# MICROSOFT EXCEL-BASED NUMERICAL SOLUTION OF LINEAR, HOMOGENEOUS 1D TRANSIENT PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT

Many transient phenomenons are mathematically described and simulated by the homogeneous, parabolic partial differential equations. Irregular and non-linear boundary conditions pose formidable difficulties to engineers to obtain closed form (exact) solutions.

In this paper, a procedure is outlined to make use of Microsoft Excel software to solve these differential equations by converting them into finite difference equations employing explicit and implicit techniques.

The advantages of this study are to apply universally available Microsoft Excel to solve fairly complex engineering problems. The method lends itself as a powerful tool to Engineering students to design and perform parametric analysis by employing simple, quick techniques without having to acquire specialized and sometimes expensive software packages.

A case study is presented where the time history of temperature in a one-dimensional heat transfer problem is analyzed. Finite difference techniques are used to solve the differential equations. Initially the equations are solved using an explicit method and the same problem is also solved using an implicit method. Step by step procedure to generate Excel worksheet is described.

Results obtained with the two numerical methods are compared with analytical results. Effects of grid size and time interval on the accuracy of the results are graphically presented.

INTRODUCTION

Exact analytical solutions of the partial differential equations describing various transient systems and processes are cumbersome and sometimes complex, and time consuming. The difficulty is compounded, if not impossible, when shapes of irregular and non-linear boundaries are encountered. Due to these complex procedures and solutions, the Engineering students, do not take full advantage of parametric study in the design and analysis of the engineering systems. Numerical solution of some such differential equations and the tools to generate these solutions without requiring large amounts of efforts and time is the primary goal of this study.

Microsoft excel is not only a powerful tool to organize and present data in various forms, latest additions to it make it a powerful tool for many scientific and engineering calculations. It is widely used for normal applications and a large portion of the student population with PCs is familiar with its usual applications. However, many do not employ Excel (Excel-solver) package to solve engineering problems. Therefore, using Excel for this purpose eliminates the need to purchase specialized software packages for various applications. The problem setup is effortless and quick, and once formulated it can be used repeatedly. As a result, it is time and cost effective. A case study of solving linear homogeneous 2D steady state partial differential equations was previously reported [1].

In this paper, a numerical solution of an unsteady state heat equation is presented. The heat transfer problem is solved by two different numerical methods to obtain temperature distribution. In the first case, the results are obtained using Euler method, which is an explicit formulation, and second method is the Crank-Nicolson implicit and explicit method and the results are compared with analytical solution. The accuracy of the numerical results is examined with various grid sizes and graphical comparison of the results is presented.

Similar methods are employed in the Heat Transfer laboratory of the Mechanical Engineering Department of the Texas A & M University to verify the experimental results [2, 3].

## MATHEMATICAL FORMULATION

In this example, one dimensional transient heat transfer with boundary and initial conditions is considered. The pictorial representation is shown the figure 1.

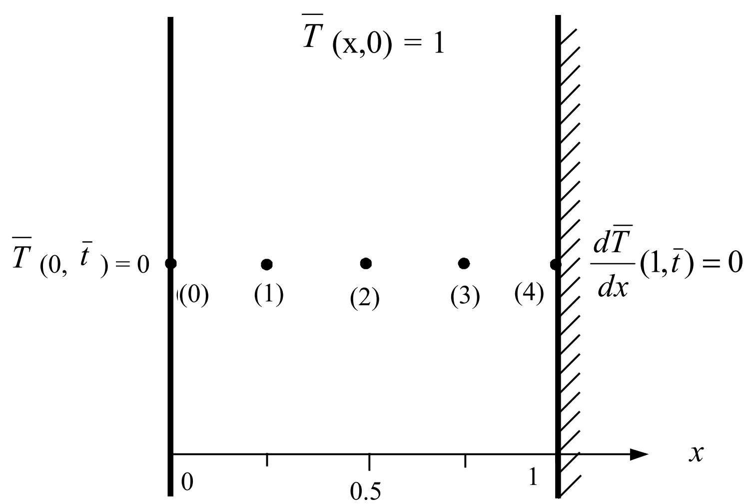


Figure 1. Finite difference nodal arrangement for temperature distribution

Mathematical description of the above problem is given by the heat equation as follows:

$$\frac{\partial \overline{T}}{\partial \bar{t}} = \frac{\partial^2 \overline{T}}{\partial x^2} \tag{1}$$

where $\overline{T}$, $x$ and $\bar{t}$ are the normalized temperature, distance and time [4].

The boundary and the initial conditions are:

$$\overline{T}(0,\bar{t}) = 0 \qquad \frac{\partial \overline{T}}{\partial x}(1,\bar{t}) = 0 \qquad \overline{T}(x,0) = 1$$

The finite difference nodes are chosen as shown in figure 1. The domain is divided into four divisions in $x$ direction. When the finite difference scheme is applied to the nodal points, the equation (1) becomes four ordinary differential equations shown below:

$$\frac{d\overline{T}_1}{d\bar{t}} = \frac{1}{(\Delta x)^2}(\overline{T}_0 - 2\overline{T}_1 + \overline{T}_2) \tag{2}$$

$$\frac{d\overline{T}_2}{d\bar{t}} = \frac{1}{(\Delta x)^2}(\overline{T}_1 - 2\overline{T}_2 + \overline{T}_3) \tag{3}$$

$$\frac{d\overline{T}_3}{d\bar{t}} = \frac{1}{(\Delta x)^2}(\overline{T}_2 - 2\overline{T}_3 + \overline{T}_4) \tag{4}$$

$$\frac{d\overline{T}_4}{d\bar{t}} = \frac{1}{(\Delta x)^2}(2\overline{T}_3 - 2\overline{T}_4) \tag{5}$$

where $\overline{T}_0$, $\overline{T}_1$, $\overline{T}_2$, $\overline{T}_3$ and $\overline{T}_4$ are the normalized temperatures at nodes 0, 1, 2, 3, and 4 respectively.

<u>Euler Method:</u> When Euler's explicit scheme is used, the equations for new temperatures after the time increment are given by:

$$\overline{T}_1^{(i+1)} = \overline{T}_1^{(i)} + \left.\frac{d\overline{T}_1}{d\bar{t}}\right|^{i} \Delta\bar{t} \tag{6}$$

$$\overline{T}_2^{(i+1)} = \overline{T}_2^{(i)} + \left.\frac{d\overline{T}_2}{d\bar{t}}\right|^{i} \Delta\bar{t} \qquad (7)$$

$$\overline{T}_3^{(i+1)} = \overline{T}_3^{(i)} + \left.\frac{d\overline{T}_3}{d\bar{t}}\right|^{i} \Delta\bar{t} \qquad (8)$$

$$\overline{T}_4^{(i+1)} = \overline{T}_4^{(i)} + \left.\frac{d\overline{T}_4}{d\bar{t}}\right|^{i} \Delta\bar{t} \qquad (9)$$

where $\overline{T}_1^{(i)}$ is normalized temperature at node 1 after i$^{th}$ iteration and $\overline{T}_1^{(i+1)}$ is the new value after one time interval.

Combining Equations (2), (3), (4), (5) with equations (6), (7), (8), (9), we obtain the following equations which will be used for Excel programming purposes.

$$\overline{T}_1^{(i+1)} = \overline{T}_1^{(i)} + \frac{1}{(\Delta x)^2}(\overline{T}_0^{\,i} - 2\overline{T}_1^{\,i} + \overline{T}_2^{\,i})\Delta\bar{t} \qquad (10)$$

$$\overline{T}_2^{(i+1)} = \overline{T}_2^{(i)} + \frac{1}{(\Delta x)^2}(\overline{T}_1^{\,i} - 2\overline{T}_2^{\,i} + \overline{T}_3^{\,i})\Delta\bar{t} \qquad (11)$$

$$\overline{T}_3^{(i+1)} = \overline{T}_3^{(i)} + \frac{1}{(\Delta x)^2}(\overline{T}_2^{\,i} - 2\overline{T}_3^{\,i} + \overline{T}_4^{\,i})\Delta\bar{t} \qquad (12)$$

$$\overline{T}_4^{(i+1)} = \overline{T}_4^{(i)} + \frac{1}{(\Delta x)^2}(2\overline{T}_3^{\,i} - 2\overline{T}_4^{\,i})\Delta\bar{t} \qquad (13)$$

In the above equations, let $p = \frac{\Delta\bar{t}}{(\Delta x)^2}$. The value of $p$ should be assigned with due consideration as it affects accuracy as well as the stability of the numerical scheme. Smaller values increase the time of computation and accuracy, whereas high value makes the computations unstable. For explicit methods $p < 0.25$ [5].

Excel Implementation

The built-in iteration feature of Excel can be used to solve the finite difference nodal equations to obtain the time history of temperatures at the nodal points.

Create a new worksheet and label as 'Explicit'. We need 6 columns. The first column indicates time; second column indicates left boundary and the other four columns define the four nodal points. Define the values of parameters such as Δt, Δx and compute the value of *p*. Enter value of Δt as "0.015625" in cell C1, that of Δx as "0.25" in cell C2. (Enter the information inside the Quotes). Enter "=C1/(C2^2)" in C3 and it should display 0.25, which is the value of *p*. Various values of *p* can be achieved by selecting appropriate values in cells C1 and C2. In the fourth row we will enter the column titles as "t sec" in cell A4, "0" in B4, "1" in cell C4, "2" in cell D4, "3" in cell E4, and "4" in cell F4. In the next row enter the starting time of "0" in A5. Entering "=A5+$C$1" in the cell A6 will increment the time by Δt. Grab the cell in A6 and drag to cell A69. You will see the time in column A incrementing in each row to 1.000000 in cell A69. In the column B enter '0' in cells B5 and drag through B69 as the initial and boundary condition. The initial nodal temperatures are entered as 1 in cells C5, D5, E5, and F5. We now need to generate the finite difference equations for the four nodal points at 1, 2, 3 and 4.

Transcribing the equations into Excel spreadsheet follows:

- In the cell C6 transform the finite difference equation (10) in to Excel equation format as:
- =C5 + $C$3*(B5 - 2*C5 + D5)
- Grab the cell C6 at the lower right corner and drag this through the column E. Notice that appropriate Excel equations (11) and (12) of the nodes 2 and 3 are automatically generated.
- Grab three cells C6 through E6 and drag down to the row 69.
- In the cell F6 transform the finite difference equation (13) in to Excel equation format as:
- =F5 + $C$3*(- 2*F5 + 2*E5)
- Grab the cell E6 and drag through the column F till row 69.

The computation process progresses and final values are displayed. A graph can be generated using Excel for temperature variation with time at any location. A graph of the temperature variation at x = 1 is plotted in figure 2.

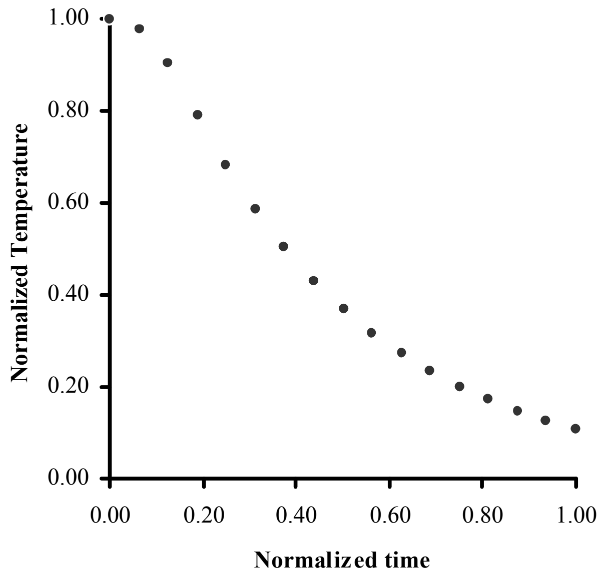


Figure 2. Temperature variation with time at x = 1 (Euler Method).

Crank-Nicolson Method: In the Euler Method, the value of the derivative at the beginning of the time interval was used to move ahead in time. Crank-Nicolson method uses average of the derivatives at the beginning and the end of time interval, which yields more accurate and faster results. This can be formulated as follows:

$$\bar{T}_n^{(i+1)} = \bar{T}_n^{(i)} + \frac{1}{2}\left[ \left.\frac{d\bar{T}_n}{d\bar{t}}\right|^{i} + \left.\frac{d\bar{T}_n}{d\bar{t}}\right|^{i+1} \right]\Delta\bar{t} \tag{14}$$

where n = 1, 2, 3, and 4.

Utilizing equations (2), (3), (4), and (5) for derivatives in equation (14) we obtain the algebraic equations for the nodal temperatures as shown below:

$$\bar{T}_1^{(i+1)} = \bar{T}_1^{(i)} + \frac{\Delta\bar{t}}{2(\Delta x)^2}\left[ (-2\bar{T}_1^{\,i} + \bar{T}_2^{\,i}) + (-2\bar{T}_1^{\,i+1} + \bar{T}_2^{\,i+1}) \right] \tag{15}$$

$$\bar{T}_2^{(i+1)} = \bar{T}_2^{(i)} + \frac{\Delta\bar{t}}{2(\Delta x)^2}\left[ (\bar{T}_1^{\,i} - 2\bar{T}_2^{\,i} + \bar{T}_3^{\,i}) + (\bar{T}_1^{\,i+1} - 2\bar{T}_2^{\,i+1} + \bar{T}_3^{\,i+1}) \right] \tag{16}$$

$$\bar{T}_3^{(i+1)} = \bar{T}_3^{(i)} + \frac{\Delta\bar{t}}{2(\Delta x)^2}\left[ (\bar{T}_2^{\,i} - 2\bar{T}_3^{\,i} + \bar{T}_4^{\,i}) + (\bar{T}_2^{\,i+1} - 2\bar{T}_3^{\,i+1} + \bar{T}_4^{\,i+1}) \right] \tag{17}$$

$$\overline{T}_4^{(i+1)} = \overline{T}_4^{(i)} + \frac{\Delta\bar{t}}{2(\Delta x)^2}\left[(2\overline{T}_3^{\,i} - 2\overline{T}_4^{\,i}) + (2\overline{T}_3^{\,i+1} - 2\overline{T}_4^{\,i+1})\right] \tag{18}$$

In the above equations, let $p = \frac{\Delta\bar{t}}{(\Delta x)^2}$. We can now write these equations to create Excel worksheet.

$$\overline{T}_1^{(i+1)} = \frac{1}{1+p}\left[(1-p)\overline{T}_1^{(i)} + \frac{p}{2}(\overline{T}_2^{\,i} + \overline{T}_2^{\,i+1})\right] \tag{19}$$

$$\overline{T}_2^{(i+1)} = \frac{1}{1+p}\left[(1-p)\overline{T}_2^{(i)} + \frac{p}{2}(\overline{T}_1^{\,i} + \overline{T}_1^{\,i+1} + \overline{T}_3^{\,i} + \overline{T}_3^{\,i+1})\right] \tag{20}$$

$$\overline{T}_3^{(i+1)} = \frac{1}{1+p}\left[(1-p)\overline{T}_3^{(i)} + \frac{p}{2}(\overline{T}_2^{\,i} + \overline{T}_2^{\,i+1} + \overline{T}_4^{\,i} + \overline{T}_4^{\,i+1})\right] \tag{21}$$

$$\overline{T}_4^{(i+1)} = \frac{1}{1+p}\left[(1-p)\overline{T}_4^{(i)} + p\,(\overline{T}_3^{\,i} + \overline{T}_3^{\,i+1})\right] \tag{22}$$

The value of $p$ should be assigned with due consideration as it affects accuracy as well as the stability of the numerical scheme. For explicit-implicit methods for steady convergence $p < 1$.

Excel Implementation

The built-in iteration feature of Excel can be used to solve the finite difference nodal equations to obtain the time history of temperatures at the nodal points.

Create a new worksheet and label as 'Implicit/Explicit'. We will need again 6 columns. The first column indicates time; second column indicates left boundary and the other four columns define the four nodal points. Define the values of parameters such as Δt, Δx and estimate the value of $p$. Enter value of Δt as "0.0625" in cell C1, that of Δx as "0.25" in cell C2. (Enter the information inside the Quotes). Enter "=C1/(C2^2)" in C3 and it should read 1, which is the value of $p$. Various values of p can be achieved by selecting appropriate values in cells C1 and C2. In the fourth row we will enter the column titles as "t sec" in cell A4, "0" in B4, "1" in cell C4, "2" in cell D4, "3" in cell E4, and "4" in cell F4. In the next row enter the starting time of "0" in A5. Entering "=A5+$C$1" in the cell A6 will increment the time by Δt. Grab the cell in A6 and drag to cell A22. You will see the time in column A incrementing in each row to 1 in cell A22. In the column B enter '0' in cells B5 through B22 as the initial and boundary condition. The initial nodal temperatures are entered as 1 in cells C5, D5, E5, and F5. We now need to generate the finite difference equations for the four nodal points at 1, 2, 3 and 4.

Transcribing the equations into Excel spreadsheet follows:

- In the cell C6 transform the finite difference equation (19) in to Excel equation format as:
- =((1-$C$3)*C6+$C$3*(D6+D7)/2)/(1+$C$3)
- In the cell D6 transform the finite difference equation (20) in to Excel equation format as:
- =((1-$C$3)*D5+$C$3*(C5+C6+E5+E6)/2)/(1+$C$3)
- Grab the cell D6 at the lower right corner and drag this through the column E6. Notice that appropriate Excel equations (20) and (21) of the nodes 2 and 3 are automatically generated.
- Grab three cells C6 through E6 and drag down to the row 22.
- In the cell F6 transform the finite difference equation (22) in to Excel equation format as:
- =((1-$C$3)*F5+$C$3*(E5+E6))/(1+$C$3)
- Grab the cell E6 and drag through the column F till row 22.

The computation process progresses and final values are displayed. A graph generated using Excel with both Euler and Crank-Nicolson method are plotted in figure 3.

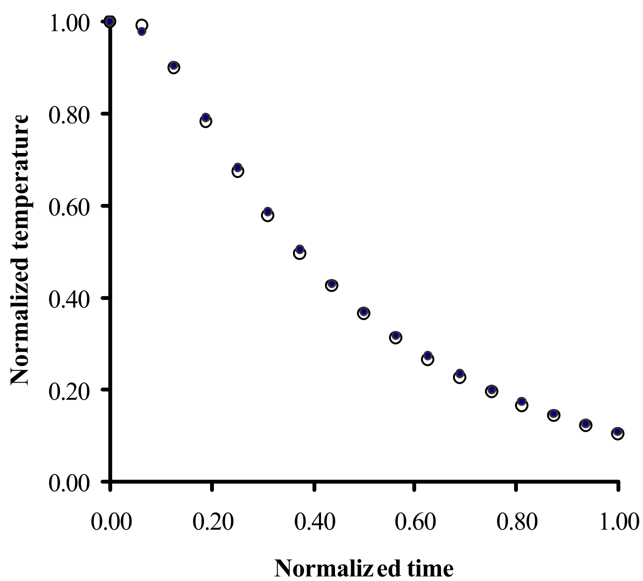


Fig. 3 Comparison of Euler (•) and Crank-Nicolson (o) Methods at x = 1

Comparison of final solutions at four nodes using Euler and Crank-Nicolson methods with an exact solution [4] is shown in Table 1. The number of iterations for Euler's method was 64 whereas Crank-Nicolson method needed only 16 iterations and produced values close to exact solution.

Table 1 Comparison of final solutions at four nodes.

| Node | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| Euler Method ($p$= 0 25) | 0.041<br>4 | 0.074<br>2 | 0.096<br>9 | 0.104<br>9 |
| Crank-Nicolson ($p$ = 1.0) | 0.041<br>9 | 0.077<br>4 | 0.101<br>2 | 0.109<br>5 |
| Exact solution | 0.042<br>1 | 0.077<br>8 | 0.101<br>6 | 0.110<br>0 |

## STUDENT ASSIGNMENTS

The important step in above case studies was forming equations suitable for Excel implementation. Formulation and implementation of such transient solution can be made two parts of a particular assignment. Once the students become familiar with creating EXCEL worksheet and run one simulation, further extensions of transient problems can be assigned. Some such examples are different boundary conditions, irregular boundaries, different materials, etc. Material properties can be included in the definition of the parameter $p$. Students can also be assigned to examine the effects of the parameter $p$ on the stability and accuracy of the solutions using both methods.

## CONCLUSIONS

The adaptation of Excel to solve partial differential equations is simple and powerful. It does not require much training and the numerical scheme can be formulated quickly. The advantages of implementation of Excel are even greater while solving complex problems with irregular boundaries.

## REFERENCES


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