad M_A(-2\,\delta\theta) + 6(12\,\delta\theta) + 2(-4\,\delta\theta) + 10\,\delta\theta = 0 \qquad M_A = 37 \qquad$ $\mathbf{M}_A = 37$ kN·m ↺

Remarks. If we are to use the **traditional method**, we refer to the *FBD* in Fig. 13 and write

- At pin B of member *ABC*, $M_B = 0$: $\quad M_A + 2A_x - 4(6) = 0 \qquad$ (i)
- At pin E of members *BGE* & *ABC*, $M_E = 0$: $\quad M_A + 2A_x - 6A_y - 4(6) + 3(4) = 0 \qquad$ (ii)
- For entire frame, $+\circlearrowleft \Sigma M_D = 0$: $\quad M_A + 6A_x - 6A_y + 3(4) + 10 - 4(2) = 0 \qquad$ (iii)

These *three* simultaneous equations yield: $A_x = -6.5$, $A_y = 2$, and $M_A = 37$.

Thus, the **traditional method** confirms the same solution: $\mathbf{M}_A = 37$ kN·m ↺

Example 7. A frame is loaded as shown in Fig. 15. Using the *virtual work method*, determine *only* the reaction moment $\mathbf{M}_A$ at the fixed support A.

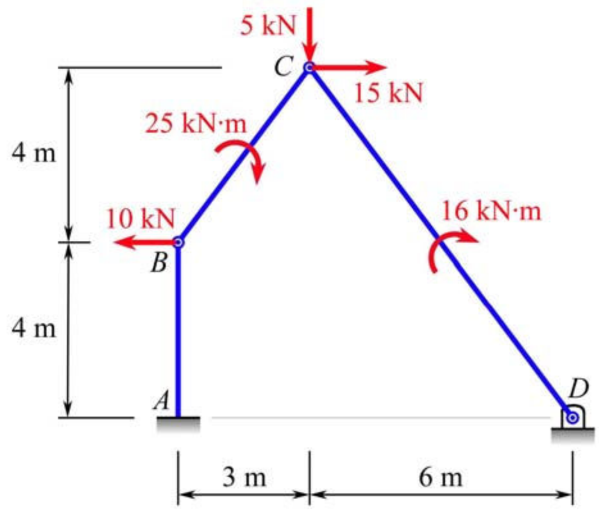

Fig. 15 A frame with fixed support at A and hinge support at D

Solution. For beginners, this is a challenging example! To avoid using differential calculus in computing the virtual displacements in the *VDD*, one may employ the concept of **displacement center**,[6,7] which will involve the use of just *geometry* and *algebra* (the *radian measure formula*). Again, we shall follow the outlined *steps* and *strategy* in the method to *directly* solve for $\mathbf{M}_A$.

Step 1: We draw the *FBD* for the frame as shown in Fig. 16.

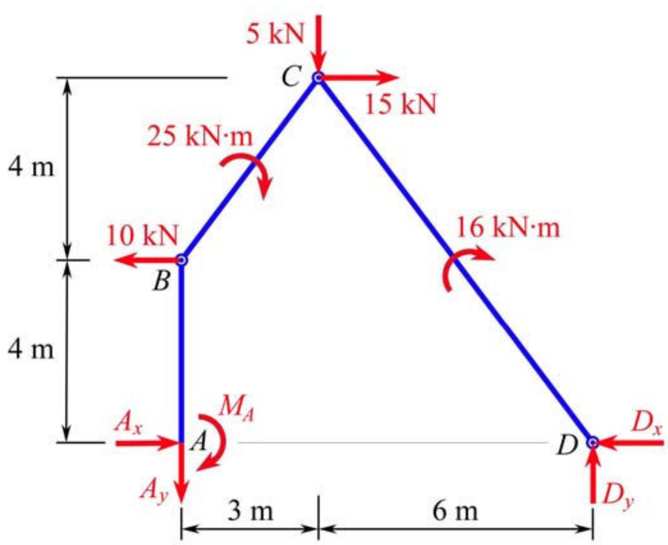

Fig. 16 *FBD* for the frame

Step 2: Keeping an eye on the *FBD* in Fig. 16, we draw a *VDD* for the frame with a **strategy** (for determining *only* $\mathbf{M}_A$) as shown in Fig. 17, where we let the *imaginary* angular virtual displacement of member *CD* be $\delta\theta$ ↺. Note that members *AB*, *BC*, and *CD* have displacement centers located at points *A*, *E*, and *D*, respectively, and we have used Eq. (4) — the *radian measure formula* — in computing virtual displacements of the points and members of the frame. Notice that the **displacement center** *E* of member *BC* is simply the point of intersection of two straight lines $\overline{BE}$ and $\overline{CE}$ drawn perpendicular to the virtual displacement vectors of points B and C (i.e.; $\overline{BE} \perp \overrightarrow{BB'}$ and $\overline{CE} \perp \overrightarrow{CC'}$). Furthermore, it may be helpful to envisage BC, CE, and BE as the three sides of a "rigid triangular plate BCE," which is rotated clockwise through the angle of $(10\ \delta\theta)/5 = 2\delta\theta$ about point E to the new position $B'C'E$. This is *why* each of the three sides of this "rigid triangular plate BCE" also rotates clockwise through the same angle of $2\delta\theta$!

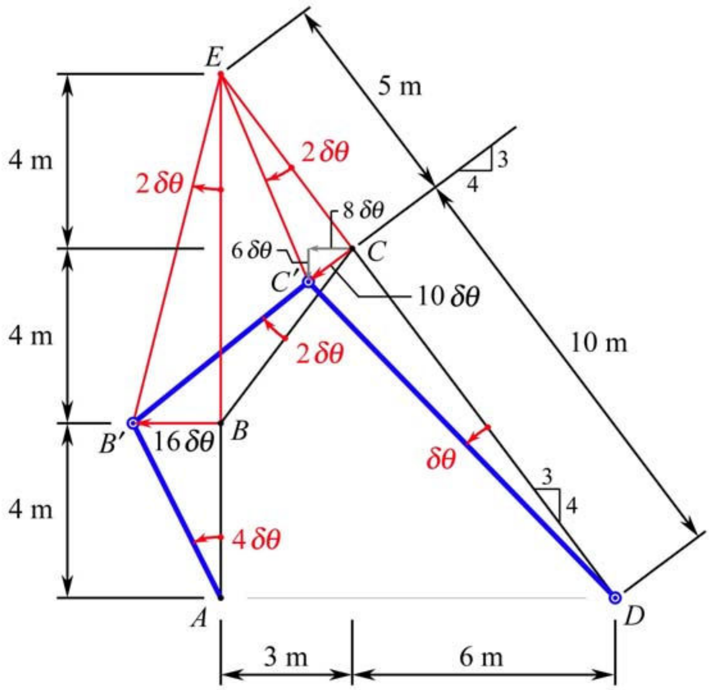

Fig. 17 *VDD* for determining only M_A

The resulting *VDD* in Fig. 17 is well done because it will allow *no* unknowns *except* M_A to be involved in the total virtual work done.

Step 3: We refer to Figs. 16 and 17 and apply Eqs. (5), (6), and (7) to write

$$\delta U = 0: \qquad M_A(-4\,\delta\theta) + 10(16\,\delta\theta) + 25(2\,\delta\theta) + 15(-8\,\delta\theta) + 5(6\,\delta\theta) + 16(-\delta\theta) = 0$$

$$M_A = 26 \qquad\qquad \mathbf{M}_A = 26\ \text{kN·m} \circlearrowright$$

Remarks. If we are to use the **traditional method**, we refer to the *FBD* in Fig. 16 and write

- At hinge B, $M_B = 0$: $\qquad -M_A + 4A_x = 0 \qquad$ (i)
- At hinge C, $M_C = 0$: $\qquad -M_A + 8A_x + 3A_y - 4(10) - 25 = 0 \qquad$ (ii)
- For the entire frame, $+\circlearrowleft \Sigma M_D = 0$:

$$-M_A + 9A_y + 4(10) - 25 - 8(15) + 6(5) - 16 = 0 \qquad \text{(iii)}$$

These *three* simultaneous equations yield:

$$A_x = 6.5 \qquad A_y = 13 \qquad M_A = 26$$

Thus, the **traditional method** confirms the same solution: $\qquad \mathbf{M}_A = 26\ \text{kN·m} \circlearrowright$

V. Concluding Remarks

Oftentimes, the virtual work method is used to treat problems involving mainly *machines*, where virtual displacements are restricted to be *consistent* with constraints at the supports. Such a restriction is generally too strong and is an *over restriction* in treating other types of problems. In particular, it prevents the virtual work method from being effectively used to treat problems involving *beams* and *frames*, and it diminishes the usefulness of the virtual work method in Statics.

The subtle shade of difference between a **compatible virtual displacement** and a **rigid-body virtual displacement** is pointed out in the presentation, although it has not been commonly recognized and emphasized in the literature. In using the virtual work method to solve problems involving beams, frames, or machines, we need to remember that all compatible virtual displacements are *acceptable* virtual displacements. There are *three major steps* and *one strategy* in the virtual work method.

Solving a *simple* equilibrium problem by the virtual work method may come across as "unconventional" or "an act of overkill," as witnessed in Example 1. Still, the virtual work method is a standing topic in Statics, and it has been shown to have advantages in solving challenging problems as illustrated in Examples 2 through 7. Naturally, teaching of such a topic in Statics is usually at the discretion of the instructors to enrich the learning of students. It may be true that the virtual work method is initially not easy to learn for many people. Nevertheless, it is surmountable when one understands the following: (*a*) the proper steps and strategy in the method, (*b*) the displacement center, (*c*) some basic geometry, and (*d*) the radian measure formula to compute virtual displacements. George Bernard Shaw once said, "You see things; and you say, 'Why?' But I dream things that never were; and I say, 'Why not?'"

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