AC 2009-1466: ON THE ANALYSIS AND DESIGN OF VEHICLE SUSPENSION SYSTEMS GOING OVER SPEED BUMPS

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Overview

In this paper, we discuss a novel framework in the form of a classroom project in which clients in first courses on vibrations would learn in an effective manner about basic elements of vibrations such as transient load, field equations, simulation, and design. We also note that the problem we present in this paper cannot be solved by existing techniques such as proportional damping through modal analysis.

In effect, to teach transient vibrations to our students in vibrations class the authors gave them a project, in which the students analyzed and studied the ensuing motions of a vehicle passing over a speed bump. Students used a 2-degree of freedom model to simulate the motion of the vehicle going over the bump. They studied the effect of vehicle speed, speed bump’s geometry on the subsequent bounce and pitch motions of the vehicle. Once they understood the underlying physical concepts of transient vibrations, the students then modified the original system’s parameters to reduce the respective amplitudes of the bounce and the pitch motions of the vehicle at a given vehicle speed and a set bump geometry.

After deriving the equations of motion, we had students use MATLAB and SIMULINK in this project to overcome the mathematical difficulties inherent in the solution of the physical problem of transient response, to simulate its behavior, and to design the corresponding system.

Problem Statement

An automobile such as the one shown in the Figure 1, exhibits bounce, pitch, and roll on top of its rigid body motion as it goes over a speed bump. In this analysis, we assume that the rolling motion compared to the two other types of oscillatory motions is negligible. Neglecting the rolling motion and mass of tires, and combining the stiffness and damping effects of tire and suspension system into an equivalent damping and stiffness system, a preliminary model for automobile’s suspension system is presented in the Figure 2. Initial values for the respective inertias, damping coefficients, and spring rates are as follows:

\[ m = 2000 \text{ kg} \quad J = 2500 \text{ kg.m}^2 \quad k_1 = k_2 = 30000 \text{ N/m} \quad c_1 = c_2 = 3000 \text{ N.s/m} \]

\[ l_1 = 1 \text{ m} \quad \text{and} \quad l_2 = 1.5 \text{ m} \]
Where \( m \) is the auto’s body mass, \( J \) is its moment of inertia about the center of mass, index 1 refers to the front suspension system whereas index 2 refers to rear suspension system, and \( l_1 \) and \( l_2 \) are the distances between the center of mass and front and rear suspensions respectively.

The car is assumed to be traveling at 20 km/hr and the speed bump is approximated as sinusoidal in cross section with amplitude of 50 mm and the length \( \lambda = 0.5 \) m.

**Figure 1**

Formulation

The governing system of differential equations which describe the bounce and pitch motions of the system shown in Figure 2 is found using Lagrange’s Equations. The generalized coordinate \( x(t) \) and \( \theta(t) \) are used to describe the bounce and pitch motion of the auto body. In formulating the problem one realizes two different time spans. A time
period \( t \), in which the automobile’s front tire is in contact with the bump. This time period is:

\[
0 < t \leq t_1 = \lambda / V ,
\]

where \( V \), is the speed of the automobile. The other time span is when the rear tire is in contact with the speed bump. This time period is:

\[
t_2 < t \leq t_2 + \frac{\lambda}{V} ,
\]

where \( t_2 \) is the time it takes for the rear tire to reach the trough of the bump. That is:

\[
t_2 = \frac{l_1 + l_2}{V} .
\]

Formulating the problem for the time period, in which the front tire is in contact with the bump, one writes the kinetic energy of the automobile as:

\[
T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J \dot{\theta}^2 .
\] (1)

The potential energy is described in Equation 2 as:

\[
U = \frac{1}{2} k_1 \left( y + x - l_1 \theta \right)^2 + \frac{1}{2} k_2 \left( x + l_2 \theta \right)^2 .
\] (2)

Rayleigh’s dissipation function describing viscous dissipation in the dampers is:

\[
Q = \frac{1}{2} c_1 \left( \ddot{y} + \dot{x} - l_1 \dot{\theta} \right)^2 + \frac{1}{2} c_2 \left( \dot{x} + l_2 \dot{\theta} \right)^2 .
\] (3)

The Lagrangian, \( L = T - U \), evaluated from (1) and (2), and together with (3) substituted in (4) and (5), one obtains the equations of motion.

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = -\frac{\partial Q}{\partial \dot{x}}
\] (4)

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = -\frac{\partial Q}{\partial \dot{\theta}}
\] (5)

The application of Equations 4 and 5 yields the coupled equations of motions for the bounce \( x \) and the pitch \( \theta \), as:

For \( 0 < t \leq t_i = \lambda / V \),
\[ m\ddot{x} + (c_1 + c_2)\dot{x} + (l_2c_2 - l_1c_1)\theta + (k_1 + k_2)x + (l_2k_2 - l_1k_1)\theta = -k_1y - c_1\dot{y} \\
J\ddot{\theta} + (c_2l_2 - c_1l_1)\dot{x} + (l_2^2c_2 + l_1^2c_1)\theta + (k_2l_2 - k_1l_1)x + (l_2^2k_2 + l_1^2k_1)\theta = k_1l_1y + c_1l_1\dot{y}. \]

The equations of motion can also be shown in matrix form as:

\[
\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} \ddot{x}(t) \\ \ddot{\theta}(t) \end{bmatrix} + \begin{bmatrix} c_1 + c_2 & l_2c_2 - l_1c_1 \\ l_2^2c_2 + l_1^2c_1 & k_1 + k_2 \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{\theta}(t) \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & k_2l_2 - k_1l_1 \\ k_2l_2 - k_1l_1 & l_2^2k_2 + l_1^2k_1 \end{bmatrix} \begin{bmatrix} x(t) \\ \theta(t) \end{bmatrix} = 0,
\]

\[
\begin{bmatrix} -k_1 & 0 \\ k_1l_1 & 0 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \end{bmatrix} + \begin{bmatrix} -c_1 & 0 \\ c_1l_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ \dot{\theta}(t) \end{bmatrix} = 0.
\]

(6)

The above equations are subject to the following initial conditions:

\[
x(0) = \dot{x}(0) = 0 \\
\theta(0) = \dot{\theta}(0) = 0.
\]

Similarly the equations of motion for the time period where the rear tire is in touch with the bump is derived as:

For \( t_2 < t \leq t_2 + \frac{\lambda}{V}, \)

\[
\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix} \begin{bmatrix} \ddot{x}_1(t) \\ \ddot{\theta}_1(t) \end{bmatrix} + \begin{bmatrix} c_1 + c_2 & -(l_2c_2 - l_1c_1) \\ -(l_2^2c_2 + l_1^2c_1) & k_1 + k_2 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{\theta}_1(t) \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -(l_2^2k_2 - l_1^2k_1) \\ -(l_2^2k_2 + l_1^2k_1) & k_2l_2 - k_1l_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ \theta_1(t) \end{bmatrix} = 0 \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \end{bmatrix} + 0 \begin{bmatrix} 0 \\ k_2l_2 \end{bmatrix},
\]

(7)

where in the above \( \begin{bmatrix} x_1(t) \\ \theta_1(t) \end{bmatrix} \) is used for the system response in this time period instead of \( \begin{bmatrix} x(t) \\ \theta(t) \end{bmatrix} \) in the first period. Equations (7) is also subject to the following initial conditions:

\[
x_1(t_2) = x(t_2); \quad \dot{x}_1(t_2) = \dot{x}(t_2) \\
\theta_1(t_2) = \theta(t_2); \quad \dot{\theta}_1(t_2) = \dot{\theta}(t_2).
\]

Solution
To find the damped natural frequencies of the system one sets:

$$\det\begin{bmatrix}
ms^2 + (c_1 + c_2)s + k_1 + k_2 & (c_2 l_2 - c_1 l_1)s + k_2 l_2 - k_1 l_1 \\
(c_2 l_2 - c_1 l_1)s + k_2 l_2 - k_1 l_1 & Js^2 + (l_2^2 c_2 + l_1^2 c_1)s + l_2^2 k_2 + l_1^2 k_1
\end{bmatrix} = 0.$$

With the aid of MATLAB and using given system’s parameters, one finds the following damped natural frequencies with their corresponding damping ratio and natural frequencies.

$$\omega_{d1} = 4.9676 \text{ rad/s}$$
$$\omega_{d2} = 6.1681 \text{ rad/s}$$

Where the first natural frequency and damping ratio from above are derived as $$\omega_{n1} = 5.1399 \text{ rad/s}$$ and $$\zeta_1 = 0.2570$$ and the second natural frequency and damping ratio are $$\omega_{n2} = 6.5251 \text{ rad/s}$$ and $$\zeta_2 = 0.3263$$. The non-dimensionalized times, defined as the ratio of the time for each tire to pass the bump over the natural periods of the vibrations, are:

$$t_1 = \frac{\lambda}{\frac{V}{2\pi}} = \frac{\lambda \omega_{n1}}{2\pi V}$$

$$t_2 = \frac{\lambda}{\frac{V}{2\pi}} = \frac{\lambda \omega_{n2}}{2\pi V}.$$

Although closed form solution for response of a 2-degree of freedom system (but not this problem) to transient excitation has been studied by a number of authors $^{1,2,3,4,5,6}$ in the case of simple second degree systems in proportional damping through modal approach, the above problem cannot be solved by that approach since the damping matrix is not proportional. Therefore students had to use a numerical approach to obtain the solution in order to gain some physical insight into the nature of transient response of the automobile. To obtain the solution numerically, SIMULINK was used because of its ease of application.

SIMULINK$^7$ is an interactive environment for system simulation and embedded system design. As a platform for multi-domain modeling and simulation, SIMULINK lets students precisely describe and explore a system’s behavior. In addition, SIMULINK provides a graphical user interface that is often much easier to use than traditional command-line programs. Integration of SIMULINK into the instruction of vibrations will therefore be of great pedagogical value.
The following will describe the SIMULINK models used to obtain the solution.

**SIMULINK Models**

First, the speed bump is modeled as the algebraic sum of two sine waves one starting at \( t_1 = \frac{\lambda}{V} \) seconds later. The frequency of the sine waves is \( \omega = \frac{\pi}{t_1} \). That is:

\[
y(t) = 0.05\sin\left(\frac{\pi}{t_1}t\right)\times u(t) - 0.05\sin\left(\frac{\pi}{t_1}t\right)\times u(t - t_1),
\]

where \( u(t) \) is the step function at \( t = 0 \). SIMULINK implementation of this bump signal is shown below,

![SIMULINK Diagram](image)

**Figure 3**

where \( t_1 = 0.09 \) at the car speed of 20 km/hr. The signal that is produced is shown below, which clearly indicates a bump with the height of 5 cm and the duration of 0.09 seconds.
Let us now use SIMULINK to see the response of the vehicle as the front tire goes over the bump. The SIMULINK model to simulate this is shown in Figure 5.
The vehicle’s bounce motion can be seen from the Scope2 output, shown in Figure 5. The pitch response for the car speed of 20 km/hr is then:

This indicates a damped free vibration of the bounce motion. Notice that the peak amplitude of this motion is -6.85 mm. The reason this number is negative is that we have taken positive direction for x(t) in Figure 2 downward (opposite y(t)). To see the frequency of the damped vibrations students looked at the output of the Power Spectral Density shown in Figure 5. This indicates a peak in the frequency amplitude around 4.9 rad/s in Figure 7. This is close to the first damped frequency of the system which we obtained previously in the above as $\omega_{d1} = 4.9676 \text{rad/s}$.
The pitch motion response is the output of Scope3. This is depicted in Figure 8.

As we can see again, the ensuing motion is one of the damped free vibrations with the frequency of vibrations shown by the Power Spectral Density 1. The Power Spectral Density for this motion indicates a peak of the frequency at about 6.2 rad/s in Figure 9. This is very close to the pitch damped natural frequency of $\omega_d^2 = \frac{\omega^2}{1681.6^2}$, obtained in the above. Notice that the peak pitch motion is about $-4.53 \times 10^{-3}$ rad. The reason that this number is negative is because the car rotates clockwise as it goes over the bump.
Students then studied the vehicle as it went over a rectangular pulse (---) speed bump of the same height and width as the sine pulse. The vehicle speed was kept at 20 km/hr. Figure 10 shows the SIMULINK model for this study. Since the area under the rectangular pulse is bigger than that under the sine pulse, one would expect higher amplitudes for both bounce and pitch motions of the car in this case than in the sine pulse instance. This is indeed what the scopes in Figure 10 indicate. The maximum amplitude for the bounce motion is -7.25 mm, as it is indicated in Figure 11. The maximum pitch amplitude for this case is slightly higher than in the sine case, and it is \(-4.63 \times 10^{-3}\) rad, as shown in Figure 12. The power spectral densities shown in Figure 10 indicate that the bounce damped free vibrations shown in Figure 11 has a damped frequency of 4.9 rad/s and the pitch damped free vibrations has a damped natural frequency of 6.2 rad/s. We have not shown these diagrams here to avoid lengthening this article; instead we have simply stated the results students obtained from these diagrams. Simply said there is no difference between the Power Spectral Densities of this case and the sine pulse’s case.
Figure 10

Figure 11

Figure 12
The next lesson for the students was the study of the effect of speed of the vehicle on the bounce and pitch motions of the vehicle as it moves over the rectangular pulse. They tried speeds of 15 km/hr, 10 km/hr, 5 km/hr, 2 km/hr, 1 km/hr, and 0.5 km/hr besides the original 20 km/hr speed. The peaks of the bounce motion (for each of these different speeds) over the height of the speed bump vs. the ratio of the time that it takes for the vehicle to pass the bump over the bounce damped natural frequency are shown in Figure 13. This graph is known as the Shock Response Spectrum. We have also shown the Shock Response Spectrum for the pitch motion.
As it is apparent from these two curves, the shock response spectrum levels off after some value of the time ratio. When the passage time over the bump is much less than the natural damped periods of the system, system sees the excitation as a shock; and the longer the duration of the shock the higher the peak response. However, for passage times longer than natural periods, system behaves as if a steady load acts on the system.

When the rear tire goes over the bump, the vibrations produced from front tire passage over the bump (if it has not died out by the time rear tire reaches the bump) are used as the initial conditions, and the same SIMULINK model is used to obtain the new oscillation.

By changing springs’ constants and damping, one can arrive at lower amplitudes of the induced vibrations. This is done by replacing the constant gain boxes with the variable gains in the SIMULINK model.

Conclusions

Students’ feedback has been very positive regarding the project. Students learned about modeling a complex problem and learned how to solve a problem, which has no closed form solution, using the available technology. They learned about the difference between shock load and steady load and the different nature of response for these two different loading solutions.

Bibliography