AC 2010-2340: ON STOCHASTIC FINITE ELEMENT ANALYSIS OF BAR STRUCTURES

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On Stochastic Finite Element Analysis of Bar structures

Abstract

The Finite Element Analysis of structures is one of the most powerful and well known methods to determine the displacements, member forces and member stresses or strains. The external loads and properties of members are generally assumed deterministic, meaning that the variation of loads are not random in time or the member properties are of constant values over time. In this paper, the analysis will be discussed on bars structures with stochastic loads and/or with random member properties. The random member properties are input with its mean values and its standard deviation (or variances) values. A simple perturbation technique is applied to develop the stochastic finite element analysis of bars and beams with applied external random loads and member properties. The resulting random displacement solution of the analysis is written also in terms of the displacement mean and variance values. Hence, member forces, stresses and strains can be computed from the computed random displacements. One bar example is used to apply the discussed stochastic analysis with a given random load or a given member random property.

Introduction

In dealing with real world problems, uncertainties are unavoidable[1]. It is well known that most engineering material properties, external applied loads on structures, and hence, the material point displacements, material element stresses or strains are not deterministic at all. In reality, these quantities are all random in nature, and some of them are more random than others. But it is easy to assume these quantities as deterministic to make our analyses simple and determination of these material property values as the mean statistical mean of the physical tests. The central variation of these quantities is ignored in the mechanical analysis of structures while computing material point displacements or material element stresses or strain. One usually wonders whether such a deterministic analysis is satisfactory in terms of reliable and safe. The engineers doing such deterministic analysis rely on the standards for safety in terms of the allowable stresses.

The sources of uncertainty may be classified into two broad types[1]: (1) Those that are associated with natural randomness, also known as 'aleatory' type of randomness, and (2) those that are associated with inaccuracies in prediction and estimation of reality, also known as 'epistemic' type of randomness. The effects of uncertainties on the design and planning of an engineering system are important, however quantification of such uncertainties and the evaluation of its effects on the system is very important and such quantification is done through the use of the concepts and methods of probability and statistics. Also, under the conditions of uncertainty, the design and planning of engineering systems involve risks, which in turn involve probability and associated consequences. The importance of the quantification of uncertainty on such calculated risks is documented by the National Research Council[2], US Department of Energy[3], NASA[4], and NIH[5].

In the age of computers, and high technology, and Uncertainty Risk Quantification initiatives by major US national agencies, it seems appropriate to use of random values of materials properties, and known random external applied loads in mechanical analyses. Three examples[1] of random variation of properties are shown as Figure 1, 2 and 3 for the bulk density of soil[9], Water-Cement ratio in Concrete[10], and Yield Strength of reinforcing bars[11], respectively. Hence, using the random properties, an assessment is made if the computed random stresses at critical structural points has the central tendency values below the allowable stresses, and the standard deviation from the central values is as minimal as can be possible. This is one motivation to introduce the random analysis of structures during the undergraduate study for an engineering degree.

The paper is written both as a tutorial of random variables study, and as a documentation to teach the stochastic bar structure analysis. The motivation discussion section identifies the reasons for extending the probability and statistical study with the random variables and stochastic analysis study during the undergraduate senior year courses, namely after the study of the statistics course and the statics course. Next, the random variables are introduced with its definition for its mean, its central tendency variation, the normal distribution, and combination of random variables examples on its outcome. An introduction to stochastic function is discussed as precursor to the bar stochastic analysis approach as the topic of this paper. Many technical details are omitted for being concise, but references are provides for further study and details of stochastic functional analysis of other engineering problems.

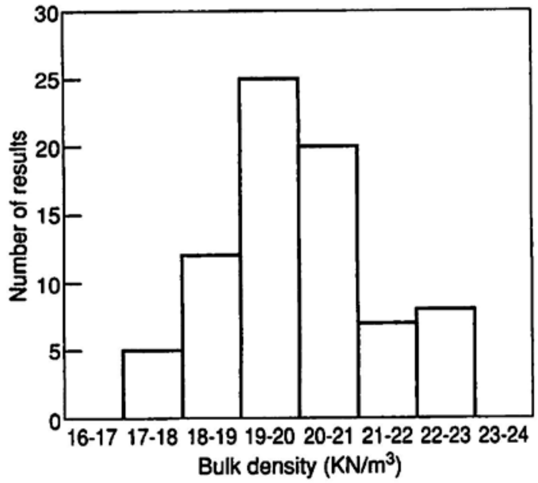

Figure 1: Bulk Density of Residual Soils[9] (after Winn et al. 2001)

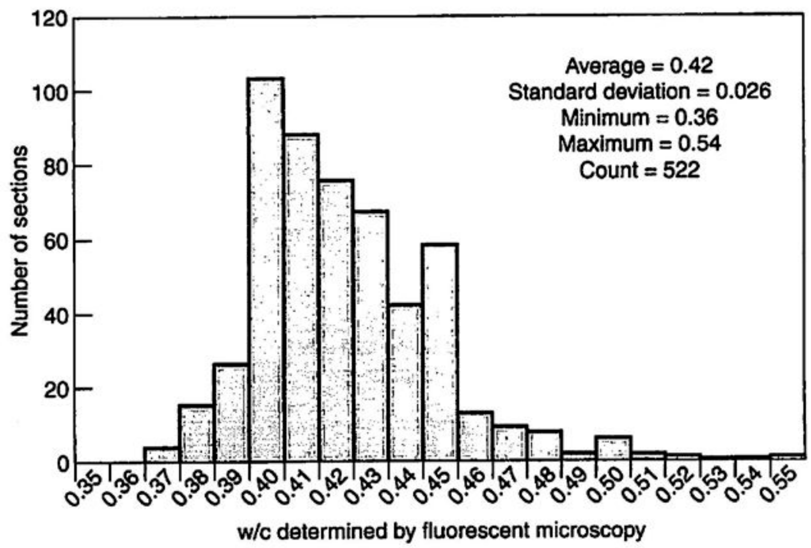

Figure 2: Water-Cement ration of Concrete[10] (after Thoft-Christensen, 2003)

Motivation to Teach Random and Stochastic Analysis

The undergraduate student does study the 'Probability & Statistics' as a course, but none of the undergraduate engineering subjects uses these random values or probability distribution concepts that he or she learns. Most (almost all) engineering subjects in the undergraduate study deal with deterministic variables, and safety factor is used in case of designs of mechanical members. However, in reality, most engineering material properties or applied loads are random in nature, as discussed in the introduction above, but they are used as deterministic to simplify our computational or design procedures. Of course, we do pay a price in terms of 'risks'[1-5] on its design, with respect to not knowing the central variation of the stresses or strains or displacements of material points due to the random variations of the engineering material properties used within the design. It is suggested in this paper that, in the age of high technology and six-sigma production procedures, one must teach the concept of random variables analysis of engineering structures, at an early stage of engineering undergraduate education, and an example of a simple bar random analysis is discussed in the paper to show how the ability to cope with the randomness of materials and nature, in terms of external loading, can be incorporated in mechanical analysis, with minimal extra efforts.

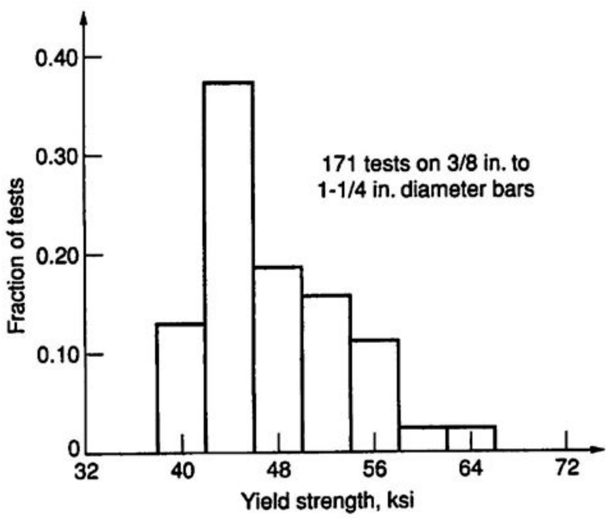

Figure 3: Yield Strength of Reinforcing Bars[11] (data from Julian, 1957)

The stretch of teaching random variables and its effect on simple bar static analysis is possible within the current undergraduate curriculum. Probability and Statistics course is taught now, and so the next logical sequence is to introduce random variables of member forces and displacements in their 'Statics' and 'Strength of Materials' courses, as add on advanced subject materials sections. Bar and Beam equilibrium equations and its formulations are taught in Statics, and the moment and shear force diagrams are taught in the strength of materials course. As an extension, in the statics course, both deterministic and random loadings should be introduced. These random load mathematical concepts of probability distributions are just an extension of the 'Probability and Statistics' course, and for ease of study, one could only discuss the normal probability distribution case in statics course. These random concepts should be extended in their strength of materials course in terms of using the load distribution, and solving for the random displacement or stresses. Of course, it would be hard initially for a new learner to appreciate the difficult random concepts, but when the student comes in touch with randomness in other aspects of life, the student would be prepared well to deal with the randomness issue in solving the engineering problem rather than brush it aside, due to ignorance, by not dealing with randomness in any quantity or with any random analysis.

For completeness, an attempt is made to discuss briefly the subject materials that would be appropriate as an extension to the Probability and Statistics taught in the undergraduate curriculum. Of course, the author tries to be coherent in the presentation. More details of these random topics in this paper can be found in many references[2,6-8].

This paper discusses first the basic definition of random variable, then defines the mean and variance of that random variable, then discusses the outcome of a sum of two random variables, leading to the discussion of the linear functions of Random variables, and finally, a bar analysis using randomness in the material properties is discussed. No specific plot or detail results are intentions of this paper, as a scientific study, but gives discussion of subject materials that the author suggests in this paper as an example for introducing the study of random concepts in engineering curriculum.

What is a RandomVariable?[2,6,7]

According to statistical definition, when the numerical value of a variable is determined by or assigned to a chance event, that variable is called a **random variable**. Continuous random variables can take on any value within a range of values. For engineering purposes, both discrete and continuous random variables are important, and discrete randomness is discussed in statistics course[6]. So the paper looks at continuous random variables.

The probability distribution of a continuous random variable is represented by an equation, called the **probability density function**[2,6,7] (pdf). All probability density functions satisfy the following conditions:

- The random variable Y is a function of X; that is, y = f(x).
- The value of y is greater than or equal to zero for all values of x.
- The total area under the curve of the function is equal to one.

Just like variables from a discrete data set, continuous random variables are also described by measures of central tendency (i.e., mean) and measures of variability (i.e., standard deviation and variance).

Mean and central Variation Measure of a continuous Random Variable

The mean of the discrete random variable X is also called the **expected value** of X. To write in notation, the expected value of X is denoted by E(X). Use the following formula to compute the mean of a discrete random variable[6]:

$$E(X) = \mu_x = \Sigma [x_i * P(x_i)]$$

where x_i is the value of the random variable for outcome i, μ_x is the mean of random variable X, and $P(x_i)$ is the probability that the random variable will be outcome i.

The central deviation tendency of a random variable from the mean is measured by the standard deviation of random variable. The standard deviation of a discrete random variable (σ) is equal to the square root of the variance of a discrete random variable (σ^2). The equation for computing the variance of a discrete random variable is shown below[6].

$$\sigma^2 = \Sigma [x_i - E(x)]^2 * P(x_i)$$

where x_i is the value of the random variable for outcome i, $P(x_i)$ is the probability that the random variable will be outcome i, E(x) is the expected value of the discrete random variable x. For continuous random variable this same central deviation from the mean uses the integral instead of the sum in the above expression of the standard deviation.

Combination of Random Variables[6,8]

Sometimes, it is necessary to add or subtract random variables. When this occurs, it is useful to know the mean and variance of the result.

Suppose you have two variables: X with a mean of μ_x and Y with a mean of μ_y. Then, the mean of the sum of these variables μ_{x+y} and the mean of the difference between these variables μ_{x-y} are given by the following equations.

$$\mu_{x+y} = \mu_x + \mu_y \quad \text{and} \quad \mu_{x-y} = \mu_x - \mu_y$$

The above equations for general variables also apply to random variables. If X and Y are random variables, then

$$E(X + Y) = E(X) + E(Y) \quad \text{and} \quad E(X - Y) = E(X) - E(Y)$$

where E(X) is the expected value (mean) of X, E(Y) is the expected value of Y, E(X + Y) is the expected value of X plus Y, and E(X - Y) is the expected value of X minus Y.

Linear functions of Random Variables[6,8]

One of the most commonly encountered transformations of the random variable X is the linear transformation given by

Y = a*X+b

where a and b are constants. The means and variances of X and Y random variables are related. Thus,

E[Y] = E[a*X+b] = a*E[X] + b or y^0 = a*x^0 + b where superscript on x and y refer to mean of random variables X and Y, respectively; and

Var[Y] = E[$(Y-y^0)^2$] = a^2*Var[X]

This can be extended to the linear transformation Y = $\sum_{i=1}^{n} a_i X_i$. Results for mean and variance are similar except the covariace between random variables play an active part in the covariance of Y. Please refer to Ref 6 or 8 for more details.

Normal Probability Density Functions[2,6]

As an example of many useful Probability Density Functions(PDF), the one commonly employed in scientific and engineering applications is the Gaussian (normally distributed) PDF. A random variable X is said to be Gaussian distributed if its PDF has the following exponential form:

$$p(x) = \frac{1}{\sigma_x \sqrt{2\pi}} \exp[-\frac{(x-x^0)^2}{2\sigma_x^2}]$$

This PDF has two parameters, namely, the mean x^0 and the standard deviation σ_x.

Bar Structure Stochastic Static Analysis[8]

Before doing any computer solutions, let us discuss a simple one element bar analysis without making any reference to any computer programs or results. Figure 4 shows a cantilever bar of length L, cross sectional area A, the material modulus of elasticity E, and the bar is subject random axial load Q. In addition, let us assume that one of the parameters of A, L E and Q is random at a time.

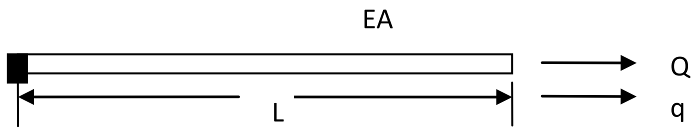

Figure 4: Cantilever Bar Subjected to an Axial Force[8]

By having the randomness in one of the material, geometry or load parameter, the finite element equilibrium equation, Ku = Q with k=AE/L, yields the following stochastic finite element equations describing the static response of the single random variable system[2]:

$$k^0 u^0 = Q^0$$

$$k^0 u^{'b} = Q^{'b} - k^{'b} u^0$$

$$k^0 u^{(2)} = (Q^{'bb} - 2k^{'b} u^{'b} - k^{'bb} u^0) \, \mathrm{Var}(b)$$

where $(.)^0$, $(.)^{'b}$,and $(.)^{'bb}$ denote the zeroth, first and second derivatives with respect to the random variables b; these functions are evaluated at the spatial expectations of b.

The second order accurate expectations and the first order accurate variance for the displacement can be expressed as

$$E[u] = u^0 + (0.5)u^{(2)}; \text{ and } \mathrm{Var}(u) = (u^{'b})^2 \, \mathrm{Var}(b)$$

Note that for bar or beam structures with many finite elements, the form of the above equations is similar in matrix notation, and one element is used here for simplicity to understand the random analysis methodology.

There are four possible cases of single random variable problem for the simple bar example. In this paper only, one case is detailed for presenting the idea, and results are stated for the other three cases.

<u>Case 1</u>: Let us assume the cross-sectional area A be a random variable, that is b in the above equations b identified as A, while E, L and Q are deterministic. Derivative of the stiffness variable k and the load variable Q are written as:

$$k^0 = EA^0/L; \; k^{'A} = E/L; \; k^{'AA} = 0$$

$Q^0 = Q; Q^{,A} = 0; Q^{,AA} = 0$

Consequently, the zeroth, first and second order SFEM equations become (see above and substitute appropriately):

$(EA^0/L)\ u^0 = Q$

$(EA^0/L)\ u^{'A} = (-E/L)\ u^0$

$(EA^0/L)\ u^{(2)} = (-2E/L)\ u^{'A}\ Var(A)$

These equations allow the zeroth, first and second order displacements as:
$u^0 = \frac{QL}{EA^0}$; $u^{'A} = -\frac{QL}{E(A^0)^2}$; $u^{(2)} = \frac{2QL}{E(A^0)^3} Var(A)$

Now, we can write the first two statistical moments of the displacement ‘u’ in the form:

Mean Value for displacement ‘u’ random variable: $E[u] = \frac{QL}{EA^0}[1 + \frac{Var\,(A)}{(A^0)^2}]$

Variance for displacement ‘u’ random variable: $Var(u) = \frac{Q^2L^2}{E^2(A^0)^4} Var(A)$

<u>Case 2</u>: In this case, the Modulus of Elasticity E is assumed to be random, that is b is identified as E, while A, L & Q are deterministic. Following steps as in the previous case, we obtain the two statistical properties of displacement in the form:

Mean Value for displacement ‘u’ random variable: $E[u] = \frac{QL}{AE^0}[1 + \frac{Var\,(E)}{(E^0)^2}]$

Variance for displacement ‘u’ random variable: $Var(u) = \frac{Q^2L^2}{A^2(E^0)^4} Var(E)$

<u>Case 3</u>: In this case, the length of bar L is assumed to be random, that is b is identified as L, while A, E & Q are deterministic. Following steps as in the previous case, we obtain the two statistical properties of displacement in the form:

Mean Value for displacement ‘u’ random variable: $E[u] = \frac{QL^0}{AE}$

Variance for displacement ‘u’ random variable: $Var(u) = \frac{Q^2}{(EA)^2} Var(L)$

Case 4: In this case, the applied load Q is assumed to be random, that is b is identified as Q, while A, E & L are deterministic. Following steps as in the previous case, we obtain the two statistical properties of displacement in the form:

Mean Value for displacement 'u' random variable: E[u] = $\frac{Q^0 L}{AE}$

Variance for displacement 'u' random variable: Var(u) = $\frac{L^2}{(EA)^2} Var(Q)$

Conclusion

The discussion can be extended to multiple-random-variable system using the matrix notations of stochastic finite element equations[8]. The random concept analysis identification and discussion towards teaching the random concept in the engineering curriculum is the essence of this paper, and it is not the intention to show any complicated technical analysis in this paper. The author believes strongly that the concept of random variables can be introduced after the probability & statistical course study, and the bar analysis (simple one bar will do the purpose for the curriculum) discussed in this paper with more details and numerical examples will serve enough as the extension of stress computations done for bar in the strength of analysis course. The author has been very careful in the bar random analysis to compute symbolically for the two statistical random displacement values of bar for 4 cases. Even so, please do each analysis identified to verify the random displacement results before using it for teaching or as such for any professional value of interest.

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