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Limits, Singularities and other concerns in the Elementary Functions of Calculus

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Limits, Singularities and other Concerns in the Elementary Functions of Calculus

Conventional calculus textbooks pointlessly introduce the concepts of singularities and limits in a strange way that confuses many students who could be practical engineers and technicians.

When the points of the elementary functions of calculus are plotted on a Cartesian coordinate plane, the functions appear as curves that are single-valued, and mostly continuous and smooth. Except for a finite number of very special separated points, the values of the functions and their derivatives can be computed and plotted. We can call the points with known values and tangent lines "normal" points. This expository paper is intended to provide a view of the exceptions called singularities, listed below, that a beginning student can grasp. The exceptional points include:

- Infinite jumps resulting from division by zero. See figures 1, 2, 3 and 4.
- Point gaps resulting when the numerators and denominators have common zeroes.
- Excluded intervals resulting from even roots of negative numbers. Modern texts would describe excluded intervals as being outside the domain of the function. See figures 15 and 16.
- Finite jumps resulting from differing function definitions on each side of the point. Examples are shown in Figures 6, 7, 8, 9 and 11.
- Points where the function has a vertical tangent indicating the function may be doubling back. See figures 13, 14, 15 and 16.
- Cusps, indicating a jump in direction. Examples are shown in Figures 10 and 17.
- Oscillating discontinuities. An example is shown in Figure 18.
- Multiple valued functions may have self-intersecting points. See figure 14.

This paper provides an intuitive, visual interpretation of the concepts of continuity, discontinuity, of right- and left-hand limits and of limits. The summary of the paper includes a discussion of the advance of mathematics over the past two centuries and how a group of the top French mathematicians, called Bourbaki, influenced math pedagogy to go beyond the grasp and needs of the ordinary engineer.

Conventional math pedagogy obfuscates the limit concepts, forcing students to memorize apparently incomprehensible material. An initially clear, visual presentation of the simple cases lets in fresh air, enabling a student intuitively to envision the arc of the study.

Continuity and Differentiability:

Continuity and differentiability are wonderful features of functions. Continuity means the function has no jumps and differentiability means the function possesses a tangent line. Differentiability means the function has a definite direction or rate of change. I will use the word "smooth" to represent the ordinary points of a function that have a well-defined position, tangent line and curvature. These points exhibit the behavior which is studied in Differential Calculus.

It should be noted that sums, differences and products of smooth functions are smooth. Quotients of smooth functions are smooth except at the isolated points where the denominator is zero.

Polynomials:

1) Polynomials are defined everywhere and are single-valued and smooth.
2) Polynomials can be written (expanded) as sums or differences of scaled non-negative whole number powers of x. In expanded form, the largest of the exponents is called the degree.
3) The number of intersections of a polynomial with a straight line cannot exceed the degree. Therefore, the number of zeros of a polynomial cannot exceed the degree.
4) For every zero of a polynomial, there is a corresponding linear factor. N zeroes coincide at x = a when the polynomial has repeated factors, $(x – a)^N$.
5) Polynomial vertical values approach + or – infinity as x approaches either + or – infinity. That is, as x grows in either direction the vertical values exceed all bounds.
6) Sums, differences and products of polynomials are polynomials. The quotients of polynomials may not be polynomials. The class of functions, where the division has a remainder, is called rational.

Example; The 5th degree polynomial $y = (x + 2)(x-2)^2(x^2+2)$ has only 3 zeroes, one zero located at x = –2, and two coincident zeroes located at x = +2.

The word "singularity" is used to describe isolated points of a function where discontinuities or cusps occur. The singularities treated in this paper are called removable, essential or poles, oscillating and finite jumps. Because polynomials are smooth everywhere, they have no singularities.

Rational Functions:

The singularities of rational functions can only occur where the denominator polynomial of the rational function has a zero. A rational function whose denominator polynomial is of degree n, can have at most n singularities.

Let us examine a rational function where the numerator and denominator polynomials have been factored. Say there is a singularity at x = a and the numerator polynomial is written as $N(x) = (x – a)^n g(x)$ and the denominator polynomial has the form $D(x) = (x – a)^m h(x)$ and neither g(x) or h(x) has (x – a) as a factor. That is, neither g(a) nor h(a) is zero.

Our rational function appears as: $y = \frac{(x-a)^n\, g(x)}{(x-a)^m\, h(x)}$ Equation 1

The common zeros are not visible when the polynomials are written in expanded form. The evaluation of the rational function at x = a results in the "undefined," $\frac{0}{0}$.

When the number of common denominator polynomial factors, m exceeds the number of numerator common factors, n the rational function appears as $y = \frac{g(x)}{(x-a)^{m-n}\,h(x)}$.

In this case at x = a, the function shown in equation 1 evaluates to $\frac{g(a)}{0}$; the function values grow without bound around x = a and the singularity is described as essential or a pole. Graphs of 1st and 2nd degree poles are shown in Figures 1, 2, 3 and 4 below.

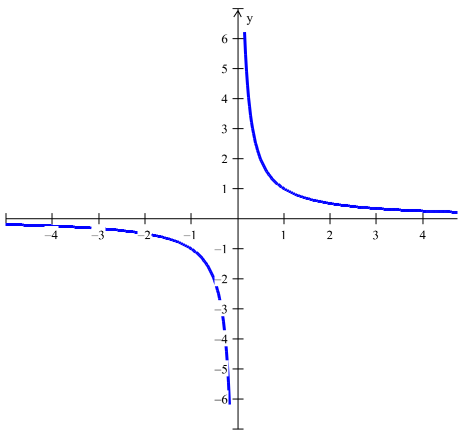

Figure 1 $y = \frac{1}{x}$

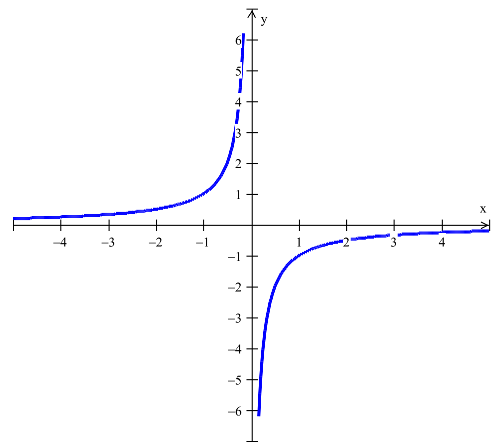

Figure 2 $y = \frac{-1}{x}$

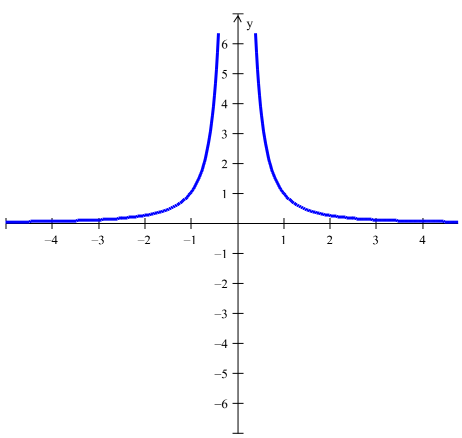

Figure 3 $y = \frac{1}{x^2}$

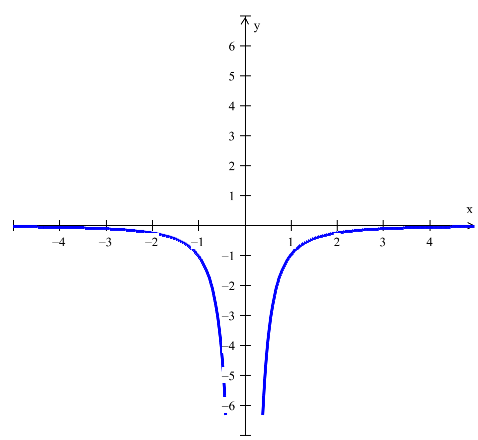

Figure 4 $y = \frac{-1}{x^2}$

Two cases remain that are called, "removable singularities"; the case where n > m and the case where n = m. In these cases, it appears as if a point was removed from the function at x= a. This phenomenon produces a "point gap" in the graph. A single point removed from a curve cannot be seen.

When the number, n, of common numerator polynomial factors in equation 1 exceeds the number of common denominator factors, m the rational function appears as y = $\frac{(x-a)^{n-m} g(x)}{h(x)}$. In this case the function evaluates to $\frac{0}{h(a)} = 0$. Now the singularity that formerly was as “undefined” reasonably can be assigned the value zero at x = a providing a value at all points surrounding the horizontal value, a. The singularity is removed and the function becomes smooth.

In the last case when n = m, the numerator and denominator factors (x – a) cancel and the rational function evaluates as the nonzero value, y(a) = $\frac{g(a)}{h(a)}$. If the original rational function is defined at x = a to be y(a) = $\frac{g(a)}{h(a)}$, the singularity is removed and the function becomes smooth.

Limits

Definition: The vertical value, $\frac{g(a)}{h(a)}$ that fills in the point gap and makes the function smooth is called the “limit” of y(x) as x approaches a. The notation for the limit is: $\lim_{x \to a} y(x)$

Examples:

a. Compute $\lim_{x \to 7} y_1(x) = \lim_{x \to 7}(x^3 - x^2 - 56x)$ Equation 2

All polynomials, including the polynomial $y_1(x)$ in equation 2, are defined, single-valued and smooth everywhere. The limit can be found by evaluating either the factored or expanded forms of $y_1(x)$:

$x^3 - x^2 - 56x = x(x - 8)(x + 7)$ at x = 7 .

The value of $\lim_{x \to 7} y_1(x)$ is found to be –98 .

Principle: No problems arise in computing limits of smooth functions.

b. Compute $\lim_{x \to 7} y_2(x) = \lim_{x \to 7}(\frac{x^3 - 14x^2 + 49x}{x^2 - 10x + 21})$, Equation 3

The first evaluation of $y_2(x)$ in equation 3 yields the undefined $\frac{0}{0}$. On the other hand, the common factors in the factored form $\frac{x(x-7)^2}{(x-3)(x-7)}$ can be cancelled to obtain $\frac{x(x-7)}{x-3}$. Substituting x = 7 in the form $\frac{x(x-7)}{x-3}$ yields $\lim_{x \to 7} y_2(x) = 0$

c. Compute $\lim_{x \to 7} y_3(x) = \lim_{x \to 7}(\frac{x^3 - 7x^2}{x^2 - 14x + 49})$ Equation 4

Again, our first evaluation of $y_3(x)$ in equation 4 at x =7, yields the undefined $\frac{0}{0}$. But when the polynomials in $y_3(x)$ are factored, the form $\frac{x^2(x-7)}{(x-7)^2}$ is obtained. The common factors can be cancelled and x = 7 can be substituted in $\frac{x^2}{x-7}$ resulting in $\frac{49}{0} = \infty$. The singularity of the rational function $y_3(x)$ at the point x = 7 is discovered to be a 1st degree pole which is not removable

A last example is:

a. Compute $\lim_{x\to 7} y_4(x) = \lim_{x\to 7}(\frac{x^3 - 2x^2 - 35x}{x^2 - 4x - 21})$ Equation 5

And here too, the evaluation of $y_3(x)$ in equation 5 at x =7, yields the undefined $\frac{0}{0}$. And now again, we factor the polynomials in $y_4(x)$ to obtain $\frac{x(x-7)(x+5)}{(x-7)(x+3)}$. After cancelling and evaluating y_3 at x=7, we obtain $\lim_{x\to 7} y_4(x) = 7*\frac{12}{10} = 8.4$.

The function $g(x) = \frac{(x-a)}{(x-a)}$ has the value 1 for every x except for x = a where it is undefined. Multiplying any continuous function, h(x) by g(x) leaves h(x) untouched everywhere except at the point x = a where the product is undefined. A point gap has been inserted into the graph of h(x). When the factors of $\frac{(x-a)}{(x-a)}$ h(x) are cancelled, the point gap is removed and the continuity of h(x) is restored.

Principle: Point gaps in continuous functions can be inserted or removed at will by multiplying or cancelling factors such as $\frac{(x-a)}{(x-a)}$ and $\frac{(x-b)}{(x-b)}$.

Summarizing rational and polynomial functions, we have seen that polynomials are smooth and defined everywhere. We have also seen that, except for the special cases where a denominator zero occurs, rational functions possess the same features. The singular points of rational functions are either removable or are poles.

Interval-defined functions

On the boundaries of adjacent intervals, interval-defined functions can be continuous or have finite jumps, point gaps, or cusps. We continue to describe the conditions when singularities on the boundaries can be removed yielding continuity across these boundaries.

Examples of interval-defined functions are the absolute value function, the bipolar step, the unit step and the unit pulse. Algebraic constructions of these functions are listed below.

The absolute value function is $y = |x| = \begin{cases} -x, & x \le 0 \\ x, & x \ge 0 \end{cases}$ See Figure 5 below.

The bipolar step is $y = \frac{x}{|x|} = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}$ See Figure 6 below.

The unit step is $u(x) = \frac{1}{2}\{\frac{x}{|x|} + 1\} = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases}$ See Figure 7 below.

The delayed unit step is $u(x-a) = \frac{1}{2}\{\frac{x-a}{|x-a|} + 1\} = \begin{cases} 0, & x < a \\ 1, & x > a \end{cases}$ See Figure 8 below.

A unit pulse of width a is $\begin{cases} 0, & x < 0 \\ 1, & 0 < x < a \\ 0, & x > a \end{cases}$ See Figure 9 below.

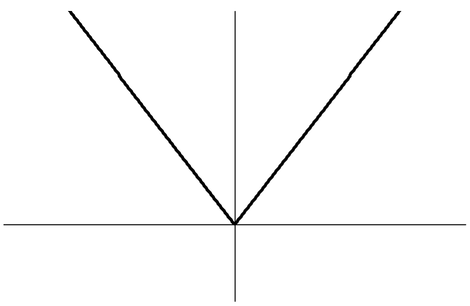

Figure 5 Absolute Value function $|x|$

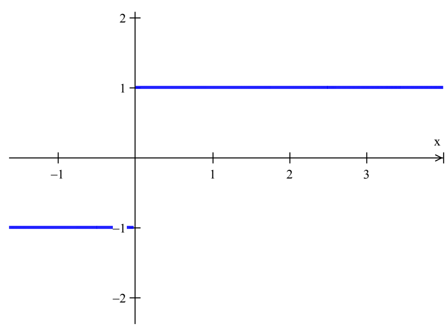

Figure 6 Bipolar step $\frac{x}{|x|}$

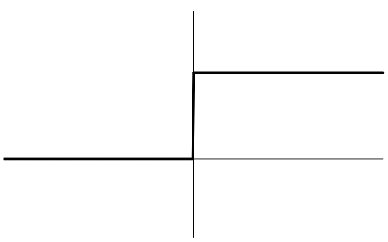

Figure 7 Unit Step $u(x) = \frac{1}{2}\{\frac{x}{|x|} + 1\}$

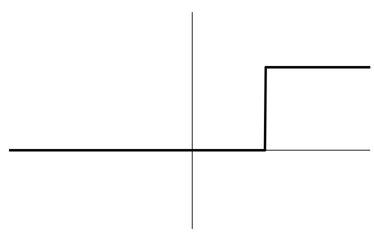

Figure 8 Delayed unit step $u(x-a)$

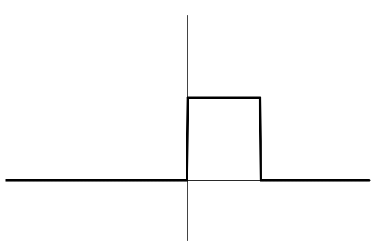

Figure 9 Unit pulse $u(x) - u(x-a)$

Consider the bipolar step function, $|x|/x$ shown above in figure 6. The function is smooth in the interiors of both the right- and left-half planes but not at the origin where the function is undefined. There is a finite jump.

Consider extending, the function $y(x) = 1$ that is defined in the right half plane, $0 < x < \infty$. to the boundary $x = 0$. The reasonable extension is $y(0) = 1$ which is called the Right Hand Limit of $y(x)$ at $x = 0$. On the other hand, consider extending, the function, $y(x) = -1$ that is defined in the left half plane, $-\infty < x < -0$. to the boundary $x = 0$. The reasonable extension is $y(0) = -1$ which is called the Left Hand Limit of $y(x)$ at $x = 0$.

When as in this case the two one-sided limits are not equal, the function has a finite jump discontinuity. When the two one-sided limits are equal, the function has a point gap. When the two one-sided limits are equal, this missing value is called the limit of the function. This language usage is clumsy but is commonly used by mathematicians. Defining the value of the function on the boundary to have the common value of both the right- and left-hand limits when they are equal makes the function continuous on the union of the adjacent intervals.

Let us consider the following examples:

a) The function, $y(x)$, defined implicitly in the upper-half plane by the equation:

$$| x | + | y | = 2 \qquad \text{See Figure 10 below.}$$

In the first quadrant both x and y are positive and therefore the Right-Hand Limit at the origin of the function $y = 2 - x$ is the value, 2. In the second quadrant, y is positive but $|x|$ equals $-x$. The Left-Hand Limit at the origin of the function $y = x + 2$ is the value 2. Both one-sided limits equal 2. If $y(0)$ is defined to be 2, the point gap is filled in and the function will be continuous on the interval $-2 < x < 2$. This function has a cusp at $x = 0$ and is excluded for values of $|x| > 2$.

b) The function, $g(x)$, constructed as the product of the parabola $y = x^2$ and the bipolar step:

.
$$g(x) = \frac{x}{|x|} * x^2 = |x|*x \qquad \text{See Figure 11 below.}$$

At $x = 0$ the Right-Hand Limit of the function $y = x^2$ is the value 0 and the Left-Hand Limit of the function $y = -x^2$ is also 0. Since the one-sided limits are equal, the point gap can be filled in with the value 0 and the function $g(x)$ can be made continuous.

c) The function, $h(x) = \frac{x-2}{|x-2|} * x^2$ is constructed as the product of the parabola $y = x^2$ and the delayed bipolar step. See Figure 12 below.

At x = 2 the Right-Hand Limit of the function $y = x^2$ is the value 4 and the Left-Hand Limit of the function $y = -x^2$ is the value –4. Since the one-sided limits are not equal, the function h(x) does not have a limit and is inherently discontinuous.

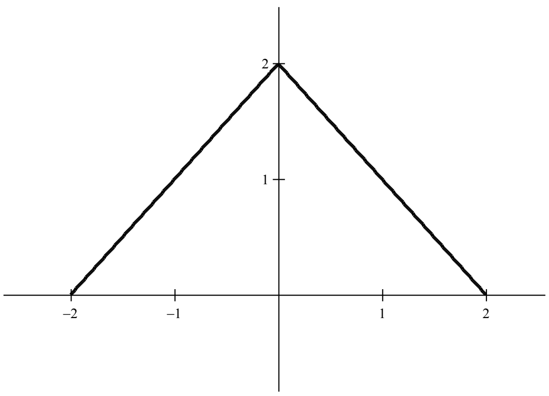

Figure 10 $|x| + |y| = 2$;

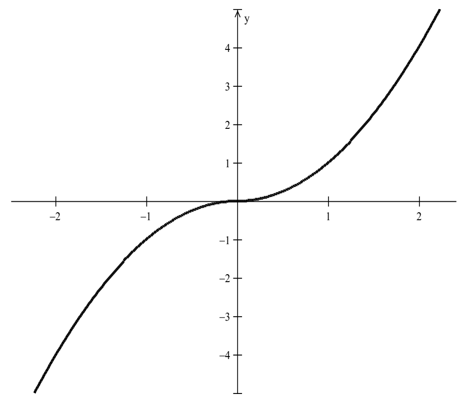

Figure 11 $g(x) = \frac{x}{|x|} * x^2 = |x|*x$

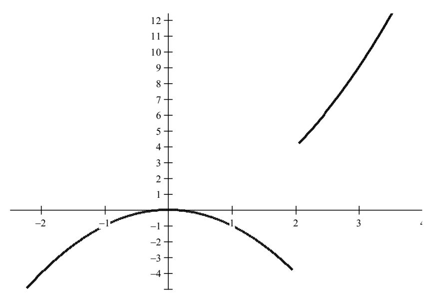

Figure 12 $h(x) = \frac{x-2}{|x-2|} * x^2$

Explicit and Implicit Forms of Equations:

When in an equation such as F(x, y) = 0, the dependent variable, y, appears only to the first power the equation can solved for y and be written in the form y = f(x) where x's appear only on the right side of the equation. This form of the equation is called explicit and y is clearly a function of x.

However, solving for y may not be possible. In this case the graph of the equation may double back or loop and y is not a function as is shown in figures 13 and 14. When F(x, y) contains only integer or fractional powers of x and y, the graph of the equation is called an algebraic curve. The equations Ax + By + C = 0 and $x^2 + y^2 = 25$ are implicit forms. The equations y = mx + b and $y = +\sqrt{(25 - x^2)}$ are explicit forms. The two forms must be treated differently. For example, the rules for translating, expanding, flipping and differentiating a curve are different for the two forms.

Algebraic Curves:

In addition to the previous computational problems, algebraic curves exhibit the following concerns.

- ❖ The graphs of algebraic curves can appear as multi-valued functions in their domains by doubling back and looping as appears in figures 13 and 14.
- ❖ The graphs of algebraic curves can exhibit excluded intervals. See figures 15 and 16.
- ❖ The graphs of algebraic curves can possess cusps. See figure 17
- ❖ The graphs of algebraic curves can contain points of self-intersection. See figure 14

The inverse functions of non-monotonic or wiggling polynomials are algebraic curves.

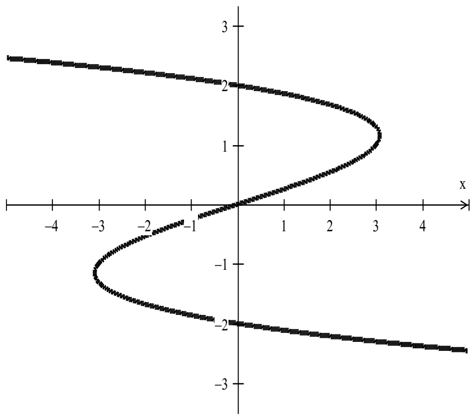

Figure 13 $x = 4y - y^3$

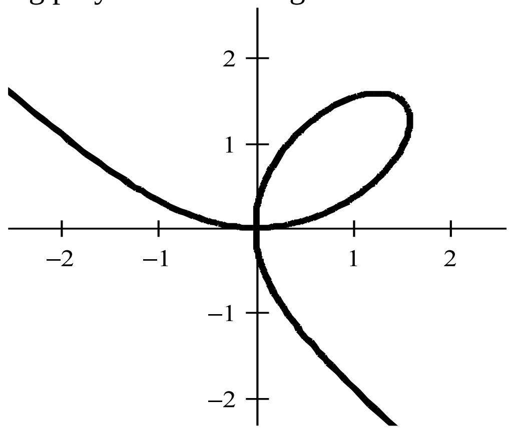

Figure 14 $x^3 + y^3 = 3xy$
The Folium of Descartes

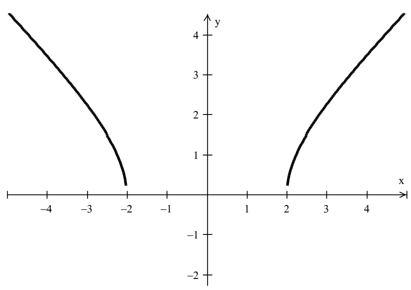

Figure 15 $y = +\sqrt{(x^2 - 4)}$

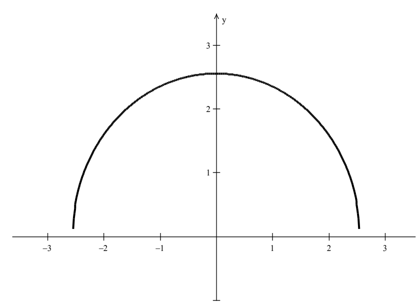

Figure 16 $y = +\sqrt{(25 - x^2)}$

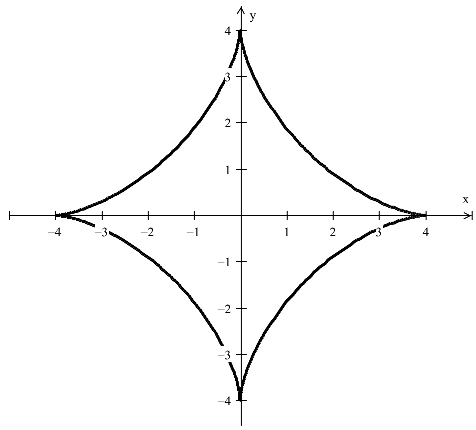

Figure 17 $x^{2/3} + y^{2/3} = a^{2/3}$
The Astroid

Transcendental Functions and Essential Singularities

Let us start with the simplest transcendental functions: the trig functions, the exponential function and the log function.

The sine and cosine and the functions, $y = e^x$ and $y = e^{-x}$ are smooth everywhere. The sines and cosines oscillate periodically. The exponential function, $y = y = e^{-x}$ rises monotonically in the upper half plane from 0 to ∞. The function $y = e^{-x}$ falls monotonically in the upper half plane from to ∞ to 0.

The remaining four trig functions have periodically spaced poles. Otherwise, they are smooth.

The logarithmic function is smooth in the right half plane, rising monotonically from $-\infty$ to $+\infty$. In the left half plane the log function is not defined. The singularity at the origin is neither removable nor a pole.

Other examples of singularities are the functions: sin(1/x), x*sin(1/x) and x^2 *sin(1/x) shown in figures 18, 19 and 20. These functions are smooth everywhere except at the origin. The function sin(1/x) oscillates infinitely often between the vertical values of +1 and – 1 in every neighborhood containing the origin. Because at the origin this function has neither a right- nor a left-hand limit, the discontinuity at the origin cannot be removed.

The function x*sin(1/x) also oscillates infinitely often between the functions y = x and y = − x in every neighborhood containing the origin. This function has both a right- and a left-hand limit equal to zero at the origin. The discontinuity can be removed by setting the value at the origin equal to 0. However, the function cannot be considered as smooth, because it does not have a tangent line at the origin.

The function x^2 *sin(1/x) oscillates infinitely often between the functions $y = x^2$ and $y = -x^2$ in every neighborhood containing the origin. This function has both a right- and a left-hand limit equal to zero at the origin. The discontinuity can be removed by setting the value at the origin equal to 0. However as can be seen in figure 20, a proof can be made that the function has a derivative equal to 0 at the origin. Most technicians may never need such functions as these last three but they serve as examples of mathematical possibilities.

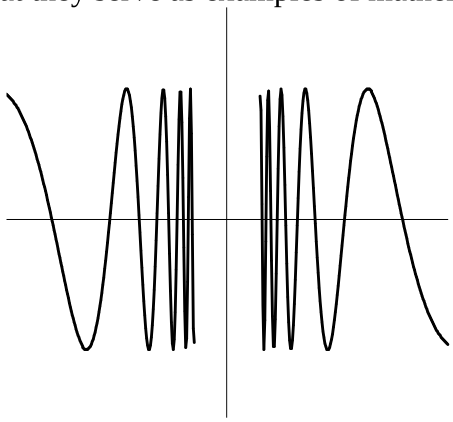

Figure 18 $y = \sin(1/x)$

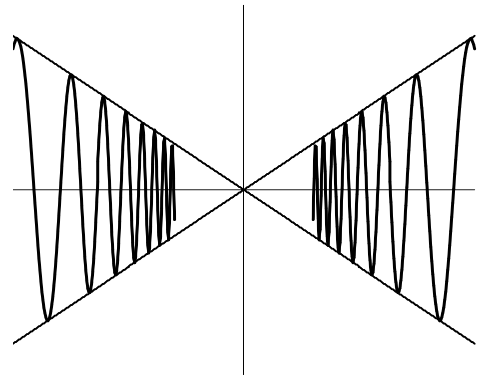

Figure 19 $y(x) = x*\sin(1/x)$

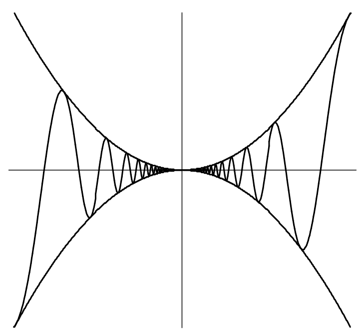

Figure 20 $y(x) = x^2 *\sin(1/x)$

Summary

This paper is designed as a measure of the effectiveness of visuals and of organizing analytical concepts for the education of technicians and engineers. It is not intended as a treatise, but as an introduction to the common singularities and computational problems of the common functions and curves of calculus. Discontinuities like those possessed by the Dirichlet function, were not included but should be delayed, if needed, for future study.

By the end of the 18th century mathematicians, led by Leonhard Euler and Daniel Bernoulli, had crystallized the calculus currently taught to engineers and physicists. The engineers and physicists of the 18th century would not have been perturbed by the previous examples. At the beginning of the 19th century, starting with Fourier, cracks were discovered in the mathematicians' logical system. The finest 19th century mathematical minds explored all the contradictions and combinations in the real number system culminating in the early 20th century Lebesgue theory of integration and measure. 20th century mathematicians, led by the secret French Bourbaki group, were ecstatic over the logic involved in the understanding of the real number system and discontinuities which then governed the words, symbols and concepts that were taught in colleges. Granting dominance to logic required the exclusion of visuals and the insight provided by visuals.

Mathematics pedagogy has other flaws. Because the organization of the topics is designed to make the proofs easy, insight is lost. Memorization, of concepts and facts that can be found easily on the Internet, is stressed. The extent of the memorized information may not be made clear. Do students who can evaluate the quadratic formula know it will not work on cubic trinomials? Are students taught the difference between conditional equations and identities and how each is used in the strategy of solutions? It appears that mathematics pedagogy needlessly creates difficulties that only the math teacher heroically can resolve.

The advanced mathematicians of every age have always been under the gun to justify new ideas which appeared crazy to the prevailing society. Zero did not need a symbol; it was nothing. No sane person in his society needed to accept negative, irrational numbers or imaginary numbers. Their names reveal how the prevailing society distrusted these strange concepts. A flat world has no need for Non-Euclidean geometry. Modern mathematicians cannot be blamed for promoting what they have been taught and mastered in their schooling.

But unfortunately, these 19th century developments are not needed by technicians and most engineers and are not required for the licensing exams. Even the engineers who work with highly discontinuous functions first need to study the elementary functions. A definition in terms of epsilons requires a proof in terms of epsilons. The function definition provided in almost all high school and college texts used in differential calculus courses includes nowhere differentiable functions. In no way is this logical. In a differential calculus course, functions should be introduced as mostly smooth curves. This would naturally lead to all the concepts needed by engineers. The functions of calculus could be introduced as smooth models of control. The horizontal variable controls the vertical variable.

The mathematics community is incapable of changing its pedagogy without outside influence. Few communities can change on their own. To their credit the mathematicians have tried; in the last seven decades, attempting quite a few math reforms. However, the pedagogical problems remain entrenched. External pressure from industry, outside academic societies like ASEE, including ETD can inform the MAA and the AMS that substantive change is needed. Of course, research mathematicians must continue to confront the real number system and discontinuities. And of course, the gifted among them will discover fantastic new truths.

It seems the math teaching community is unaware that engineers require a different presentation of analytic course material. Mathematics pedagogy is pointlessly turning off many of our nation’s youth who would enjoy the study and seek the rewards of STEM careers. I have no doubt that a math faculty, apprised of the shortcomings of the current pedagogy, would present more understandable course material that their students would find invaluable in their lives and careers. I can imagine no greater gift that the math teaching community could make to their students and to society.

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