

Learning Virtual Work Method in Statics in a Nutshell: Demystifying It as a Magic Black Box

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Abstract

Statics is a fundamental course in mechanics and is taken by students in most engineering curricula. At University of Arkansas, learning the energy approach using virtual work is given significant emphasis soon after learning the traditional approach using force and moment equilibrium equations. The transition from the traditional approach to the energy approach requires that students learn a number of relevant key concepts and strategies. Such a transition is often a challenging experience to many students. This paper presents a roundup of relevant basic concepts and the rudiments for effective learning of the virtual work method and aims to demystify the perception of this method as a magic black box. In a nutshell, there are *three major steps* for applying the virtual work method and *a guiding strategy* in choosing the virtual displacement for determining the specified unknown. The proposed steps and strategy for implementing the virtual work method have led to much better understanding and more effective learning for students

I. Introduction

The virtual work method has many applications, and it is often more powerful than the traditional method in solving problems involving frames or machines. The virtual work method may initially come across as a magic black box to students, but it generally kindles great curiosity and interest in students of Statics. The drawing of compatible virtual displacements for frames and machines involves basic geometry and requires good graphics skills. These aspects provide opportunities for students to reinforce their skills in geometry and graphics. Thus, the teaching and learning of the virtual work method result in equipping students with an added powerful analytical method and helping them enhance skills in reading drawings and presenting technical conceptions.

Work is energy in transition to a body due to force or moment acting on the body through a displacement. Work, as well as heat, is dependent on the path of a process. Like heat, work crosses the system boundary when the system undergoes a process. Thus, work is a boundary phenomenon. Unlike kinetic energy and potential energy, work is not a property possessed by a system. A *virtual displacement* of a body is an imaginary first-order differential displacement, which does not actually occur. A *virtual work* is the work done by force or moment during a virtual displacement of the system. By letting the free body of a system undergo a strategically chosen compatible virtual displacement in the virtual work method, we can solve for one specified unknown at a time in many complex as well as simple problems in Statics without having to solve coupled simultaneous equations. It is the aim of this paper to: (a) summarize relevant basic concepts needed in learning the virtual work method, (b) utilize just algebra and geometry (rather

than differential calculus) as the prerequisite mathematics to compute virtual displacements, (c) present three major steps for implementing the virtual work method, and (d) propose a guiding strategy in choosing the virtual displacement for determining the specified unknown.

II. Relevant Basic Concepts

In mechanics, a body receives work from a force or a moment that acts on it if it undergoes a displacement in the direction of the force or moment, respectively, during the action. It is the force or moment, rather than the body, which does work. In teaching and learning the virtual work method, it is well to refresh the following relevant basic concepts:

■ Work of a force

If a force \mathbf{F} acting on a body is *constant* and the displacement vector of the body from position A_1 to position A_2 during the action is \mathbf{q} , then the *work* $U_{1 \rightarrow 2}$ of the force \mathbf{F} on the body is¹⁻⁴

$$U_{1 \rightarrow 2} = \mathbf{F} \cdot \mathbf{q} = Fq_{\parallel} \quad (1)$$

where F is the magnitude of \mathbf{F} , and q_{\parallel} is the scalar component of \mathbf{q} parallel to \mathbf{F} . If the force is not constant, then integration may be used to compute the work of the force.

■ Work of a moment

If a moment \mathbf{M} (or a couple of moment \mathbf{M}) acting on a body is *constant* and the angular displacement of the body from angular position θ_1 to angular position θ_2 in the direction of \mathbf{M} during the action is $\Delta\theta$, then the *work* $U_{1 \rightarrow 2}$ of the moment \mathbf{M} on the body is¹⁻⁴

$$U_{1 \rightarrow 2} = M(\Delta\theta) \quad (2)$$

where M is the magnitude of \mathbf{M} . If the moment is not constant, then integration may be used to compute the work of the moment.

■ Rigid-body virtual displacement

In Statics, all bodies considered are rigid bodies or systems of pin-connected rigid bodies that can rotate frictionlessly at the pin joints. A *displacement* of a body is the change of position of the body. A *rigid-body displacement* of a body is the change of position of the body without inducing any strain in the body. A **virtual displacement** of a body is an imaginary first-order differential displacement, which is possible but does not actually take place. Furthermore, virtual displacements can be either *consistent* or *inconsistent* with constraints of the body.¹⁻⁵ A **rigid-body virtual displacement** of a body is a rigid-body displacement as well as a virtual displacement of the body, where the body undergoes a first-order differential deflection to a neighboring position, but the body experiences *no axial strain* at all. This is illustrated in Fig. 1 for a single body AB , and in Fig. 2 for a hinged body ABC , which is composed of two rigid members AB and BC that are hinged together at B . (Note that the body ABC is a system of pin-connected rigid bodies.)

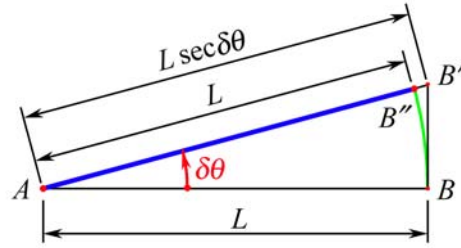


Fig. 1 Body AB undergoing a **rigid-body virtual displacement** to position AB''

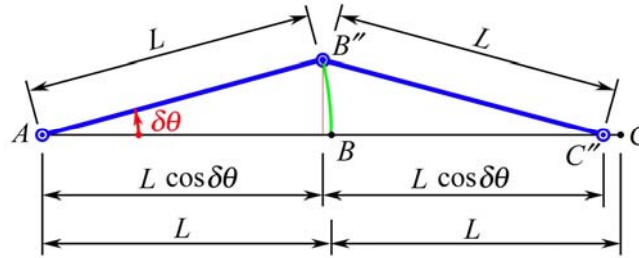


Fig. 2 Hinged body ABC undergoing a **rigid-body virtual displacement** to position $AB''C''$

Using series expansion in terms of the *first-order* differential angular displacement $\delta\theta$, which is *infinitesimal*, we find that the distance between B'' and B' in Fig. 1 is

$$\overline{B''B'} = L \sec \delta\theta - L = L \left[1 + \frac{1}{2}(\delta\theta)^2 + \frac{5}{24}(\delta\theta)^4 + \frac{61}{720}(\delta\theta)^6 + \dots \right] - L \approx \frac{L}{2}(\delta\theta)^2 \quad (3)$$

In a similar manner, we can show that the distance between C'' and C in Fig. 2 is

$$\overline{C''C} = 2L(1 - \cos \delta\theta) = 2L \left\{ 1 - \left[1 - \frac{(\delta\theta)^2}{2!} + \frac{(\delta\theta)^4}{4!} - \dots \right] \right\} \approx L(\delta\theta)^2 \quad (4)$$

Thus, the length $\overline{B''B'}$ and $\overline{C''C}$ are of the second order of $\delta\theta$ and are *negligible* in the virtual work method.

■ Compatible virtual displacement

In general, the term “body” may refer to a particle, a rigid body, or a set of pin-connected rigid bodies. A **compatible virtual displacement** of a body is an imaginary *first-order* differential displacement of the body, where the body undergoes a **first-order differential deflection** to a neighboring position, and the body may experience, at most, a **second-order infinitesimal axial strain**. A compatible virtual displacement of a body must conform to the integrity (i.e., no breakage or rupture) of its free body within the framework of *first-order* differential change in its position. A *compatible virtual displacement* of a body is **compatible** with what is required in the virtual work method; it differs from a rigid-body virtual displacement of the body by an amount no more than a *second-order* differential change in geometry. Note that a second-order differential change is a *great deal smaller than* the first-order differential change and is negligible in the limit. This is illustrated for a single body AB in Fig. 3 and for a connected body ABC in Fig. 4.

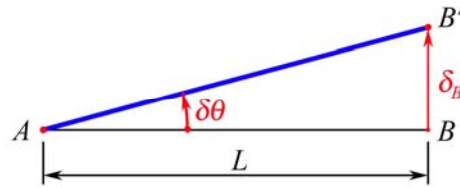


Fig. 3 Body AB undergoing a **compatible virtual displacement** to position AB'

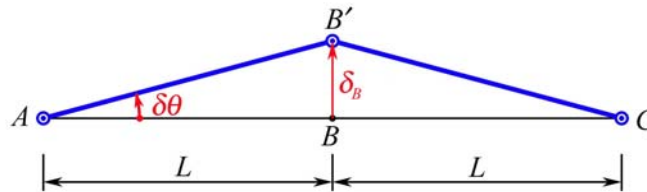


Fig. 4 Hinged body ABC undergoing a **compatible virtual displacement** to position $AB'C$

The *compatible virtual displacement* of point B in Figs. 1, 3, and 4 is from B to B' . We find that

$$\overline{BB'} = \delta_B = L \tan \delta\theta = L \left[\delta\theta + \frac{1}{3}(\delta\theta)^3 + \frac{2}{15}(\delta\theta)^5 + \frac{17}{315}(\delta\theta)^7 + \dots \right] \approx L \delta\theta \quad (5)$$

In Fig. 1, the lengths of the chord $\overline{BB'}$ and the arc $\widehat{BB''}$ can be taken as equal in the limit since the angle $\delta\theta$ is infinitesimally small. Therefore, the magnitude of the compatible linear virtual displacement of point B , as given by Eq. (5), may indeed be computed using the *radian measure formula* in calculus; i.e.,

$$s = r\theta \quad (6)$$

where s is the arc subtending an angle θ in radian included by two radii of length r . In virtual work method, all *virtual displacements* can be *compatible virtual displacements*, and these two terms can be interchangeable.

■ Displacement center

Relations among the virtual displacements of certain points or members in a system can be found by using *differential calculus*, or the *displacement center*,⁵ or both. The **displacement center** of a body is the point about which the body is perceived to rotate when it undergoes a virtual displacement. There are n displacement centers for a system composed of n pin-connected rigid bodies undergoing a set of virtual displacements; i.e., each member in such a system has its own displacement center. Generally, the **displacement center** of a body is located at the point of intersection of two straight lines that are drawn from two different points of the body *in the initial position* and are perpendicular to the virtual displacements of these two points, respectively.⁵ This is illustrated in Fig. 5, where the body AB is imagined to slide on its supports to undergo a virtual displacement to the position $A'B'$, and its displacement center C is the point of intersection of the straight lines AC and BC that are drawn from the initial positions of points A and B and are perpendicular to their virtual displacements $\overline{AA'}$ and $\overline{BB'}$, respectively.

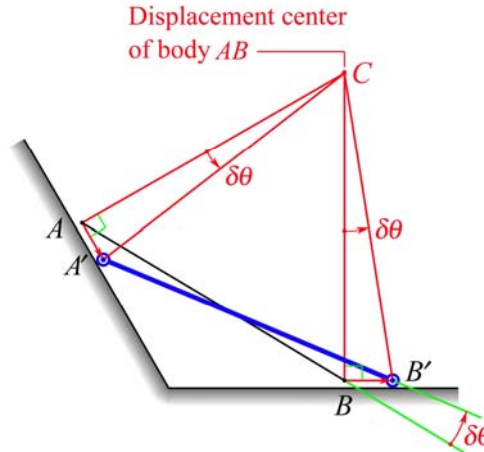


Fig. 5 Virtual displacement of body AB to position $A'B'$ with **displacement center** at C

It is often helpful to perceive the situation illustrated in Fig. 5 as an event where the body AB and its displacement center C form a “rigid triangular plate” that undergoes a rotation about C through an angle $\delta\theta$ from the initial position ABC to the new position $A'B'C$. In this event, all sides of this “rigid triangular plate” (i.e., the sides AB , BC , and CA), as well as any line that might be drawn on it, will and must rotate through the *same* angle $\delta\theta$, as indicated.

Sometimes, it is not necessary to use the procedure illustrated in Fig. 5 to locate the displacement center of a body. If a body undergoes a virtual displacement by simply rotating about a given point, then the **displacement center** of the body is simply located at the given point of rotation. This is illustrated in Figs. 6 and 7.

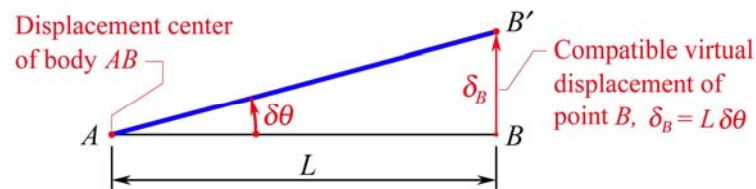


Fig. 6 Virtual displacement of body AB to position AB' with **displacement center** at A

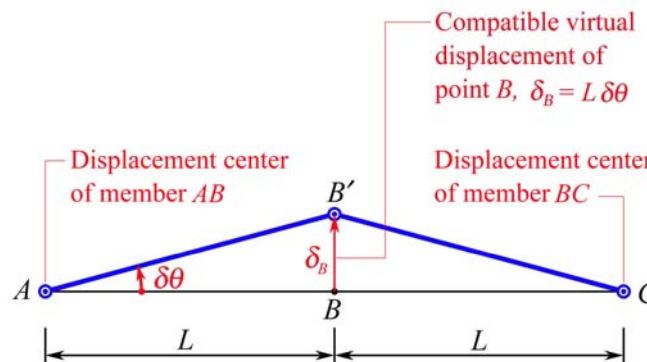


Fig. 7 Virtual displacement and the *two* **displacement centers** for the hinged body ABC

■ Principle of virtual work

Bodies considered in Statics are rigid bodies or systems of pin-connected rigid bodies. The term “force system” denotes a system of *forces* and *moments*, if any. The work done by a force system on a body during a *virtual displacement* of the body is called the **virtual work** of the force system. By Newton’s third law, internal forces in a body, or a system of pin-connected rigid bodies, must occur in pairs; they are equal in magnitude and opposite in directions in each pair acting on a particle in the rigid body. The forces in each pair act through the same displacement of the particle as the rigid body moves. Clearly, the total virtual work done by the internal forces during a virtual displacement of a body, or a system of pin-connected rigid bodies, must be zero. When a body, or a system of pin-connected rigid bodies, is in equilibrium, the resultant force and the resultant moment acting *on* its free body must both be zero.

The total virtual work done by the force system acting *on* the free body of a body is, by the distributive property of *dot product of vectors*, equal to the total virtual work done by the resultant force and the resultant moment acting on the free body, which are both zero if the body is in equilibrium. Therefore, we have the **principle of virtual work** in Statics, which may be stated as follows: *If a body is in equilibrium, the total virtual work of the external force system acting on its free body during any compatible virtual displacement of its free body is equal to zero, and conversely.*¹⁻⁴ Note that the body in this principle may be a particle, a set of connected particles, a rigid body, or a system of pin-connected rigid bodies (e.g., a frame or a machine). Using δU to denote the total virtual work done, we write the equation for this principle as

$$\delta U = 0 \quad (7)$$

■ Conventional method versus virtual work method

With the **conventional method**, equilibrium problems are solved by applying two basic equilibrium equations: (a) *force equilibrium equation*, and (b) *moment equilibrium equation*; i.e.,

$$\Sigma \mathbf{F} = \mathbf{0} \quad (8)$$

$$\Sigma \mathbf{M}_p = \mathbf{0} \quad (9)$$

With the **virtual work method**, equilibrium problems are solved by applying the *virtual work equation*, which sets to zero the total virtual work δU done by the force system on the free body during a chosen *compatible virtual displacement* of the free body; i.e.,

$$\delta U = 0 \quad (\text{Repeated}) \quad (7)$$

III. Virtual Work Method in a Nutshell: Examples

In a nutshell, there are **three major steps** in using the *virtual work method*. **Step 1:** *Draw the free-body diagram (FBD).* **Step 2:** *Draw the virtual-displacement diagram (VDD) with a guiding strategy.* **Step 3:** *Set to zero the total virtual work done.* The **guiding strategy** in step 2 is to give the free body a compatible virtual displacement in such a way that the *one* specified unknown, but *no other unknowns*, will be involved in the total virtual work done. That is it: **three major steps** and **one guiding strategy** in the virtual work method! The implementation of these steps with a guiding strategy in the virtual work method is illustrated in the following four examples.

Example 1. Using virtual work method, determine the vertical reaction force A_y at the roller support A of the simple beam loaded as shown in Fig. 8.

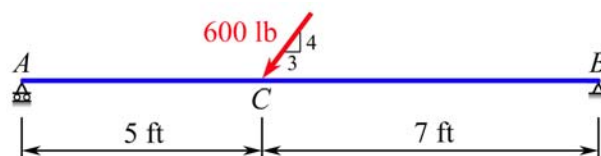


Fig. 8 A simple beam carrying an inclined concentrated load

Solution. For **step 1** in the solution, we draw the *FBD* for the beam as shown in Fig. 9, where we have replaced the 300-lb force at C with its rectangular components.

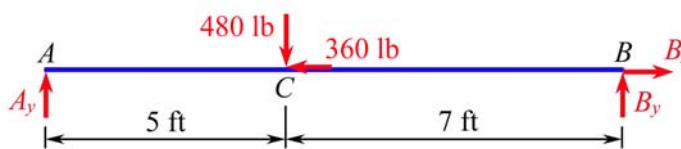


Fig. 9 Free-body diagram for the beam

For **step 2** in the solution, we keep an eye on the *FBD* in Fig. 9 and draw the *VDD* for the beam with a **strategy** in Fig. 10, where we rotate the beam through an angular displacement $\delta\theta$ about point B. Thus, the **displacement center** of the beam is located at point B. The resulting *VDD* will involve the unknown A_y , but *exclude all other unknowns* (i.e., B_x and B_y), in the total virtual work done.

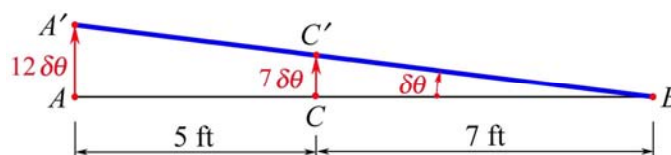


Fig. 10 Virtual-displacement diagram to involve A_y in $\delta U = 0$

For **step 3** in the solution, we refer to Figs. 9 and 10 and apply Eqs. (1) and (7) to write

$$\delta U = 0: \quad A_y (12\delta\theta) + 480(-7\delta\theta) = 0 \quad \therefore A_y = 280$$

$$A_y = 280 \text{ lb } \uparrow$$

Remarks. With the **conventional method**, we would refer to the *FBD* in Fig. 9 and write

$$+\circlearrowleft \sum M_B = 0: \quad -12A_y + 7(480) = 0 \quad \therefore A_y = 280$$

$$A_y = 280 \text{ lb } \uparrow$$

Thus, both the virtual work method and the conventional method yield the same solution. However, the virtual work method is best used to solve more complex, rather than simple, problems. The superiority of the virtual work method is manifested in the next three examples.

Example 2. Using virtual work method, determine the vertical reaction force K_y at the fixed support K of the combined beam (called a *Gerber beam*) loaded as shown in Fig. 11.

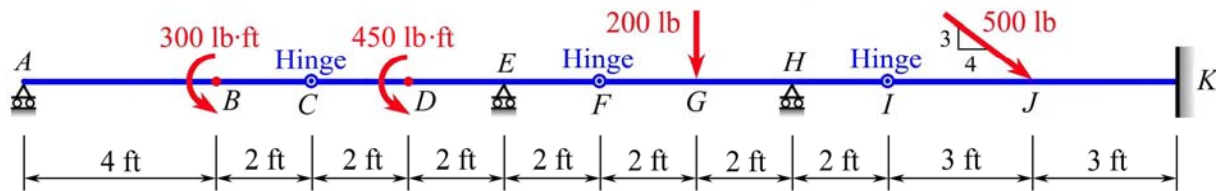


Fig. 11 A combined beam with hinge connections at C , F , and I

Solution. For **step 1** in the solution, we draw the *FBD* for the beam as shown in Fig. 12.

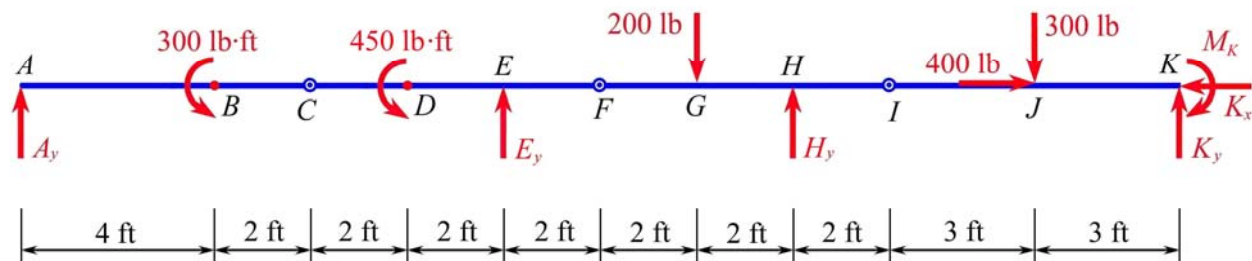


Fig. 12 Free-body diagram for the combined beam

For **step 2** in the solution, we keep an eye on the *FBD* in Fig. 12 and draw the *VDD* for the beam with a **strategy** in Fig. 13, where the segments ABC , $CDEF$, $FGHI$, and IJK have **displacement centers** located at points A , E , H , and at ∞ , respectively. The resulting *VDD* will involve the unknown K_y , but *exclude all other unknowns* (i.e., A_y , E_y , H_y , K_x , and M_K), in the total virtual work done.

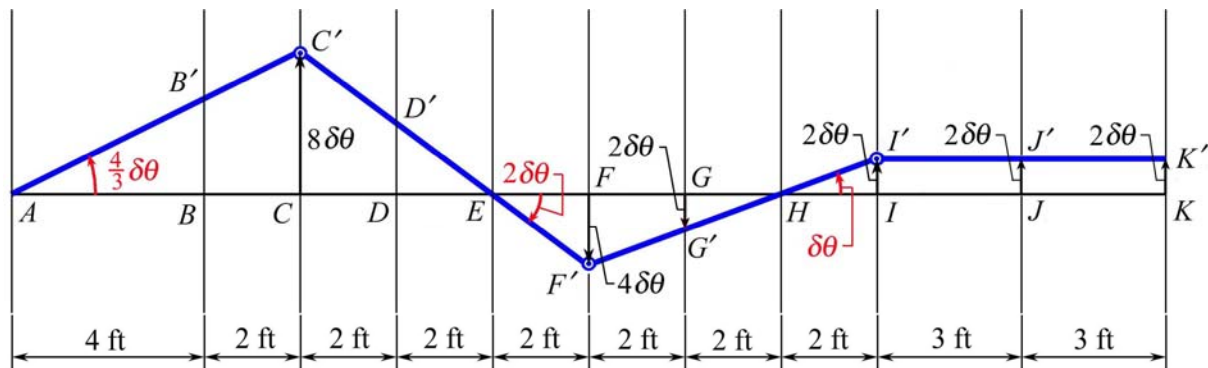


Fig. 13 Virtual-displacement diagram for the combined beam to involve K_y in $\delta U = 0$

For **step 3** in the solution, we refer to Figs. 12 and 13 and apply Eqs. (1), (2), and (7) to write

$$\delta U = 0: \quad 300\left(\frac{4}{3}\delta\theta\right) + 450(-2\delta\theta) + 200(2\delta\theta) + 300(-2\delta\theta) + K_y(2\delta\theta) = 0$$

$$K_y = 350$$

$$K_y = 350 \text{ lb } \uparrow$$

Remarks. With the **conventional method**, we would refer to the *FBD* in Fig. 12 and write

- At hinge C , $M_C = 0$: $-6A_y + 300 = 0$ (i)
- At hinge F , $M_F = 0$: $-12A_y + 300 + 450 - 2E_y = 0$ (ii)
- At hinge I , $M_I = 0$: $-18A_y + 300 + 450 - 8E_y + 4(200) - 2H_y = 0$ (iii)
- For the entire beam, $+\uparrow \Sigma F_y = 0$: $A_y + E_y + H_y + K_y - 200 - 300 = 0$ (iv)

These four simultaneous equations yield: $A_y = 50$, $E_y = 75$, $H_y = 25$, and $K_y = 350$. Thus, the **conventional method** eventually yields the same solution: $\mathbf{K}_y = 350 \text{ lb } \uparrow$

Example 3. Using virtual work method, determine the reaction moment \mathbf{M}_K at the fixed support K of the combined beam loaded as shown in Fig. 11 earlier.

Solution. For **step 1** in the solution, we draw the *FBD* for the beam as shown in Fig. 12 earlier.

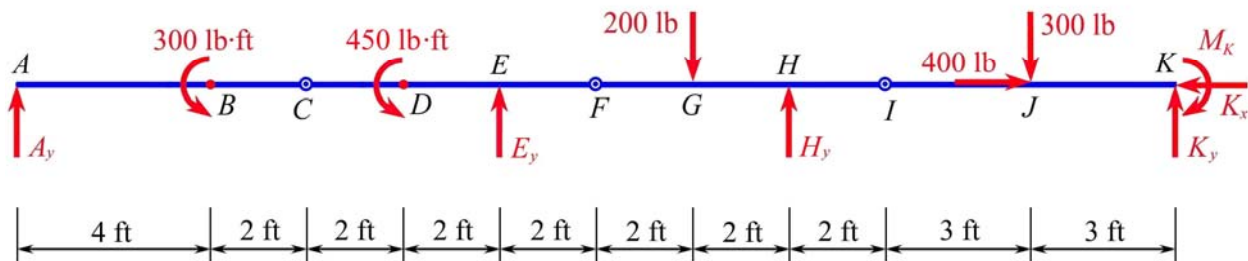


Fig. 12 Free-body diagram for the combined beam (Repeated)

For **step 2** in the solution, we keep an eye on the *FBD* in Fig. 12 and draw the *VDD* for the beam with a **strategy** in Fig. 14, where the segments ABC , $CDEF$, $FGHI$, and IJK have **displacement centers** located at points A , E , H , and K , respectively. The resulting *VDD* will involve the unknowns M_K , but *exclude all other unknowns* (i.e., A_y , E_y , H_y , K_x , and K_y), in the total virtual work done.

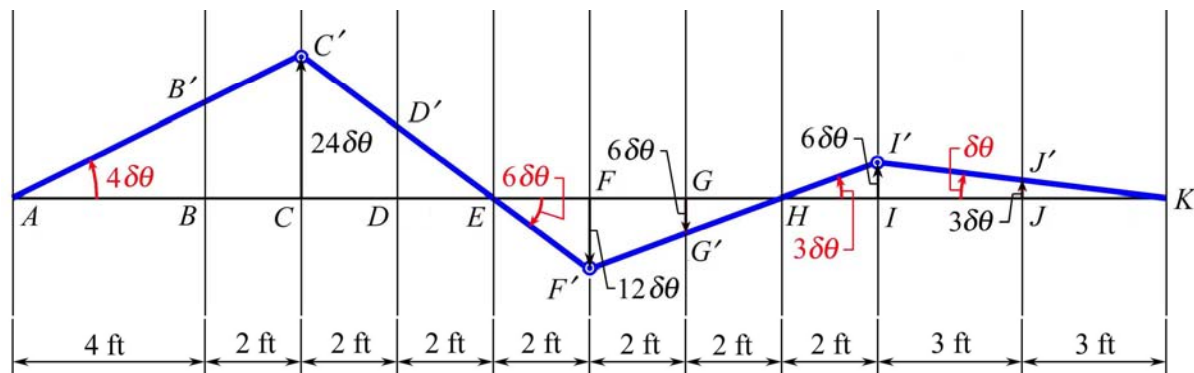


Fig. 14 Virtual-displacement diagram for the combined beam to involve M_K in $\delta U = 0$

For **step 3** in the solution, we refer to Figs. 12 and 14 and apply Eqs. (1), (2), and (7) to write

$$\delta U = 0: \quad 300(4\delta\theta) + 450(-6\delta\theta) + 200(6\delta\theta) + 300(-3\delta\theta) + M_K(\delta\theta) = 0$$

$$M_K = 1200$$

$$\mathbf{M}_K = 1200 \text{ lb}\cdot\text{ft } \curvearrowright$$

Remarks. With the **conventional method**, we would refer to the *FBD* in Fig. 12 and write

- At hinge C , $M_C = 0$: $-6A_y + 300 = 0$ (i)
- At hinge F , $M_F = 0$: $-12A_y + 300 + 450 - 2E_y = 0$ (ii)
- At hinge I , $M_I = 0$: $-18A_y + 300 + 450 - 8E_y + 4(200) - 2H_y = 0$ (iii)
- For the entire beam, $+\circlearrowleft \Sigma M_K = 0$:
 $-24A_y + 300 + 450 - 14E_y + 10(200) - 8H_y + 3(300) - M_K = 0$ (iv)

These four simultaneous equations yield: $A_y = 50$, $E_y = 75$, $H_y = 25$, and $M_K = 1200$. Thus, the **conventional method** eventually yields the same solution: $M_K = 1200 \text{ lb}\cdot\text{ft}$ \circlearrowleft

Example 4. Using virtual work method, determine the reaction moment M_A at the fixed support A of the frame loaded as shown in Fig. 15.

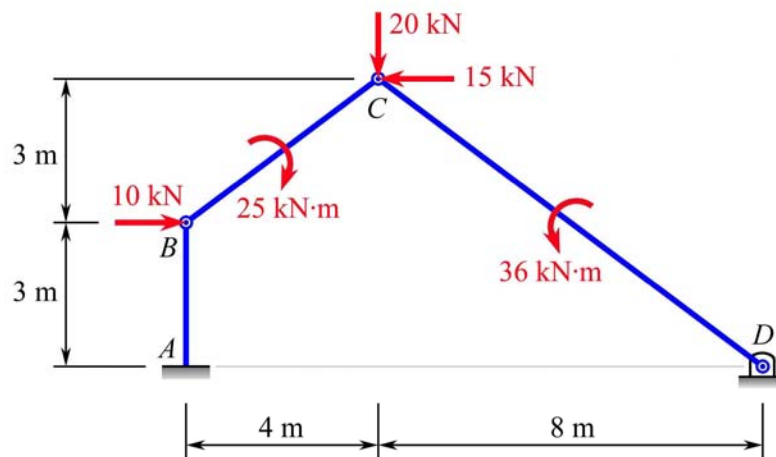


Fig. 15 A frame with hinge support at A and fixed support at D

Solution. For **step 1** in the solution, we draw the *FBD* for the frame as shown in Fig. 16.

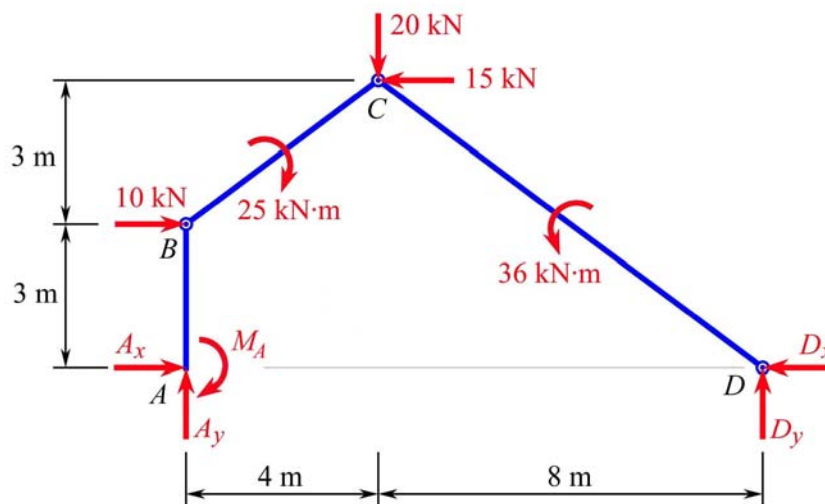


Fig. 16 Free-body diagram for the frame

For **step 2** in the solution, we keep an eye on the *FBD* in Fig. 16 and draw the *VDD* for the frame with a **strategy** in Fig. 17, where we rotate member *CD* through an angle $\delta\theta$ about point *D*, and members *AB*, *BC*, and *CD* have **displacement centers** located at points *A*, *E*, and *D*, respectively. The resulting *VDD* will involve the unknown M_A , but *exclude all other unknowns* (i.e., A_x , A_y , D_x , and D_y), in the total virtual work done. In Fig. 16, pay special attention to the following:

- The compatible virtual displacement $\overline{CC'}$ of point *C* is such that $\overline{CC'} \perp \overline{CD}$ and $\overline{CC'} = 10\delta\theta$.
- Each of the three sides (i.e., *BC*, *CE*, and *EB*) of the “rigid triangular plate” *BCE* rotates clockwise through the same angle of $2\delta\theta$, which is determined by applying the radian measure formula in Eq. (6) as follows:

$$\overline{CC'} / \overline{CE} = (10\delta\theta) / 5 = 2\delta\theta$$

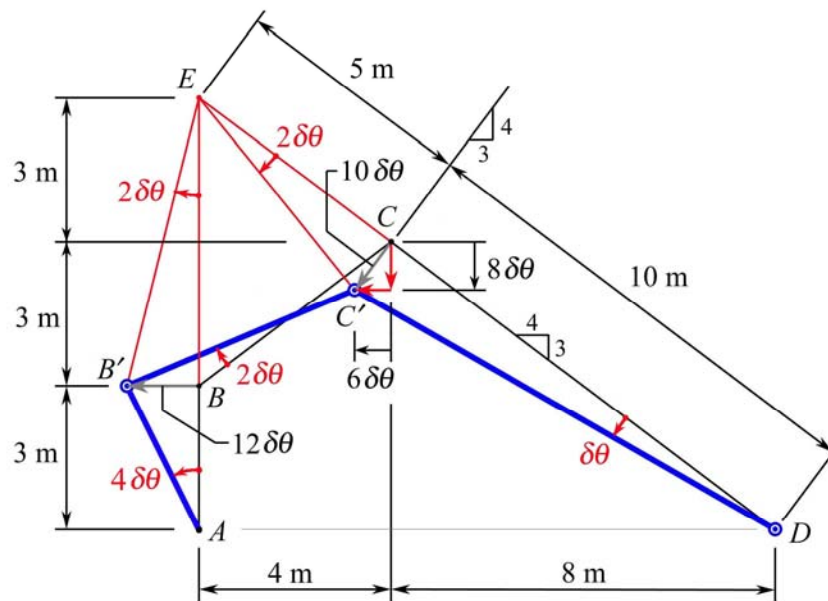


Fig. 17 Virtual-displacement diagram to involve M_D in $\delta U = 0$

For **step 3** in the solution, we refer to Figs. 16 and 17 and apply Eqs. (1), (2), and (7) to write

$$\delta U = 0: \quad M_A(-4\delta\theta) + 10(-12\delta\theta) + 25(2\delta\theta) + 20(8\delta\theta) + 15(6\delta\theta) + 36(\delta\theta) = 0$$

$$M_A = 54$$

$$M_A = 54 \text{ kN}\cdot\text{m} \curvearrowright$$

Remarks. With the **conventional method**, we would refer to the *FBD* in Fig. 16 and write

• At hinge *B*, $M_B = 0$: $-M_A + 3A_x = 0$ (i)

• At hinge *C*, $M_C = 0$: $-M_A + 6A_x - 4A_y + 3(10) - 25 = 0$ (ii)

• For the entire frame, $+\curvearrowright \sum M_D = 0$:

$$-M_A - 12A_y - 3(10) - 25 + 6(15) + 8(20) + 36 = 0 \quad \text{(iii)}$$

These three simultaneous equations yield: $M_A = 54$, $A_x = 18$, and $A_y = 14.75$. Thus, the **conventional method** eventually yields the same solution: $M_A = 54 \text{ kN}\cdot\text{m} \curvearrowright$

IV. Concluding Remarks

Any equilibrium problem solvable by the conventional method is also solvable by the virtual work method, although solutions for *simple* equilibrium problems by the virtual work method may come across as “unconventional” or “overkill” when compared head to head with the rather straightforward solutions by the conventional method, as seen in Example 1. Nevertheless, when it comes to solutions for more complex problems, as illustrated in Examples 2, 3, and 4, the virtual work method surely emerges as a much more superior and powerful method than the conventional method. The advantages of the virtual work method lie in its conciseness in the principle, its visual elegance in the formulation of the solution via virtual-displacement diagrams drawn with a guiding strategy, and its ability to save algebraic effort by doing away with the need to solve large sets of simultaneous equations in complex problems. The virtual work method may amaze beginning students, but its superior and powerful features witnessed by students are sparks that kindle their interest in learning the virtual work method in particular and the subject of Statics in general. Readers interested in more in-depth study of the virtual work method may refer to the textbooks, particularly the one by Jong and Rogers,⁴ in the references.

In a nutshell, there are **three major steps** and **one guiding strategy** in the virtual work method, as described and illustrated in Section III. Implementation of these steps and strategy has greatly helped students understand the method and demystify it as a magic black box. The knowledge of the location of *displacement center*⁵ for each member in a system is what makes possible the use of just algebra and geometry (rather than differential calculus) as prerequisite mathematics for the teaching and learning of the *principle of virtual work* in Statics. It is true that the drawing of compatible virtual displacements for frames and machines involves basic geometry and requires good graphics skills. These aspects do present some degree of challenges to a number of beginning students. Nevertheless, the learning of the virtual work method is an excellent training ground for engineering and technology students to develop their visual skills in reading technical drawings and presenting technical conceptions.

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