On Calculating the Slope and Deflection of a Stepped and Tapered Shaft

Dr. Carla Egelhoff, Montana Tech of the University of Montana

Dr. Egelhoff teaches courses that include petroleum production engineering, oil property evaluation and capstone senior design within the Petroleum Engineering program at Montana Tech of the University of Montana.

Dr. Edwin M. Odom, University of Idaho, Moscow

Dr. Odom teaches courses that include introductory CAD, advanced CAD, mechanics of materials, machine design, experimental stress analysis and manufacturing technical electives within the Mechanical Engineering program at the University of Idaho.

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Introduction

As this is written there are natural gas-fired power plants that include a bottoming cycle achieving 45% thermodynamic efficiency. There is ongoing development of a gear driven compressor for an aircraft engine that could reduce fuel consumption by 15%. Additionally, there are a number of automobiles using hybrid power trains in the marketplace and there are eight and nine speed automobile transmission designs that maximize the fuel economy. It is easy to focus on these sophisticated applications and marvel at the systemic design, and not think about the basic components deep inside. One of those components is the shaft which may locate bearings, gears and couplings while transmitting power and motion.

The design considerations of a shaft can be broken down into three areas, fatigue, deflection, and critical frequency. During operation it can be subject to minimum and maximum axial, transverse and torsional loads leading to mean and alternating stress states. These stresses can be addressed during a fatigue analysis which is well covered in texts on machine component design and governing standards. Critical frequency prediction is reasonably straightforward once the deflection of the shaft is known along with the attendant masses.

As long as the loading is not complicated and the shaft has a constant diameter, determining the deflections of a shaft is straight forward and well covered in texts on mechanics of materials and machine component design. However, when the shaft cross section becomes *practical* it includes changes of diameter to provide steps that can be used to accurately mount bearings and gears. It can have overhanging ends and tapered cross sections. The need for finding the deflection and slope of these types of shaft geometries and loadings is timeless. Each generation of engineers has used that part of mechanics of materials theory that fit the calculating capability available to them at that particular time. The method presented here is offered in that vein.
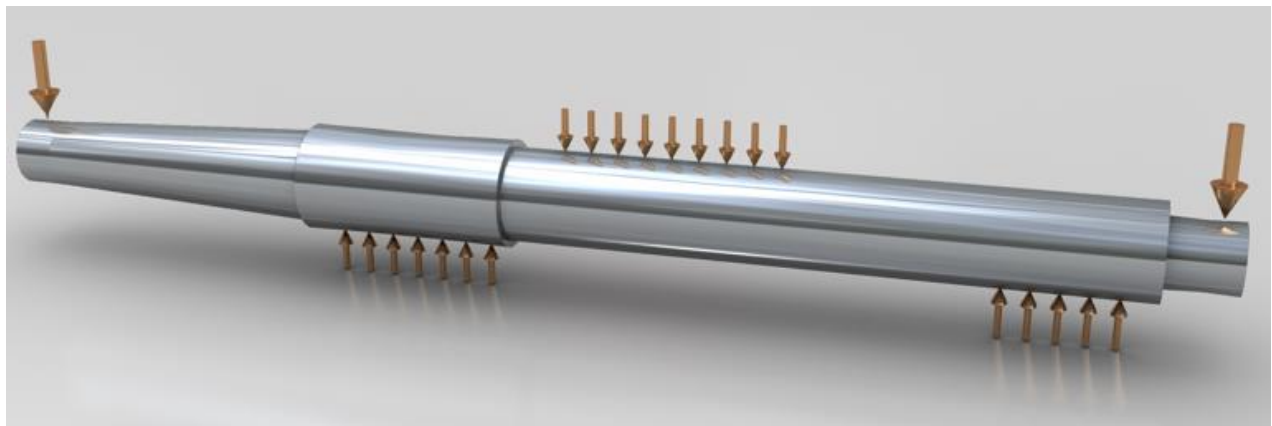

Figure 1. Machine Designer Walter Schroeder of the Cincinnati Milling Machine Co. was interested in the deflection of the stepped shaft loaded as shown.[1] To avoid binding at the bearing ends, their locations were of critical importance.

Background

The literature search is purposefully limited to methods that have been previously used for finding deflections of stepped shafts. An article by Professor C.W. Bert in 1960 entitled "Deflection of Stepped Shafts" [2] used Castigliano's theorem to find the deflection of a simply supported grinding machine spindle with two intermediate masses for the purpose of calculating the critical frequency of the shaft. In this article Professor Bert also reviewed the other methods available at the time to find deflections. These included: (a) the graphical funicular polygon method [1] (still presented in some literature [3]), (b) the moment area-integration method, (c) the finite difference method, (d) the relaxation method, (e) the conjugate beam method, (f) the matrix method, (g) the Laplace transform method and (h) the Hetenyi trigonometric-series method. Additional methods that can be added to this list could include those based on the use of Macaulay functions [4-6], singularity functions as well as finite element analysis. All of these methods can provide numerically accurate results and there are undoubtedly certain shaft geometries and loadings that might be more amenable to one method or the other. Some methods were appropriate for the classroom such as the graphical methods when drafting was still taught, but they are more difficult to use today.

The method presented here is based on the work of Professor F.D. Ju as presented in his 1971 article "On the Constraints for Castigliano's Theorem" [7] and the notes of one of the authors as a student in Professor Ju's class in the mid 1980's. In his article Professor Ju provides two extensions to the application of Castigliano's theorem. First, it is shown how to incorporate constraints in the form of the equations of equilibrium (e.g., $\Sigma F=0$ and $\Sigma M=0$) by way of Lagrangian multipliers into the Castigliano's theorem resulting in a "generalized form of Castigliano's theorem." For typical statically determinate problems such as the example presented in this article, there is no need for incorporating the equilibrium constraints. For statically indeterminate structures, this method can be quite effective. Second, in his article Professor Ju also incorporates the use of dummy loads to find the displacement at the location of the dummy load. A second virtual axis that tracks the location of the dummy load is also incorporated into the analysis. Additionally, Heaviside step functions were used to write the continuous load (moment, torque) expressions thus allowing a continuous displacement function. This means that when the closed-form analysis is completed, the deflection anywhere along the structure from beginning to end can be calculated.

Professor Ju concludes his article by presenting the closed form solution of the deflection of a semi-circular beam (Figure 2) of constant cross section, built-in at one end with supports at the opposite and half way position loaded uniformly perpendicular to the axis of the beam. A uniform distributed load, p_0, is applied along the length. Position C is built in loosely so as "to allow no resistance to twist."

The initial portion of Professor Ju's article is very theoretical and presented in indicial notation. If a reader is interested in the deflection of a beam, this presentation and its example problem can be a challenge. When the authors began their
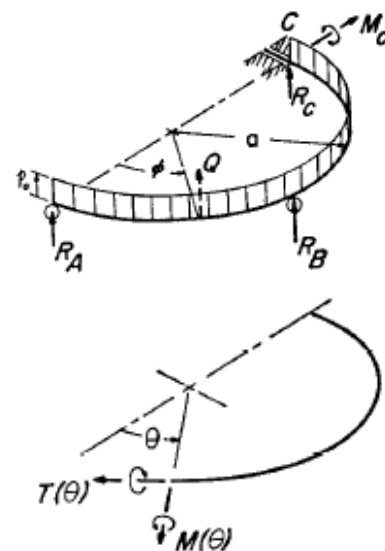

Figure 2. From Ju [7] curved beam.

study of this subject, a literature review found no one had ever referenced this article. Other than the authors that is still true. The only supplemental information available was the classroom notes. The method presented here takes advantage of these notes and uses a portion of Professor Ju's work.

Context

The method presented has been tuned to fit within the undergraduate Mechanical Engineering curriculum. Our assumption is that students have completed classes in statics and mechanics of materials and they are ready to learn this approach in their study of machine component design. We have reviewed the Machine Design textbooks and found they all provide the following: a review of free body diagrams, statics, and determination of reactions for simple beam-load configurations, a section on the use of singularity functions, writing shear and moment equations, and strain energy methods. Finally, we also assume students have access to an equation solver. The authors use TK Solver™ and EES© but our students and colleagues have produced solutions using Mathematica®, Matlab® and MathCad®. In deference to the faculty who might be interested in this method, we selected a very complex shaft geometry and loading. Additionally, our complete solution provided in this paper may be more than is needed in a shaft design problem. The typical textbook problem involves a simply supported shaft with one concentrated load between the supports complicated by numerous changes in cross sectional dimensions. A bare-bones deflection solution to such a problem using this method requires about a half dozen lines of code and a table function. Exploring this solution method began as a curiosity and was very slowly introduced into the classroom over a number of semesters. To date over 450 students at the University of Idaho and 130 students at the United States Coast Guard Academy have been introduced to this method and only about a dozen, overall, failed to master the process and produce virtually perfect analysis and results.

The Method

The method stays generalized, using an engineer's knowledge of free body diagrams, writing moment equations, and Castigliano's theorem to set up the problem solution into a form that is solved in an engineer's favorite computer program.

Beginning with Castigliano's Theorem, the strain energy, U, stored in a structural member due to its bending is written as:

$$U = \int_0^L \frac{M^2}{2EI} dx \qquad \text{Eq.1}$$

where M is the moment along the length, L, of the beam, E is the modulus of elasticity and I is the second moment of the area. Castigliano's second theorem relates deflection at a point to the partial derivative of the strain energy with respect to a load applied at that point. If an external load is not present at the point of interest, then a dummy load can be applied there for the purpose of deflection determination. After the partial derivative is calculated with respect to the dummy load, that dummy is set to zero in the moment equation. In equation form, we write:

$$\delta_Q = \frac{\partial U}{\partial Q} = \int_0^L \frac{M_{Q=0}}{EI} \frac{\partial M}{\partial Q} \, dx \qquad\qquad \text{Eq.2}$$

The variable Q is used to delineate the dummy load. It should be noted here that variables I, M, and the partial derivative are all functions of x.

Since designing engineers are also acutely interested in shaft slope at key locations such as at bearings or overhangs, a similar process can be used. Castigliano's Theorem for slope at a point-of-interest along the beam is:

$$\theta_m = \frac{\partial U}{\partial m} = \int_0^L \frac{M_{m=0}}{EI} \frac{\partial M}{\partial m} \, dx \qquad\qquad \text{Eq.3}$$

The variable m represents a dummy moment located at the point where the slope, θ, is desired. For determination of slope, the partial derivative is taken with respect to the dummy moment.

Solving Eq.2 and Eq.3 directly yields the deflection and slope of any shaft or beam at any chosen location along the length. If each term of the integrand can be correctly written, then an equation solver provides the numerical muscle needed. Consider each of the terms in the integrands. The modulus, E, is constant for most cases so it can be moved outside of the integral. The moment of inertia, I, is a function of diameter which is defined within the equation solving software chosen. It remains, then, to insert a dummy load, Q, and a dummy moment, m, on the shaft and write a moment equation for the entire length. Determine two partial derivatives of the moment equation, one with respect to the dummy load, Q, and one with respect to the dummy moment, m. Finally, re-write the moment equation for use in the integrand (set $Q,m=0$). Then x is used as the integrating variable while a secondary axis, ξ, serves to track the location along the shaft where the deflection is being calculated (Figure 3). Writing the moment equation for the entire beam is accomplished efficiently by introducing a Heaviside step function to serve the same purpose as Macaulay brackets [8] in discontinuity functions taught in mechanics of materials class.

Although in 1947 Walter Schroeder had no spreadsheet or equation-solving software, he articulates clearly the type of real-life problem needing to be solved: "those cases where loading is manifold and arranged at random, where beam cross section is not constant but varying, and where deflections at special points or over the full length of the beam are desired."[1] Such is the shaft shown in Figure 1. It has several steps and one taper in its diameter. Supported by two bearings (upward distributed loads), the shaft accommodates three external loads, one of which is distributed. Schroeder's design criteria incorporated slope at each bearing end and smallest possible deflection everywhere.

Traditionally in challenging deflection problems, distributed loads are modeled as concentrated loads for simplicity with the assumption that concentrated loading will be "close enough" to the actual distributed loading for determination of deflection. In the example which follows, the analysis begins with the treatment of distributed loads as concentrated loads. Then, because the method shown is readily repeated using distributed loading, we can assess whether the simplification is sufficient.

Overall, the analysis method consists of the following steps:

    (1) apply a dummy load/moment, and solve for static support reactions,

(2) write a moment equation in Macaulay form augmented with Heaviside step function variables,

(3) take a partial derivative of the moment equation with respect to the dummy load and a second partial derivative with respect to the dummy moment,

(4) re-write the moment equation to eliminate the dummy load/moment and finally,

(5) use the results of steps 3 and 4 to develop the deflection calculation via Castigliano's Theorem applied parametrically to create a deflection curve for the entire length of the beam.

Figure 3 shows the example shaft having several steps and one taper in its diameter. Three loads are applied, one of which is distributed (3450-lb over 8-inches), and the shaft is supported by two rigid bearings (left support, R_L, 3500-lb over 6-inches; right support, R_R, 1600-lb over 4-inches). The free-body diagram is augmented with dummy-load, Q, and dummy-moment, m, and the concomitant secondary axis, ξ. Diameter measurements are indicated; distances from $x=0$ to load locations are shown on the middle axis. Distances from $x=0$ to diameter changes are shown on the lowermost axis. All distances are measured in inches; loads are in lbs. The tapered section begins at $x=1$ and ends at $x=12$ inches. The left bearing begins at $x=13$ and ends at $x=19$ inches from the left. Both x and ξ are zeroed at the same left position where the 900-lb overhang concentrated load is applied. The entire length of shaft in the analysis is 45-inches.
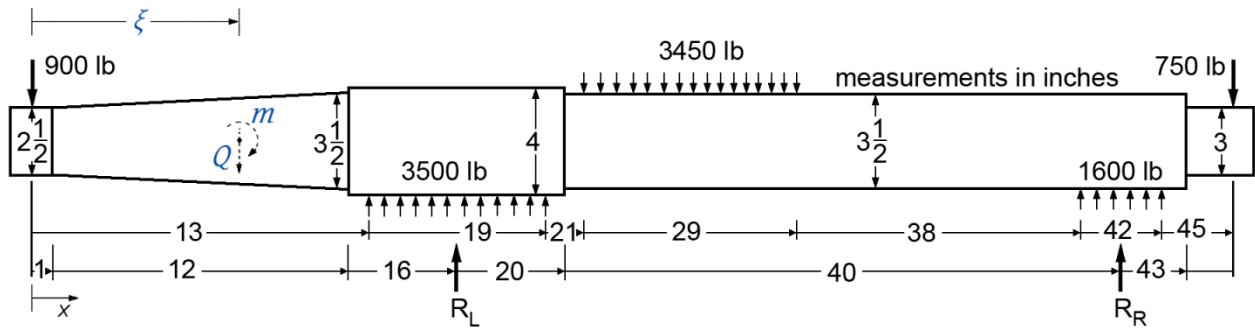

Figure 3. Example problem shaft (after Schroeder [1]). For the machine component designer the shaft deflection and rotation is important at the bearings so that clearance is provided to prevent binding.

Concentrated load assumption

As shown, a dummy-load (Q) and dummy-moment (m) are applied to the free body diagram at the arbitrary location indicated by the secondary axis, ξ. For only the dummy load and dummy moment, reactions, R_L and R_R are determined using statics:

$$R_L = \frac{40 - \xi}{24}Q - \frac{m}{24} \qquad\qquad R_R = \frac{\xi - 16}{24}Q + \frac{m}{24} \qquad\qquad \text{Eq.4}$$

Before proceeding to write the moment equation, we need to define the Heaviside function

$$H(a,b) = \begin{cases} 0 \ if \ a < b \\ 1 \ if \ a \geq b \end{cases} \qquad\qquad \text{Eq.5}$$

When used to write moment equations Heaviside step functions serve the same purpose as a singularity function or Macaulay function (the Heaviside step function is used here in deference to Professor Ju [7]). Instead of using pointed brackets we use regular parenthesis followed by the Heaviside step function which operates as a switch to activate the term. Treating all distributed loads as concentrated loads, the moment equation is:

$$M(x,\xi) = -900x - Q(x - \xi)H(x,\xi) + mH(x,\xi) + R_L(x - 16)H(x,16)$$
$$+ 3500(x - 16)H(x,16)$$
$$- 3450(x - 25)H(x,25) + R_R(x - 40)H(x,40)$$
$$+ 1600(x - 40)H(x,40)$$

Eq.6

where R_L and R_R are defined in Eq.4. The terms in this equation are in the order encountered from left to right in Figure 3. The term $-Q(x-\xi)H(x,\xi)$ is the moment caused by the dummy load, Q, when coordinate x becomes greater than the point-of-interest coordinate, ξ. The moment arm is $(x - \xi)$ and the term is not active as long as $x < \xi$. The term representing the 750-lb load at the right end is omitted because we consider 45-inches to be the end and do not integrate beyond that location.

Determine the partial derivative with respect to the dummy-load, Q.

$$\frac{\partial M(x,\xi)}{\partial Q} = -(x - \xi)H(x,\xi) + \frac{40 - \xi}{24}(x - 16)H(x,16) + \frac{\xi - 16}{24}(x - 40)H(x,40)$$

Eq.7

The partial derivative with respect to the dummy-moment, m, will be used to determine slope and is included here since it conveniently follows Eq.7.

$$\frac{\partial M(x,\xi)}{\partial m} = H(x,\xi) - \frac{1}{24}(x - 16)H(x,16) + \frac{1}{24}(x - 40)H(x,40)$$

Eq.8

Rewrite the moment equation setting $Q,m=0$.

$$M(x,\xi)_{Q,m=0} = -900x + 3500(x - 16)H(x,16) - 3450(x - 25)H(x,25) +$$
$$1600(x - 40)H(x,40)$$

Eq.9

Before a solution can be accomplished the area moment of inertia term, $I(x)$, will need to be defined as a function of shaft diameter and location (x) for integration.

$$I(x) = \frac{\pi[dia(x)]^4}{64}$$

Eq.10

The shaft diameter can be defined according to the equation solver chosen. Figure 4A shows the list function used by TKSolver™ which serves as a look-up table and Figure 4B shows EES© code for the user-defined function which produces the same result. While we are aware that integrating across a discontinuity can be problematic for numerical tools, we have found convergence to be extremely rapid.

LIST FUNCTION: dia		
Comment:		
Domain List:	Distance	
Mapping:	Linear	
Range List:	Diameter	
Element	**Domain**	**Range**
1	0	2.5
2	1	2.5
3	11.99	3.5
4	12	4
5	19.99	4
6	20	3.5
7	43	3.5
8	43.01	3
9	45	3

(A)

"Define the diameter as a function of x"

```
function dia(x)
    if(x<1) then dia:=2.5 else
        if(x<12) then dia:=(2.5+(x-1)/11) else
            if(x<20) then dia:=4 else
                if(x<43) then dia:=3.5 else
                    if(x<=45) then dia:=3 else
                    endif
                endif
            endif
        endif
    endif
end
```

(B)

Figure 4. (A) User-created TKSolver™ list function defining the shaft diameter along the length. (B) User-created EES© code also defines shaft diameter. Both serve as look-up tables, determining diameter for any location, x.

Construct the correct integral (using Castigliano's Theorem) which will be solved by the software of choice. Combine Eq.7, Eq.9 and Eq.10 with Eq.2 (repeated here for convenience) or combine Eq.8, Eq.9 and Eq.10 with Eq.3 (also repeated). Eq.11 is coded into the software of choice to determine deflection and Eq.12 to determine slope.

$$\delta_Q = \frac{\partial U}{\partial Q} = \int_0^L \frac{M_{Q=0}}{EI(x)} \frac{\partial M}{\partial Q} dx = \frac{1}{E} \int_0^L \frac{(Eq.9)}{(Eq.10)} (Eq.7) dx \qquad \text{Eq.11}$$

$$\theta_m = \frac{\partial U}{\partial m} = \int_0^L \frac{M_{m=0}}{EI(x)} \frac{\partial M}{\partial m} dx = \frac{1}{E} \int_0^L \frac{(Eq.9)}{(Eq.10)} (Eq.8) dx \qquad \text{Eq.12}$$

There are several checks which might be performed as the solution proceeds. First, the geometry is easily checked by creating a plot such as Figure 5. Similarly, a check of the moment diagram would support confidence in the solution if done prior to attempting the repeated integrations. At a minimum, calculating deflection at one point (such as at supports) where the value is known would be an excellent task before moving on to multiple repeated integrations.

The critical action is to integrate with respect to x but to create a series of solutions (such as in a parametric table) using ξ as the indexing variable. For this shaft, selecting 91 positions of ξ will give reasonable smoothness for the tapered section geometry as well as smooth deflection and slope curves.
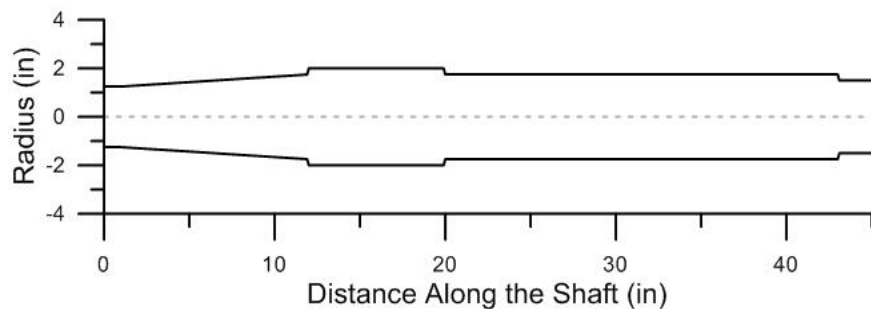

Figure 5. Equation-solver graph used to ensure the geometry function worked as intended.

Moment Comparison

The solution development for distributed loads is provided in Appendix A. Here we compare the results from the concentrated load and the distributed load approach. Figure 6 shows the differences for the moment along the shaft length. As expected the moment curve compares exceptionally well with [1].
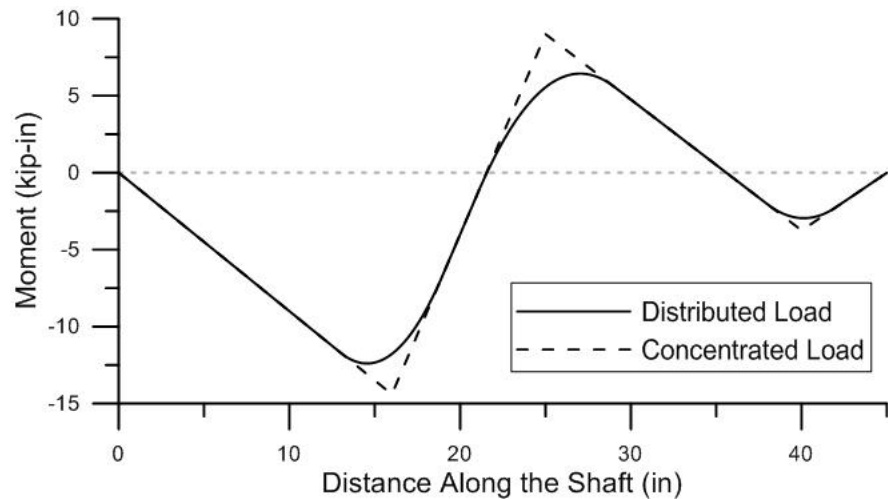

Figure 6.   Comparison of the moment along the length of the shaft.

Deflection Comparison

The deflection curve shows how much difference it makes to treat the distributed loads precisely. As it turns out, the deflection is less (better clearance) at the critical points of interest (ends of bearings) than predicted using concentrated loads. The distributed load shows less deflection resulting at the midway external load than that predicted by concentrated loads, but minimally different. So, at least in the case of this shaft, the simplification of concentrated loading for calculations of deflection is reasonable.
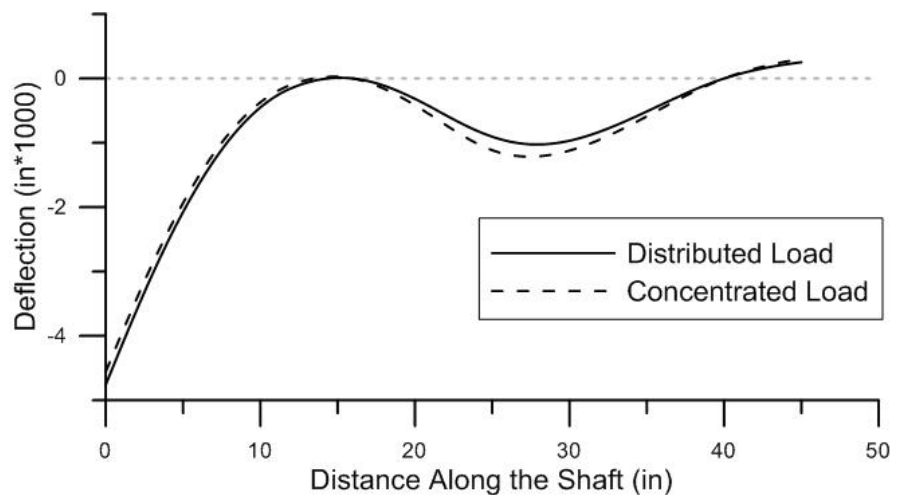

Figure 7. Comparison of deflection for concentrated load and distributed load.

Slope Comparison

Figure 8 shows the excellent comparison of slope as determined using concentrated loads for all loads versus using the more precise distributed load where it applies. Clearly the assumption that concentrated loads are sufficient is exemplified in the graph, since both curves are very close together and the y-axis units are thousands of radians. Historically, slope was not determined *per se*; rather it was inferred by visual inspection of the deflection curve characteristics.
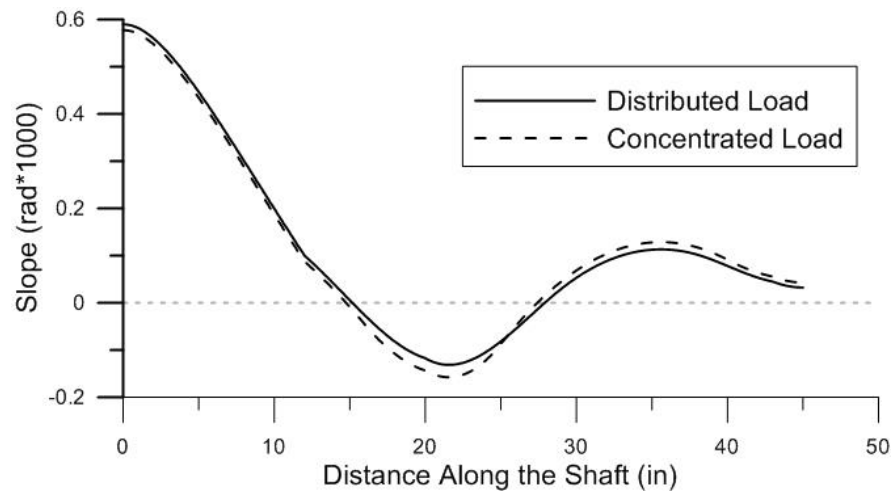

Figure 8. Comparison of slope for distributed load and concentrated load.

Discussion

In today's undergraduate Machine Design textbooks, we see few general approaches to the solution of deflection for stepped or tapered shafts; one approach is graphical and other approaches use some form of discontinuity equations [9-13]. These approaches work well for a simply supported stepped shaft with a single load.

By any measure, the Schroeder shaft is complicated. It is also a *real* shaft whose deflection and slope are of primary interest to the engineer. The method presented here offers a roadmap to the determination of deflection and slope whether or not one elects to assume distributed loads as fungible with concentrated loads. The method presented relies on basic engineering skills such as solving statics, writing moment equations and determining partial derivatives. Senior undergraduate students should have no difficulty with this level of problem-solving. Because individuals select an equation-solving tool of personal choice, difficulties with coding and syntax are mitigated. The method presented here allows for visual inspections along the way using knowledge of paper-and-pencil moment diagrams. Depending on the software selected, less than one page of code need be created, even for a complicated problem such as this one where the equations get lengthy. The method can be extended to any degree of indeterminacy using Lagrange multipliers. The method can also be applied to any geometry; curved beam or variable cross-section beam deflections benefit from this same simple, structured problem-solving approach. The authors and their students have benchmarked the method against a dozen published solutions [14-21] as well as closed form solutions and found the method is accurate. Few numerical difficulties have been encountered during the several years of our use; the method and solutions are robust.

Assessment of the method over several years in multiple institutions has shown that virtually every student can determine deflection "everywhere" along a beam regardless of the complexity of loading or changing cross-section.

Concluding Remarks

Each generation of engineers has used that part of mechanics of materials theory that fit the calculating capability available to them at that particular time. As long as the loading is not complicated and the shaft has a constant diameter, determining the deflections of a shaft is straight forward and well covered in texts.

We have presented by way of example, an analysis of distributed versus concentrated load modeling for supports and applied loads. We found the traditional simplifying assumption to use concentrated loading is a good one.

When the shaft cross section becomes practical it includes changes of diameter to provide steps that can be used to accurately mount bearings and gears. It can have overhanging ends and tapered cross sections. The need for finding the deflection and slope of these types of shaft geometries and loadings is timeless. We have presented a solution method which stays generalized, using an engineer's knowledge of free body diagrams, writing moment equations, and Castigliano's theorem to set up the problem solution into a form that is solved in an engineer's favorite computer program.

Acknowledgement

Bibliography

[1] Schroeder, Walter, "Beam Deflections," *Machine Design*, p. 85-90, January 1947.

[2] Bert, Charles W., "Deflections in Stepped Shafts," *Machine Design*, p. 128-133, November 24, 1960.

[3] Hall, Allen S., A. R. Holowenko and H.G. Laughlin, *Schaum's Outline Series Theory and Problems of Machine Design*, McGraw-Hill, Inc., New York, New York, 1961.

[4] Stephen, N.G., "Macaulay's Method for a Timoshenko Beam," *International Journal of Mechanical Engineering Education*, Vol. 35, No.4, 2007.

[5] Wittrick, W.H., "A Generalization of Macaulay's Method with Applications in Structural Mechanics," *AIAA Journal*, Vol. 3, No.2, February, 1965.

[6] Cueva-Zepeda, Alfredo, "Deflection of Stepped Shafts Using Macaulay Functions," *Computer Applications in Engineering Education*, Vol. 4(2), p. 109-115, 1996.

[7] Ju, F.D., "On the constraints for Castigliano's theorem," *Journal of the Franklin Institute*, Volume 292, Issue 4, October 1971, Pages 257-264.

[8] Macaulay, W.H., "note on the Deflection of Beams," *Messenger of Mathematics*, Vol.48, pp. 129-130, 1919.

[9]   Collins, Jack A., Henry Busby and George Staab, *Mechanical Design of Machine Elements and Machines*, 2nd Edition, John Wiley & Sons, Hoboken, NJ, 2010.

[10]   Spotts, Merhyle F., Terry E. Shoup and Lee E. Hornberger, *Design of Machine Elements*, 8th Edition, Pearson Prentice Hall, Upper Saddle River, NJ, 2003.

[11]   Mott, Robert L., *Machine Elements in Mechanical Design*, 4th Edition, Pearson Prentice Hall, Upper Saddle River, NJ, 2004.

[12]   Juvinall, Robert C. and Kurt M. Marshek, *Fundamentals of Machine Component Design*, 4th Edition, John Wiley and Sons, Inc., Hoboken, NJ, 2006.

[13]   Budynas, Richard G. and J. Keith Nisbett, *Shigley's Mechanical Engineering Design*, 8th Edition, McGraw-Hill, New York, NY, 2008.

[14]   Hopkins, R. Bruce, "Calculating deflections in Stepped Shafts and Nonuniform Beams," *Machine Design*, p. 159-164, July 6, 1961.

[15]   Hopkins, Bruce R., *Design Analysis of Shafts and Beams*, McGraw-Hill Book Company, New York, NY, 1970.

[16]   Cowie, Alexander, "A tabular method for Calculating Deflections of Stepped and Tapered Shafts," *Machine Design*, p. 111-118, August 9, 1956.

[17]   Umasanker, G. and C.R. Mischke, "A simple Numerical Method for Determining the Sensitivity of Bending Deflections of Stepped Shafts to Dimensional Changes," *Journal of Vibration, Acoustics, Stress, and Reliability in Design*, Vol. 107, p 141-146, 1985.

[18]   Shigley, Joseph E. and Charles R. Mischke, *Classic Mechanical Engineering Design*, 5th Edition, McGraw-Hill, New York, NY, 1989.

[19]   Mischke, C. R., "An Exact Numerical Method for Determining the Bending Deflection and slope of Stepped Shafts," in *Advances in reliability and stress analysis*; presented at the ASME winter annual meeting, San Francisco, California, December 1978.

[20]   Nobel, William, "The Critical or Whirling Speed of Shafts," Association of Engineering and Shipbuilding Draughtsmen, Onslow Hall, Little Green, Richmond, Surry, 1957.

[21]   Church, Irving P., *Mechanics of Internal Work*, John Wiley & Sons, New York, 1910. Davit of circular and variable cross-section p. 120.

Appendix A

Compare concentrated-load versus distributed load representations

Once the deflection and slope calculations have been set up and completed using the simpler concentrated load in lieu of the distributed loads on the shaft, it is rather straight-forward to solve the same problem without making the simplifying assumption. The distributed load terms are easily developed for use in the moment equation and the solution structure is already in place.

The only change is in the moment equation (Eq.9) where three terms will need to be replaced. Many sophomore level mechanics of materials texts offer excellent content on discontinuity functions [13] and a quick reference table is certainly useful [13]. For the Schroeder shaft, using the method proposed herein, the distributed load terms take the form of $\frac{w_0}{2} \langle x - a_1 \rangle^2$ where w_o is the magnitude per unit length of the load, x is any location along the beam and a_1 is the leftmost point at which the distributed load is applied. Unless the distributed load extends to the right end, a companion term is required to "turn off" the distributed load at an appropriate location, a_2. Table I summarizes the moment equation terms needed to represent the distributed loads and Eq. 13 shows the resulting moment equation. The pointed Macaulay brackets are replaced with regular parentheses and each term is augmented with a Heaviside function to serve as the "switch" to activate the term depending on the location being calculated.

Table I. Representing Distributed Loads

Load	Force (lb)	Length (in)	Start (in)	Stop (in)	Terms representing distributed load for moment equation
Left Bearing	3500	6	13	19	$+\frac{3500}{(6)(2)}(x-13)^2 H(x,13) - \frac{3500}{(6)(2)}(x-19)^2 H(x,19)$
Mid	3450	8	21	29	$-\frac{3450}{(8)(2)}(x-21)^2 H(x,21) + \frac{3450}{(8)(2)}(x-29)^2 H(x,29)$
Right Bearing	1600	4	38	42	$+\frac{1600}{(4)(2)}(x-38)^2 H(x,38) - \frac{1600}{(4)(2)}(x-42)^2 H(x,42)$

$$
\begin{aligned}
M(x,\xi)_{Q,m=0} = -900x &+ \frac{3500}{(6)(2)}(x-13)^2 H(x,13) - \frac{3500}{(6)(2)}(x-19)^2 H(x,19) \\
&- \frac{3450}{(8)(2)}(x-21)^2 H(x,21) + \frac{3450}{(8)(2)}(x-29)^2 H(x,29) \\
&+ \frac{1600}{(4)(2)}(x-38)^2 H(x,38) - \frac{1600}{(4)(2)}(x-42)^2 H(x,42)
\end{aligned}
\qquad \text{Eq.13}
$$

Of note is the happy condition that both partial derivatives (i.e. with respect to Q and with respect to m) remain exactly as they were under the concentrated loading case. And since no other relationships are altered for the distributed load case, the equations to enter into the software are summarized in Eq.14 and Eq.15 where the only substitution is Eq. 13 for Eq. 9.

$$
\delta_Q = \frac{\partial U}{\partial Q} = \int_0^L \frac{M_{Q=0}}{EI(x)} \frac{\partial M}{\partial Q} dx = \frac{1}{E}\int_0^L \frac{(Eq.13)}{(Eq.10)}(Eq.7)dx \qquad \text{Eq.14}
$$

$$
\theta_m = \frac{\partial U}{\partial m} = \int_0^L \frac{M_{m=0}}{EI(x)} \frac{\partial M}{\partial m} dx = \frac{1}{E}\int_0^L \frac{(Eq.13)}{(Eq.10)}(Eq.8)dx \qquad \text{Eq.15}
$$