On Implementing General Modal Analysis within the Mathcad® Software Package

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I. Abstract

A general Mathcad [1] model is presented to simulate the suspension dynamics of a small off-road vehicle designed for the SAE Mini-Baja collegiate competition. The model uses the method of Modal Analysis to solve the multiple degree-of-freedom dynamic system. Model variations addressing both front quarter car dynamics (with tire stiffness effects) and half car pitch/heave dynamics are discussed, each subject to arbitrary forcing. The model presented, generated as part of a student project within the senior Machine Vibrations class at Oklahoma Christian University, allows the students to integrate several analytical techniques into a single computational design tool. With the simulation process automated within Mathcad, the student designer is free to concentrate on parametric studies and optimization of the suspension response. Moreover, the presented model allows the designer to consider realistic (i.e. arbitrary) off-road ground profiles. The educational strengths of this integrated Mathcad model are discussed.

II. Background

Most practical dynamic systems incorporate multiple degrees-of-freedom and are subject to complex, if not random, excitation. Hand analysis of such a system in an undergraduate setting is a daunting task. In an effort to reduce the complexity of the analyses, greatly simplified academic models often replace realistic systems. Such academic models, though simpler, are less interesting and fail to expose students to the challenge of more complicated systems.

An approach for analyzing multiple degree-of-freedom dynamic systems, subject to arbitrary excitation, is herein proposed. The proposed model implements the method of Linear Modal Analysis[2] within the Mathcad software suite. Modal analysis, as outlined in the next section, requires many calculations to determine the response of the system. As such, it is prohibitive for student to consider design variations via hand calculations using this approach. The ideal balance is for the student, once he or she understands the methodology and required calculation steps, to implement the model within a computational environment that facilitates "what if" experimentation. Mathcad is ideally suited for this task as it allows the analyst to focus on the implementation, not the mathematical manipulations.

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III. Modal Analysis Overview

Modal analysis begins by defining the mass matrix (**M**) and the stiffness matrix (**K**) of the dynamic system. The size of the system may in general be N-by-N (N degrees-of-freedom).

$$\mathbf{M}\,\mathbf{x}'' + \mathbf{K}\,\mathbf{x} = \mathbf{F(t)}$$

By substituting $\mathbf{x(t)} = \mathbf{M}^{-1/2}\,\mathbf{q}(t)$ and pre-multiplying each side by $\mathbf{M}^{-1/2}$, the system is converted to the equation shown below, where Υ is the mass normalized stiffness matrix.

$$\mathbf{I}\,\mathbf{q}'' + \Upsilon\,\mathbf{q} = \mathbf{M}^{-1/2}\,\mathbf{F(t)} \qquad\qquad \Upsilon = \mathbf{M}^{-1/2}\,\mathbf{K}\,\mathbf{M}^{-1/2}$$

The eigenvalue problem associated with the mass normalized stiffness matrix is next solved to determine the eigenfrequencies and eigenvectors for the system. A new matrix (**P**), called the modal matrix, is constructed using the orthonormal eigenvectors associated with the mass normalized stiffness matrix.

$$(\Upsilon - \lambda\mathbf{I})\mathbf{v} = 0 \qquad\qquad \mathbf{P} = [\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{v}_3\ \ \ldots.\ \ \mathbf{v}_4]$$

The governing equations and the initial conditions are next decoupled by converting to the "modal coordinates", denoted **r**(t). This is accomplished by substituting $\mathbf{q(t)} = \mathbf{P}\,\mathbf{r(t)}$ and pre-multiplying by $\mathbf{P}^{\mathbf{T}}$. In the equation below, Λ is symmetric.

$$\mathbf{I}\,\mathbf{r}'' + \Lambda\,\mathbf{r} = \mathbf{P}^{\mathbf{T}}\,\mathbf{M}^{-1/2}\,\mathbf{F(t)}$$

Energy dissipation may be present in the original system of governing equations (usually denoted by a damping matrix, **C**). The modal matrix does not in general decouple the damping matrix. In the development herein, modal damping will be implemented. The alternative is to utilize proportional damping. The reader is referred to any quality vibration textbook to learn of the advantages and disadvantages of each option [2].

The analysis above is readily performed using the matrix capabilities standard within Mathcad. To solve the decoupled system, however, the analyst must proceed according to the nature of the forcing **F(t)**. For generality, arbitrary forcing is considered herein. Once the decoupled modal equations are solved, the modal solution **r(t)** is transformed back to the original variable **x(t)** by once again using the modal matrix.

$$\mathbf{x(t)} = \mathbf{M}^{-1/2}\,\mathbf{P}\,\mathbf{r(t)}$$

IV. Example Implementations based on the SAE Mini-Baja West Event

A realistic dynamic system of interest to many students is the off-road vehicle designed and built each year for the SAE Mini-Baja West competition [3]. This vehicle, powered by a 10 horsepower Briggs & Stratton engine, must be capable of withstanding severe abuse to successfully compete. Extremely rough terrain is the norm for the competition. Figure 1 presents a photo typical of the annual competition. Two student developed suspension models, suitable for conceptual design of the Mini-Baja, are now presented.

Figure 1: Oklahoma Christian University SAE Mini-Baja West Entry

The first model considers only the front quarter of the vehicle. This model focuses on simulating the dynamics of the front suspension with the effects of tire stiffness and damping included. Figure 2 presents both the actual system and the idealized mathematical model simulated by the quarter car suspension.

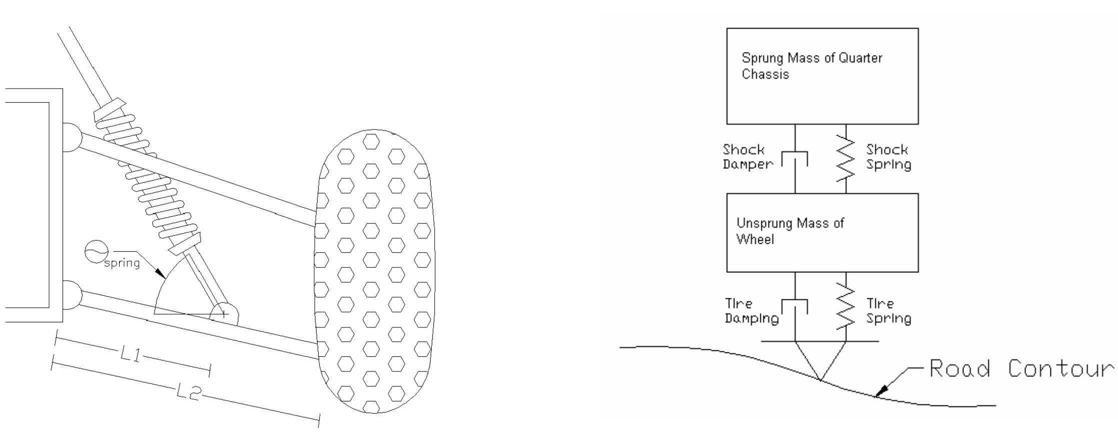

Figure 2: Front Quarter Car Suspension Model

The governing equations of motion are linearized about the suspension equilibrium position to allow application of the linear modal analysis methodology presented in Section III. This step limits the validity of the model predictions to small motions about the equilibrium state. As the response amplitude increases, nonlinearities associated with changes in the shock absorber angle will degrade accuracy.

The impulse convolution method has been selected for solving the system of decoupled equations resulting from the quarter car suspension model. The impulse convolution method mathematically replaces the continuous forcing with an infinite sequence of impulses. The response associated with each individual impulse is determined and “summed” using the convolution integral. The strength of the impulse convolution technique rests upon the ability to consider arbitrary road profiles. This is a key requirement as the vehicle will routinely strike oddly shaped structures, such as logs and/or rocks. Figures 5 through 9 at the end of this paper present the entire Mathcad model for the front quarter car suspension.

A second model of interest addresses the heave/pitch dynamics of the vehicle as it travels along an arbitrarily periodic road surface. This model considers the car as viewed from the side (shown in Figure 3) with both the front and rear suspension elements considered. The effects of tire stiffness and damping may be excluded if a simpler model is desired.

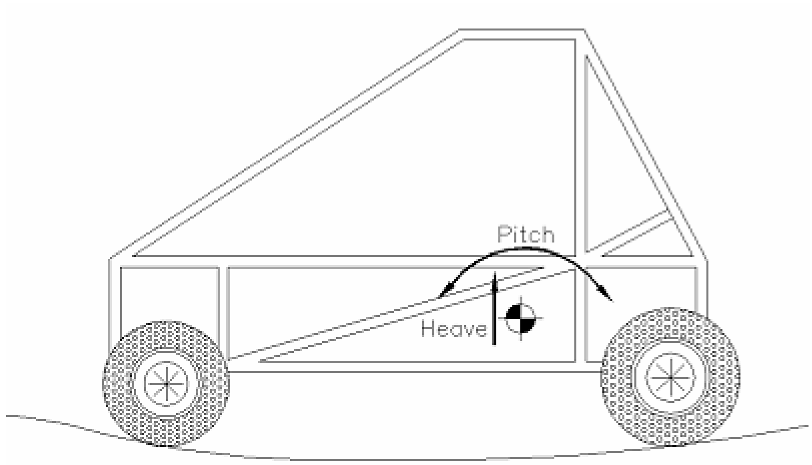

Figure 3: Side Heave and Pitch Vehicle Suspension Model

To enhance student learning, a Fourier solution technique was assigned for this model (instead of the impulse convolution method). The Fourier technique is restricted to periodic excitation; though the nature of the periodicity may be arbitrary. This technique consists of decomposing the excitation into a truncated Fourier series that accurately recreates the actual forcing. Figure 4 demonstrates the Mathcad implementation of the Fourier decomposition.

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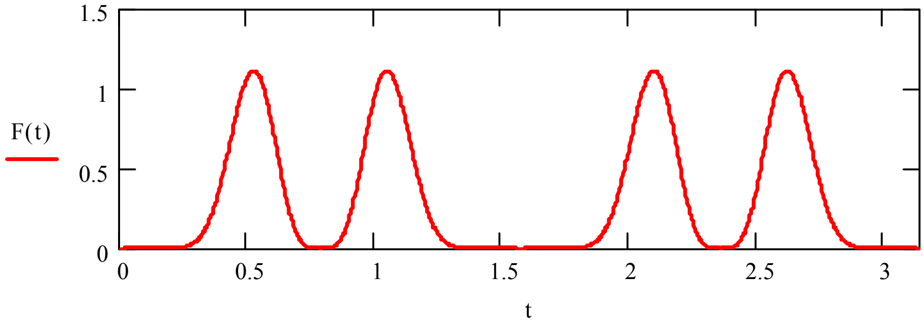
DEFINE THE FORCING (ARBITRARY, BUT PERIODIC)

$T := \pi \qquad \omega := 2\cdot\frac{\pi}{T} \qquad F(t) := 100\left(\sin(\omega\cdot t)^3\cdot\cos(\omega\cdot t)\right)^4$ **<== Nonlinear Forcing**



DETERMINE THE TRUNCATED FOURIER SERIES COEFFICIENTS

$nmax := 20 \qquad n := 0..\,nmax$

$$a0 := \frac{2}{T}\cdot\int_0^T F(t)\,dt \qquad a_n := \frac{2}{T}\cdot\int_0^T F(t)\cdot\cos(n\cdot\omega\cdot t)\,dt \qquad b_n := \frac{2}{T}\cdot\int_0^T F(t)\cdot\sin(n\cdot\omega\cdot t)\,dt$$

CHECK THE FOURIER REPRESENTATION OF THE EXCITATION

$$G(t) := \frac{a_0}{2} + \sum_{n=1}^{nmax}\left(a_n\cdot\cos(n\cdot\omega\cdot t) + b_n\cdot\sin(n\cdot\omega\cdot t)\right)$$ **<== Linear combination of simple harmonics**

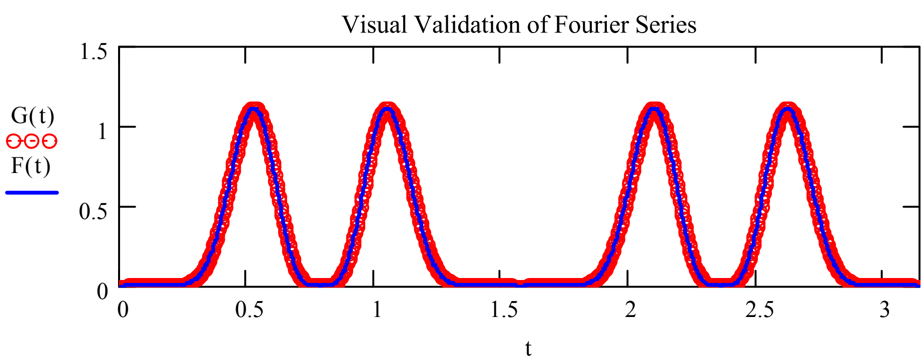


Figure 4: Mathcad Implementation Example of Fourier Decomposition

Once the decomposition of the forcing is complete, the excitation becomes the linear combination of harmonic terms. Harmonic solution techniques, routinely addressed in Differential Equations at the sophomore level, may be used to determine the solution by exercising superposition. Implementation of this solution process is left to the reader.

V. Educational Strengths and Student Testimonials

It is common to teach the method of Linear Modal Analysis within the scope of a typical undergraduate Machine Vibrations class. Too often, however, Modal Analysis is viewed by the students as a distinct method, separate and apart from the more fundamental techniques of Impulse Convolution and Fourier Decomposition. In fact, these three methods build upon each other. The merger of these techniques using Mathcad demonstrates to the students that simple theoretical models may be extended and integrated to solve complex design scenarios. Student feedback related to the project indicates the value of this educational practice. Comments from two engineering students regarding the models discussed herein follow.

"Developing the MathCAD model for the Mini-Baja suspension was one of the most rewarding projects that I participated in during my undergraduate education. It allowed me to see how a complex theoretical process like modal analysis could be used in a practical application to solve real world problems. Using a software tool like Mathcad allowed us to easily vary the parameters of the simulation to study the outcome. This allowed us to study a wide variety of possibilities and then allowed us to make the best design decision. Without the computational power of Mathcad, this wouldn't have been practical."
(Caleb Chitwood, Oklahoma Christian University, Mechanical Engineering Student)

"The Mathcad model allowed me to quickly create several forcing scenarios and design constraints with a minimal amount of hand calculations. After developing a design model of this type, I clearly see the benefit of using computational resources. The computational model increases the number of analytical designs that I can explore and enabled me to bridge the gap between the prototype design and the analytical design. Thus the Mathcad model serves as the culmination of my research and understanding of the mathematics and theories used for modal analysis."
(Ira Lockwood, Oklahoma Christian University, Mechanical Engineering Student)

VI. Summary

Mathcad is proposed as an effective tool to enhance student understanding of dynamic systems by including more complicated and realistic loading conditions within multiple degree-of-freedom simulations. The models presented utilize the method of Linear Modal Analysis combined with either Impulse Convolution or Fourier Decomposition techniques to simulate the response of an off-road vehicle subject to realistic road profiles. Student testimonials are presented that indicate enhanced learning as a result of both increased student interest and increased student confidence associated with the user-friendly Mathcad environment. Details of the Mathcad implementation are presented.

VII. Bibliography

1. Mathcad Professional Edition 2001, Mathsoft, 2001.

2. Inman, Daniel J., Engineering Vibrations, 2nd Edition, Prentice-Hall, 2001.

3. SAE Mini-Baja West Collegiate Competition, www.sae.org.

VIII. Biographical Information

BYRON L. NEWBERRY
Dr. Byron L. Newberry is an Assistant Professor of Mechanical Engineering at Oklahoma Christian University. He holds B.S., M.S., and Ph.D. degrees in Mechanical Engineering (advanced degrees from the University of Michigan). His areas of interest include structural analysis, thermal stress, linear and nonlinear oscillations, and engineering design.

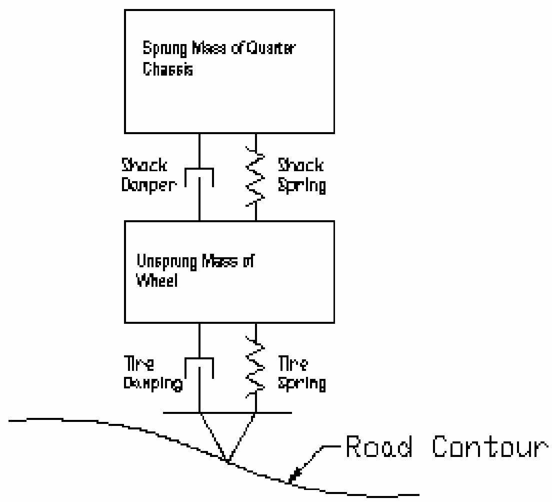
JOHN CALEB CHITWOOD
Caleb Chitwood will graduate from Oklahoma Christian University in April of 2003 with a B.S. in Mechanical Engineering and a minor in Business Management. Following graduation, he will begin working for the Battelle Memorial Institute in the Equipment Development product line. His interests include dynamic systems, systems design, and numerical analysis.

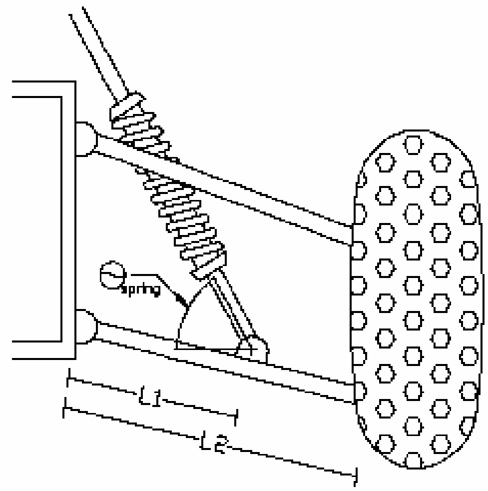
IRA LOCKWOOD
Ira Lockwood is a senior at Oklahoma Christian University majoring in Mechanical Engineering and Bible & Ministry. He is interested in designing products for the automotive or aerospace industry, especially in areas of vibration analysis or aerodynamic design.

MODAL ANALYSIS VIA MATHCAD

(C. Chitwood and I. Lockwood - Oklahoma Christian University)

This Mathcad sheet implements Modal Analysis as outlined in Engineering Vibration, 2nd Edition, by Inman. The specific system being solved is a quarter car model for an SAE Mini-Baja racer. The model simulates the dynamics of the front quarter section of the car with the dynamics of the tire being considered.





Suspension System with Parameters Labeled *Diagram of Mass-Spring-Damper System*

System Parameters

Shock Spring Coefficient $k_{shock} := 200 \cdot \frac{lbf}{in}$

Tire Spring Coefficient $k_{tire} := 75 \cdot \frac{lbf}{in}$

Unsprung Mass $m_u := .30 \cdot slug$

Sprung Mass $m_s := 5.4 \cdot slug$

Length to Lower A-arm Mount $l_1 := 8 \cdot in$

Length of Lower A-arm $l_2 := 14 \cdot in$

Mounting Angle of Shock $\theta_{spring} := 56 \cdot \frac{\pi}{180}$

Wheel Radius $wheel\_radius := 0.5 \cdot ft$

Figure 5: Mathcad Implementation of Modal Analysis with Impulse Convolution (1 of 5)

Assumed damping coefficients

Shock damping $c_s := 700 \cdot \dfrac{kg}{s}$

$\zeta := \begin{pmatrix} .5 \\ .15 \end{pmatrix}$

Tire Damping $c_t := 110 \cdot \dfrac{kg}{s}$

Description of the Road Profile and Vehicle Speed

Speed of Car $\text{speed} := 10 \cdot \dfrac{mi}{hr}$

Period of Peaks (distance) $\text{Period} := 10 \cdot ft$

Half-Peak Amplitude $\text{Ampl} := .3 \cdot ft$ $\omega_b := \dfrac{\text{speed}}{\text{Period}} \cdot 2 \cdot \pi$

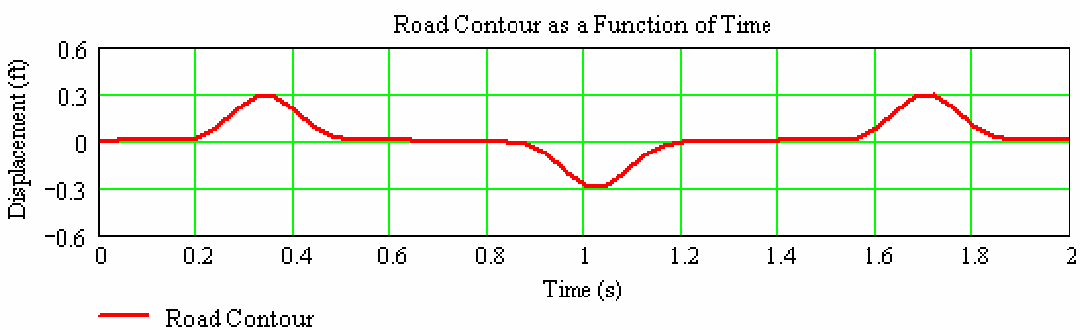
Road Profile Function (any arbitrary function is allowed)

$$y(t) := \text{Ampl} \cdot \sin\left(\frac{2 \cdot \pi}{\text{Period}} \cdot \text{speed} \cdot \frac{t}{2}\right)^{11} \cdot \Phi\left(\frac{t}{s}\right) - \text{Ampl} \cdot \sin\left(\frac{2 \cdot \pi}{\text{Period}} \cdot \text{speed} \cdot \frac{t}{2}\right)^{11} \cdot \Phi\left(\frac{t}{s} - 4\right)$$

Visualization of Excitation:

-decrease the interval to decrease the computation time at the cost of accuracy

$t := 0s, .04s .. 2s$ $X(t) := \text{speed} \cdot t$ $\text{Displacement}(X) := y\left(\dfrac{X}{\text{speed}}\right)$



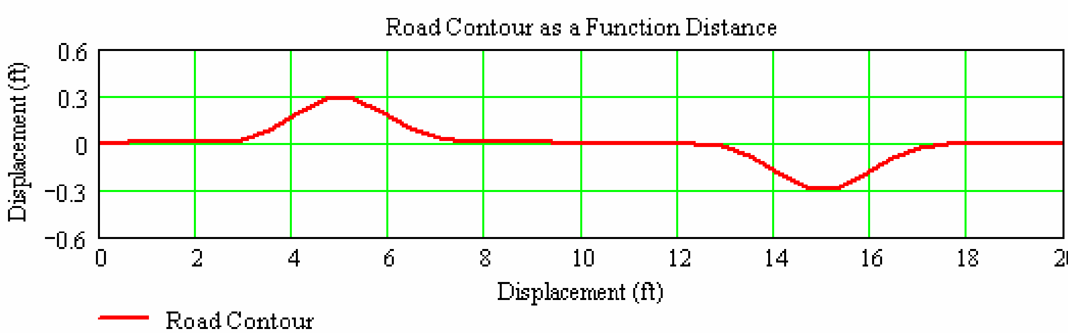


Figure 6: Mathcad Implementation of Modal Analysis with Impulse Convolution (2 of 5)

Calculate Modal Analysis Parameters:

Applied Force in Vertical Direction $k_s := \frac{l_1}{l_2} \cdot k_{shock} \cdot \cos(\theta_{spring})$

Mass Matrix $mass := \begin{pmatrix} m_s & 0 \\ 0 & m_u \end{pmatrix}$

Stiffness Matrix $k_m := \begin{pmatrix} k_s & -k_s \\ -k_s & k_s + k_{tire} \end{pmatrix}$

Force Vector $F(t) := \begin{bmatrix} 0 \\ c_t \cdot \left(\frac{d}{dt} y(t)\right) + k_{tire} \cdot y(t) \end{bmatrix}$

Inverted Mass Root Matrix $mass_i := \begin{pmatrix} \frac{1}{\sqrt{mass_{0,0}}} & 0 \\ 0 & \frac{1}{\sqrt{mass_{1,1}}} \end{pmatrix}$

Mass Normalized Stiffness Matrix $k_f := mass_i \cdot k_m \cdot mass_i$

Solve for Eigenvalues

$\lambda_i := \text{eigenvals}(k_f)$ $\omega_i := \sqrt{\lambda_i}$ $u_i := \text{eigenvecs}(k_f)$ $p_i := u_i$

Calculate Mode Shapes

$shape := mass_i \cdot p_i$ $shape = \begin{pmatrix} 0.112 & -0.012 \\ 0.052 & 0.475 \end{pmatrix} kg^{-0.5}$

Transform forcing into modal coordinates

$F_r(t) := p_i^T \cdot mass_i \cdot F(t)$

Figure 7: Mathcad Implementation of Modal Analysis with Impulse Convolution (3 of 5)

Calculate damping frequencies and solve for modal response

$\omega_{d1} := \omega_{i_0} \cdot \sqrt{1 - \zeta_0}$ $\omega_{d2} := \omega_{i_1} \cdot \sqrt{1 - \zeta_1}$

Solution in Modal Coordinates using Impulse Convolution:

$$r_1(t) := \frac{1}{\omega_{d1}} \cdot \int_0^t F_r(t - \tau)_0 \cdot \exp\left(-\zeta_0 \cdot \omega_{i_0} \cdot \tau\right) \cdot \sin\left(\omega_{d1} \cdot \tau\right) d\tau$$

$$r_2(t) := \frac{1}{\omega_{d2}} \cdot \int_0^t F_r(t - \tau)_1 \cdot \exp\left(-\zeta_1 \cdot \omega_{i_1} \cdot \tau\right) \cdot \sin\left(\omega_{d2} \cdot \tau\right) d\tau$$

Calculate Response in Linear Coordinates

Transformation of Modal Response to Actual Response $x(t) := shape \cdot \begin{pmatrix} r_1(t) \\ r_2(t) \end{pmatrix}$

Position of Lower A-arm Mounting Point $a(t) := \frac{l_1}{l_2} \cdot x(t)_1$

Position of Chassis $b(t) := x(t)_0 + .75 \cdot ft$

Equilibrium Point $e(t) := .75 \cdot ft$

Maximum Shock Extension Point $u(t) := .75 \cdot ft + \frac{2}{12} \cdot ft$

Road Profile $y_p(t) := y(t)$

Wheel Center Motion $wheel(t) := x(t)_1 + wheel\_radius$

Shock Extension $shock(t) := b(t) - a(t)$

Information of Interest:

Speed of car: $speed = 10 \frac{mi}{hr}$

Spring Constant of Shock: $k_{shock} = 200 \frac{lbf}{in}$

Distance From One Road Peak to Next: $Period = 10\,ft$

Half-Peak Amplitude: $Ampl = 0.3\,ft$

Figure 8: Mathcad Implementation of Modal Analysis with Impulse Convolution (4 of 5)

Solution Plot:

The following plot, reconstructed physical coordinates from the modal coordinates, displays the motion of the chassis, the motion of the wheel, the input ground profile, the displacement (i.e. stroke) of the shock, and the maximum shock extension.

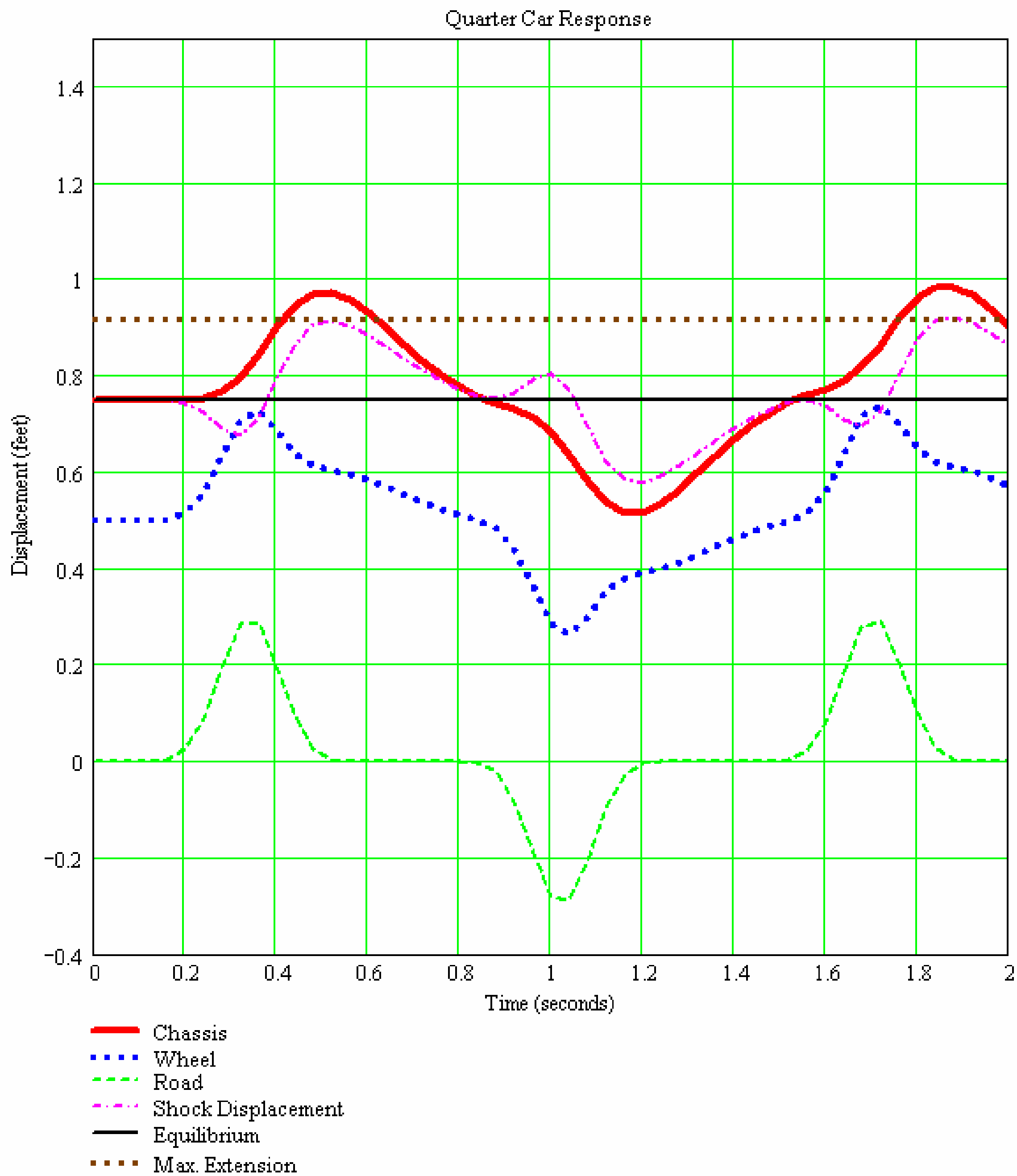


Figure 9: Mathcad Implementation of Modal Analysis with Impulse Convolution (5 of 5)