On the effectiveness of designing didactical situations targeting R^n to teach the concept of subspace in linear algebra

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Abstract

The concept of subspace, a subset of a vector space with two well-defined operations (sum and product by a scalar), can be considered as one of the most abstract, formal, and difficult concepts to learn by students, in an undergraduate linear algebra course for Science and Engineering programs. This paper presents the results of a case study in which we investigated how by using R^n to teach the concept of subspace, student learning performance can be improved. Our case study involved 36 students in 2 sections of a linear algebra course. One section in which the instructor used a conventional way to approach the subject by using traditional textbook-based lectures and examples, and a second section that uses the approach proposed in this paper. We evaluated the effectiveness of the proposed approach through qualitative evidence gathered by an external observer, the reflection made by the instructor himself, and through two tests that allowed us to assess student performance. The first test (diagnostic) was applied before the intervention to measure the level of attainment of prerequisites. The second test (closure) was applied after the intervention to measure the level of attainment of the learning objectives addressed in the case study. Findings demonstrate that our proposed approach effectively contributed to improve the level of attainment of learning objectives associated with the concept of subspace in linear algebra.

1. Introduction

One of the most difficult courses for science and engineering students is linear algebra [1]. The case study presented in this paper focuses on studying the impact of implementing a novel approach to teach vector spaces, in particular subspaces. The concept of subspaces can be considered as one of the most abstract topics to learn in an undergraduate linear algebra course [1-3]. To the best of our knowledge, effectively teaching this subject to engineering students is still an open challenge, despite several contributions that have addressed this problem, particularly from the perspective of educational models such as active learning (e.g., [4-10,18]). Less contributions have been made to explore the usage of concrete strategies that aim at using other simpler mathematical concepts as tools to reduce the level of abstraction required from the students to study subspaces. In particular, [1], [8] and [21] present how geometry can be used to this end. Also, [23] uses the geometry of R^2 and R^3, as an introductory example to vector spaces. However, the approach proposed in

[23] uses simultaneously other abstract spaces such as P_n and the space of real valued functions, as it is usually done in most linear algebra textbooks.

This paper presents the results of a data-based case study we conducted to document the impact of using the properties of R^n, to facilitate the attainment of learning objectives that involve the subspace concept in an engineering linear algebra course. This work is part of a wider research project that focuses on studying the relevance of supporting math instructors in the process of improving student learning.

The methodology applied in this case study relies on qualitative data to analyze the impact of using the proposed approach when teaching the studied subject. This qualitative evaluation is complemented with quantitative data about the level of attainment of learning objectives before and after addressing the studied topic with the students. Although this is a case study that focuses on the results observed by the instructor proposing the approach, in what respects to the learning process of his own students, we made a qualitative comparison against another section in which the instructor used a more traditional approach, as suggested in most linear algebra books. This is, the instructor introduces the concept of vector spaces with several examples, which usually require a level of abstraction that is highly difficult to most students. Furthermore, even when using polynomials and matrices, spaces that are supposed to be more familiar to the students, the burden of writing the algebraic formulation of concepts, such as linear independence and span, may blur the correct conceptualization of them.

The course that applied the traditional approach is called traditional course, while the other one is called novel course. This case study involved 36 students in both sections. To gather quantitative data, we measured the level of attainment by students in the corresponding learning objectives through two tests. One that was applied before the intervention and focused on the previous knowledge related to the learning objectives addressed with the new strategies (diagnostic test), and another one that was applied after the intervention to measure the attainment of learning objectives (closure test). The obtained results show that the students enrolled in the novel course demonstrated a better performance in the attainment of four out of the five learning objectives addressed in the case study.

The main contribution of this paper resides on the impact that our approach may have on the attainment of learning objectives related to subspaces, since they very often result highly difficult for most engineering students [2-4]. This paper is organized as follows. Section 2 presents the context of the case study. Section 3 explains the approach proposed to address the concept of subspaces. Section 4 presents the methodology and evaluation protocol followed in the case study. Section 4 discusses relevant results. Finally, Section 5 concludes the paper.

2. Context

In our country, the low performance of students in math courses is one of the main causes of academic attrition in undergraduate engineering programs. Moreover, academic desertion has a high impact on our society, as well as institutions and student families, considering that most students come from low-income families and depend on financial loans to access higher education. One important reason for these high attrition levels among engineering students is that the national education system does not focus on the development of STEM competencies. As a result, the motivation of engineering students in mathematics courses is continuously hampered because of deficiencies in prerequisites.

The case study presented in this paper is part of a wider project conducted at our institution. The project involved several math courses taken by first and second year engineering students. The main objective of this project was to support mathematics instructors in the process of contributing to improve student learning, by continuously reflecting on the effectiveness of the pedagogical practices that are applied inside and outside the classroom, while adopting a continuous improvement culture that benefits student learning, accordingly. The motivation behind this project, which was conducted in collaboration with professors from the Math Department, which is part of the School of Education, and the School of Engineering, is the need to improve the retention of engineering students, which in our University is about 35%, and about 50% in our country.

Each instructor involved in the project conducted a self-reflection exercise that allowed her/him to identify improvement opportunities aiming at helping students to better attain learning objectives in the impacted courses. In the linear algebra case study reported in this paper, this self-reflection exercise generated a positive impact on the learning environment, and motivated students to boost their willingness to learn. On the one hand, critical reflection, an exercise to create skills as social mediator, learning facilitator and reflective practitioner [16], implies a disciplined way to analyze how we can improve our pedagogical practices and find effective mechanisms to execute these improvements. On the other hand, reflective practice is a way of understanding and learning from experiences, it is a resource for helping instructors reflect on their pedagogical strategies, as well as to improve their personal professional practice [1,2]. Authors such as [11] state that it is possible to teach reflectively, while focusing on the basic elements of the classroom process and that reflection. Most importantly, self-awareness and self-learning always will be more important for experienced leads than the simple acquisition of information. We concur on these premises and observed a great benefit from a continuous critical reflection praxis in the process of improving our pedagogical practices.

The reflection carried out by the professor of the linear algebra course, together with a literature revision and his more than 15 years teaching this subject, led him to identify that the learning objectives that are associated with the concept of subspaces are among the most difficult to achieve by engineering students. As a result, this case study focuses on proposing and validating a novel method to address this topic in an engineering linear algebra course.

To design the new learning experience, the instructor participated in different workshops and individual sessions that helped him consider mathematical education practices that he had not used before. In particular, those related to making a continuous critical reflection on pedagogical practices [11,20], to formulate learning objectives aligned with disciplinary competencies [14], and to consider the integration of technological resources in the proposed learning activities [17].

3. The proposed approach

3.1 Overview

The concept of vector space, a set of elements endowed with two well-defined operations (sum and product by a scalar), is one and probably the most abstract and formal of a linear algebra course for undergraduate engineering students [2,3,9]. The relationship between the concept of vector space and subspace is simple, any subset of a vector space that is itself a vector space, is a subspace. Since it is much easier to establish when a subset of a known vector space is a subspace, we are interested in investigating how to present the concept of subspace, generator sets, linearly independent sets and bases of a subspace, in such a way that students can understand and use these concepts to solve problems related to the four fundamental subspaces of a matrix [19,20]. Furthermore, by using R^n, which geometric properties are relatively easy to explore for n=2,3, we introduced, established and applied the concept of subspace [21, 23]. We wanted to establish how effective is to use R^n to introduce the concept of subspace, as an effective tool for facilitating the learning of subspaces in an engineering linear algebra course.

We hypothesize that using R^n can make it easier to present the concept and properties of a vector space, such as closure of the sum and the product by a scalar, linear independence, span, bases and dimension. By using the geometry of the spaces R^2 and R^3, it is possible to introduce the main ideas in a less abstract way, based on objects previously discussed in this and other courses, such as straight lines and planes in space. Our goal here is to construct a path that goes from the analysis of contexts to its mathematical interpretation.

We designed four different didactical learning experiences or situations to guide students in the process of being able to:

- Identify when an operation is closed (or which sets are not closed under an operation). An operation (*) is closed if given two elements a,b, of that set, the result of operating them, a*b, belongs to the set. Through given examples of certain sets in R^2 or in R^3, in which the sum or the product for a scalar is not closed, it is sought that the student discover, among other things, why the bounded sets cannot be subspaces, and why zero has to be an element of every subspace.
- Identify linearly independent sets (in R^2 and R^3), that is, those non-zero unitary sets of vectors, or those with two vectors that belong to non-parallel lines, or those of three vectors that do not belong to the same plane. Using a pair of didactic experiences supported on the use of GeoGebra, it is sought that the student uses the concept of linear combination, previously introduced in the course, to show that the only linearly independent sets in R^2 are the non-null unitary sets and those that have two non-collinear elements. It is proposed as a similar exercise for students to reach similar conclusions in R^3.
- Use the previous two properties to create a base of a subspace. For example, recognizing that the linear system associated with the combination of base vectors has a unique solution. Using the geometry of R^2 and R^3, we hope to deduce the quantity of vectors necessary and sufficient to generate subspaces of dimension one (straight lines) and two (planes), in order to define the concept of base and dimension, and then generalize it to n-dimensional sets.
- Use Gaussian elimination to build bases. To finalize the experience, we present the most efficient way to find subspace bases in R^n, the Gaussian elimination algorithm. Mainly focused on the association of the pivots in the echelon form with the choice of the elements in a base.

The ability to use the geometry of R^2 and R^3 in this context allows us to provide a less abstract view of their main properties and patterns [21]. We encouraged the use of GeoGebra to manipulate and visualize these spaces to contribute to the theoretical formulation of the concepts. In particular, we found that the level of abstraction of the concepts at stake, which are mostly new to students, is the main obstacle. The others are almost habitual in the basic courses of mathematics, for instance: i) the little or no relation that is made of the change from one mathematical register to another; ii) difficulties found when identifying and analyzing the mathematical concept (e.g., subspace, linear independence, span, among others), within the rigor of symbolic and contextual generality; and iii) the impossibility of knowing with certainty when the solution to the problem being solved with the studied concepts has been found.

3.2 Work guides, in-classroom activities and learning progress

We were looking to design learning activities that result suitable to tackle the new concepts to be presented on vector spaces. These activities would allow the students to feel familiar with the ideas and tools at hand. We wanted to configure a learning environment to engage students in the proposed activities in such a way that, hopefully, all of them participate actively. For every session there was a work guide, three for this experience, with clearly stated learning objectives involving the concept at glance and the required mathematical skills to address it, use it and apply it. We encouraged the use of ICT and not so much of traditional textbooks, although the latter were allowed to be used. Through short tasks that were designed to be initiated and terminated during in-classroom sessions, mostly to be worked in peers, we cover three general moments in the designed learning experience: a short introduction, a discussion and a closure argument with general feedback by the end of each assignment.

Each guide has different learning objectives:

1. Guide one:
    a. To distinguish sets that may or may not be subspaces of R^2 or R^3, using GeoGebra for visualization.
    b. To construct linear combinations in R^2 and R^3 and identify the geometrical locus they define.
    c. To determine which vectors belong to the span of S for some set S of R^n.
2. Guide two:
    a. To solve linear systems to determine when a set of vectors is linearly independent.
    b. To review the geometry of a subspace generator sets in R^2 and R^3.
    c. To interpret the concept of linear independence in terms of linear combinations.
3. Guide three:
    a. To solve linear systems to determine when a set of vectors is a base for a subspace in R^n.
    b. To build bases of subspaces defined algebraically (in R^2 and R^3).
    c. To Determine why a set of k vectors of R^ncannot be a base of R^n if $k \neq n$

The guides were designed to be used in three to four sessions trying to cover the subject of subspaces. Each guide had between two to three tasks with several exercises, most of them to be completed in the classroom. The purpose of each task was to help the student reach the learning objectives. More relevance was given to geometrical analysis, critical thinking and interpretation of results than to solving properly the linear systems associated with most of the computation work required to reach a conclusion. The work dynamics in each session was set to work on every exercise of the assigned tasks as follows. First, the instructor revisited previous concepts and their relationship with new concepts to be addressed in the session. Next, students start working on the corresponding guide by using digital tools that allow the professor to verify the progress of each student. For this, students work individually, for about 5 to 10 minutes, on a real time questionnaire proposed by the instructor. Then, the answers to the questionnaire drove peer-based discussions about the studied concepts. The professor motivates students to comment on their answers and justify their individual choices These discussions were socialized among all the students, with the help of the instructor, until reaching a consensus. Once reached a consensus, or time has run out and further analysis is suggested by the teacher, the students were asked to continue with the next question. By the end of the session, the instructor highlights the main ideas discussed by the students and gave the required explanations regarding non-consensus or misconceptions.

The main idea behind the proposed strategies is to motivate students to conduct deep interpretation and reflection processes. A continuous interaction among students and instructor is more often possible when students feel familiar with the working examples of subspaces in R^2 or R^3 than with those presented outside this framework, in this case on $R^n, n > 3$. Moreover, the proposed sequence of learning moments allows the instructor to be more effective in supporting the individual learning process of each student. It was noticed that students feel more motivated to learn from mistakes when ICT tools are involved, probably because these tools support cooperative learning. In the case of GeoGebra, students used it to play with the geometrical ideas that support the underlying concepts of subspaces, which allows them to discuss and conclude by themselves in small groups. Finally, to close each session, the professor helps the students reflect on how well they have attained the involved learning objectives through the activities conducted in the class. Based on this reflection, students, supported by the professor, should identify what to do to improve the attainment of learning objectives, if applicable.

Last but not least, the assessment is based on an intensive use of rubrics. These rubrics are discussed with the students at the beginning of the learning unit. The professor explains to the students how to exploit these rubrics to monitor their learning process and act accordingly (metacognition). Rubrics are revisited when discussing learning progress both during in-classroom sessions and after delivering tests results to the students. Of course, rubrics are carefully aligned to learning objectives, activities and evaluations.

4. Case study: Subspaces in a Linear Algebra course

4.1 Hypothesis

R^2 and R^3 not only are spaces familiar to engineering and science students, but also offer us the possibility to exploit their geometry as a resource to show the main ideas behind subspaces. We claim that by using R^n as the main tool to present subspaces, we could more effectively help students in the learning of concepts such as linear combination, linear independence and span set, which are in general too abstract in other spaces. This hypothesis is supported by the fact that students are more familiar with the geometrical and algebraic concepts associated with these spaces (from their previous work on other courses such as pre-calculus), which eases their application to more general spaces.

4.2 Methodology

The methodology that guided the development of the reported case study is explained as follows. First, the instructor conducted a literature review looking for evidence about the most common epistemological challenges [13] associated with engineering students' learning processes in the case of linear algebra [1-3]. At this stage, the learning objectives to be addressed in the case study were selected. Second, by means of an individual and group effort coordinated by math education experts, the instructor reflected on opportunities to improve his teaching practice with the goal of helping students increase the level of attainment of the corresponding learning objectives. Third, the instructor designed and implemented new learning experiences, as a result of the self-reflection process developed in previous stages. Finally, we assessed the impact of the proposed strategy qualitatively, through interviews applied to the students, and two tests related to the involved learning objectives.

4.3 Evaluation

Originally, we wanted to conduct a randomized controlled trial, where a comparison between two groups could validate the hypothesis of this paper (one group guided by the teacher proposing the novel strategy presented in this paper, and another group with a traditional teaching strategy). However, this approach could not be applied because we could not guarantee randomized selection. There was self-selection bias in group samples: the students from both groups decided by themselves in which group to enroll, by taking into account their preferences in schedule, assigned instructor, and classmates. Furthermore, we found a significant difference between group sizes (eleven in the novel group and twenty-five in the traditional group), probably because the schedule of the traditional section resulted to be more convenient to the students, and also because several students considered easier to pass the course with the instructor in charge of the section following the traditional approach. Thus, in the case of the linear algebra course, we limited our research to the development of a case study supported by a qualitative analysis of the gathered data, the evidence collected by an external observer, and the instructor's reflection.

To evaluate the effectiveness of the proposed approach, we conducted the following steps. i) the students of both groups were assessed about the prerequisites associated with the concepts addressed in the case study. With this instrument we intended to identify any differences between the two groups of students regarding prerequisites. The results indicated that there were no relevant differences; ii) a math education expert observed the classroom experiences in both groups; iii) the students from both groups were assessed at the end of the thematic unit to measure the level of attainment of the learning objectives addressed in the case study; and iv) the instructor proposing the novel approach registered his reflections throughout the learning experience.

5. Results

5.1 The redesign of the learning experience

With the proposed learning approach, we intended to better support students in making the necessary connections to show whether a set meets the properties required to be a subspace; for example, when that set is generated by some of its elements, how many do you need to generate the entire space, among others.

The redesign focused on addressing the difficulties that most students commonly face when learning concepts associated with subspaces, seeking to formulate the learning objectives in terms of the mathematical skills and competencies that are intended to be reached with each proposed activity. For this purpose, some of the seven mathematical competencies proposed by PISA were considered. These competencies are: Reasoning and argumentation, communication, modeling, problem solving, representation, use of symbolic language, and use of technological tools (TT) [12,14,15].

Next, we highlight the mathematical activity present in the learning tasks that were part of the redesign of the learning experience, as well as the resources and teaching tools that promoted the understanding of the concepts related to subspaces through processes of generalization and visualization. According to [22], generalization processes allow us to treat the concepts from three fundamental aspects: to see, to say and to register. To see refers to the identification of patterns or relationships, followed by saying (to say) in natural language which ones are recognized. Finally, to register makes the language visible for symbolic and written communication. Regarding visualization, [21] establishes that the appropriation of mathematical concepts from a geometric point of view has positive effects on student learning, allowing them to establish connections and describe concepts from different types of representations.

In this way, one of the elements that were considered relevant within the proposed activities was the linking among subspace concepts in the classroom, where the algorithmic processes were relegated and priority was given to the geometry required for understanding the properties involved in vector spaces. The latter being one of the biggest challenges that have been identified among students of linear algebra. The mathematical objects under study and their properties were approached with GeoGebra, as a didactic resource for mediation and verification that facilitated the understanding of the concept from its geometric register through richer visualization processes. This helped students to appropriate the semantics of algebraic symbols, as well as to make associations and conjectures between graphic and symbolic representations, which in general is crucial for interpreting the mathematical properties of subspaces.

The following are the learning objectives involved in the redesigned learning experience:

LO1: Interpret the concept of subspaces geometrically in R^2 and R^3.
LO2: Express, orally and in writing, the relationship between mathematical concepts and algorithmic processes to solve problem situations (ability to record and talk about what has been learned).
LO3: Build sets of linearly independent vectors, span and bases.
LO4: Use technological tools (GeoGebra) as a support resource in the visualization processes of the mathematical subspace object.
LO5: Read theorems, identify their hypotheses and theses, and explain the ideas of each result in natural language

For this redesign, the central interest was given to the competencies of modeling, communication, argumentation and use of digital resources. Table 1 summarizes the learning situations defined for the redesigned learning experience, and their relationship with corresponding learning objectives. These learning situations were designed to encourage and favor the learning of the concepts of interest, as presented in Section 3. Situation I describes the elements of space R^n and attempt to contextualize them in different representations and registers, for example, through geometry and use of GeoGebra. Situations II and III take up the learnings of the section on lines and planes and seek to familiarize the student with the properties of the span and linear independence. Finally, situation IV looks to determine whether the student can formalize and generalize the concepts established in the two previous sections to any vector space of dimension n.

The didactical strategies focused on collaborative work, individual assignments, as well as communication and validation supported by discursive processes in the classroom. Special attention was given to the evaluation process to promote effective feedback from the instructor.

Table 1 – Learning activities and their relationship with learning objectives

Situations	Learning Objectives (LO)	Learning Activity
S1.	LO1, LO3	Build linear combinations in R^2 and R^3 and identify the geometric region they define.
	LO1, LO4	Distinguish sets that may or may not be subspaces of R^2 and R^3 by using GeoGebra as a source of visualization.
S2.	LO2, LO3	Differentiate the elements (vectors) dependent on a set of vectors.
	LO2	Interpret the concept of linear dependence in terms of linear combination.
S3.	LO2, LO3, LO5	Use the definition of span S set to build subspaces of R^2 and R^3
	LO3, LO4	Identify the linearly independent elements of a set of vectors S and associate them with the set span S
S4.	LO2, LO3	Find bases and the dimension of a subspace
	LO2, LO4, LO5	Argue with respect to the number of elements of a set of vectors when one of the two properties: generation or linear independence fails

Table 2 next presents a parallel between the novel and the traditional sections, in what respects to the proposed learning activities and used resources.

Table 2 – Learning strategies and resources: a parallel between the novel and the traditional approaches (evidence gathered by the external observer)

Novel course	*Traditional course*
Use of GeoGebra as a mediation and verification tool for understanding the concept of subspaces.	Absence of the use of educational tools or resources that promote an approach to understanding the concept of subspaces.
Formal linkage of mathematical concepts throughout validation, interpretation and argumentation processes.	The linkage of mathematical concepts is carried out in an arbitrary way, it lacks validation, interpretation and argumentation processes.
Use of visualization processes, diversity of mathematical representations.	Absence of visualizations in the representation of the mathematical object.
Use of semantics over syntactics.	Use of syntactics over semantics.
The proposed learning activities focus on argumentation and validation.	The proposed learning activities focus on the use of techniques and algorithms.
Design and adaptation of new learning activities.	Literal use of learning activities as proposed in the textbook.
The proposed learning tasks are coherent with the progress of the student.	The learning tasks do not meet expected levels of progress.

5.2 Impact on the attainment of learning objectives

We designed, adapted and implemented learning activities focused on the development of mathematical skills, such as argumentation and interpretation that, in addition to the proposed generalization and visualization processes, contributed to the effectiveness of the approach proposed to help students in their learning process.

The diagnostic and closure tests were carried out to evaluate five learning objectives (cf. Tables 3 and 4 below). The diagnostic test allowed us to measure the level of performance ("low", "basic", "high" and "higher") of each student in prerequisites associated with the addressed learning objectives. The closure test allowed us to measure the level of performance of each student for each learning objective (LO), at the end of the intervention.

Table 3 – Student attainment level in the course that used the novel approach

Proportion of students with an attainment level of High or Higher in the learning objectives associated with the redesigned learning experience		
Learning Objective	*Diagnostic test*	*Closure test*
LO1	0,00%	36,36%
LO2	0,00%	18,18%
LO3	9,09%	81,82%
LO4	9,09%	63,64%
LO5	0,00%	54,55%

Proportion of students with an attainment level of deficient in the learning objectives associated with the redesigned learning experience		
Learning Objective	*Diagnostic test*	*Closure test*
LO1	90,91%	9,09%
LO2	90,91%	36,36%
LO3	81,82%	18,18%
LO4	72,73%	18,18%
LO5	81,82%	0,00%

LO1: Interpret the concept of subspaces geometrically in R^2 and R^3.
LO2: Express, orally and in writing, the relationship between mathematical concepts and algorithmic processes to solve problem situations
LO3: Build sets of linearly independent vectors, span and with both properties
LO4: Use technological tools as a support resource in the visualization processes of the mathematical subspace object
LO5: Read theorems, identify their hypotheses and theses, and explain the ideas of each result in natural language

Table 4 – Student attainment level in the course that used the traditional approach

Proportion of students with an attainment level of High or Higher in the learning objectives associated with the redesigned learning experience		
Learning Objective	*Diagnostic test*	*Closure test*
LO1	0,00%	29,17%
LO2	0,00%	20,83%
LO3	4,17%	16,67%
LO4	0,00%	16,67%
LO5	0,00%	8,33%

Proportion of students with an attainment level of deficient in the learning objectives associated with the redesigned learning experience		
Learning Objective	*Diagnostic test*	*Closure test*
LO1	91,67%	29,17%
LO2	100,00%	45,83%
LO3	91,67%	50,00%
LO4	91,67%	41,67%
LO5	95,83%	62,50%

LO1: Interpret the concept of subspaces geometrically in R^2 and R^3.
LO2: Express, orally and in writing, the relationship between mathematical concepts and algorithmic processes to solve problem situations
LO3: Build sets of linearly independent vectors, span and with both properties
LO4: Use technological tools as a support resource in the visualization processes of the mathematical subspace object
LO5: Read theorems, identify their hypotheses and theses, and explain the ideas of each result in natural language

Table 5 below summarizes the evidence gathered by the external observer to explain the results obtained for learning objectives LO3, LO4 and LO5, which correspond to those for which the proposed approach demonstrated a considerably higher positive impact.

Table 5 – Effects of the redesign on some of the learning objectives compared to the traditional section (evidence gathered by the external observer)

Learning objective	*Findings from closure test*	*Evidence*
LO3: Build sets of linearly independent vectors, span and bases.	81.82% of the students in the novel group were placed at a high or higher performance level, compared to 16.67% of the students in the traditional group. 18.18% obtained a very low level of performance, compared to 50% of the students in the traditional group.	**(Novel)** Students are able to represent the mathematical object, making use of the properties present in the subspaces. This indicates that one departs from establishing relationships between concepts, in order to build a generality. **(Traditional)** The instructor departs from a generality in the development of the algebraic construction. Geometric records of the mathematical object are not established.
LO4: Use technological tools as a support resource in the visualization processes of the mathematical subspace object	63.64% of the students in the novel course were placed at a high or higher performance level, compared to 16.67% of the students in the traditional course. 18.18% obtained a very low level of performance, compared to 41.67% of the students in the traditional course.	**(Novel)** Students use GeoGebra as a mediation tool, which allows them to verify the properties linked to the mathematical concept of subspaces. The main focus of the activities linked to this learning objective refers to the processes of visualization of the mathematical object and the use of different types of representation. **(Traditional)** The subject is approached from an axiomatic perspective, relying on the memorization of the properties of the subspaces. **(Traditional)** Students do not use a verification tool, nor another resource that allows them to zoom in on the mathematical object and therefore have a better understanding of the concept of subspaces.

Learning objective	*Findings from closure test*	*Evidence*
LO5: Read theorems, identify their hypotheses and theses, and explain the ideas of each result in natural language	54.55% of the students in the novel group were placed at a high or higher performance level, compared to 8.33% of the students in the unaccompanied group. There was no evidence of students presenting a very low level of performance, compared to 62.50% of the students in the traditional group.	**(Novel)** Most students manage to appropriate the abstraction of mathematical notations in their natural language. **(Traditional)** No treatment of the meaning of mathematical properties is established, which makes it almost impossible to verbally communicate the understanding of the concept.

5.3 Reflections by the instructor proposing the novel approach

We divided the reflections made by the instructor into three categories according to the phases of the case study: design, implementation, and evaluation. The instructor reflected on the main aspects of each stage. First, he highlights the importance of going over the main issues students have on the learning subjects and how to improve our teaching practices to help them in the learning process:

> *"It was a constructive and very demanding experience that allowed me to have more clarity about the teaching-learning process. For the design, the main problems associated with subspace learning, which are indicated in the redesign document, were taken as a starting point."*

Based on his previous experience, the instructor addressed all the aspects he considered must be addressed for students to succeed on attaining the learning objectives associated with the concept of subspaces:

> *"The idea was to approach them from another perspective and build experiences or activities that motivated students to explore more deeply the most important concepts and associate them, when possible, with the geometry of the spaces R^2 and R^3."*

Finally, although the process at the first time was very time consuming, it was very important to prepare proper material to be addressed by the students through the proposed in- and out-of- classroom activities along their learning process. With respect to the implementation stage, the instructor emphasizes the impact of the proposed classroom activities based on peer learning, the use of technology and the work guides prepared by him. The reflection of the instructor was as follows:

> *"The implementation was well received by the students, who were always willing and interested in participating in the proposed activities. The use of study guides, instead of the traditional textbook, proved to be an excellent support for the students, increasing their individual and autonomous work, and for the teacher, being able to more easily guide discussions with subgroups and the whole group."*

The instructor's reflections about the evaluation confirms a tacit fact that we should evaluate based on what we teach, and we must teach what is needed to address competencies and their learning objectives. A fair evaluation should be close to what the student was supposed to learn given the proper tools to do it. The reflection was the following:

> *"For the evaluation of the classroom experience, the design of the guides was used as the main input and of course the rubric. The questions, based mainly on the exercises of the guides, aimed to determine what specific components of the expected competencies were achieved by students: In most cases, a direct relationship could be established between the results of the evaluation and the performance or behavior of the students during the experience.*

6. Conclusions

We conducted a case study on how using R^n as the main resource in an active learning environment impacts the ways to teach a critical subject in linear algebra. Based on qualitative and quantitative evidence, we established the effects on student learning resulting from a redesigned pedagogical practice based on our proposed approach. We measured the student level of attainment of learning objectives. Findings show that the students that benefited from our proposed approach had a significant improvement in most of the evaluated learning objectives.

The proposed intervention had a positive effect on student learning. First, regarding the geometrical interpretation of the concept of subspaces in R^2 and R^3 (LO1), it is relatively clear that most of the students in the group reached similar levels of development. This may be, as mentioned before, due to the previous work on straight lines and planes in R^3. However, due to lack of basic geometry prerequisites, it is still hard for most students to get a good performance when required to use properties or to imagine objects in space. It is important to point out that all concepts in this experience were covered by using geometrical examples thus the intervention had more impact on those students with better performance in this prerequisite.

Learning objectives related to the construction of linearly independent vectors, span and bases (LO3), the use of technological tools (LO4), and the explanation of ideas from mathematical to natural language (LO5) were considerably better attained after the intervention. We consider that this result was a consequence of the proposed activities that leveraged the geometry of R^2 and R^3 to present the concepts and to build examples that students had to show and explain to their classmates. They were encouraged to use technological support, and to effectively use the board with algebraic developments or geometrical sketches that approximate the ideas behind the examples. When the students were able to connect the algebraic work with the involved geometry, more clarity was achieved by the students and the instructor was able to better identify hidden epistemological obstacles.

Finally, with respect to the learning objective that focuses on promoting the ability to communicate the relationship between mathematical concepts and algorithmic processes to solve problem situations (LO2), we see not much improvement in the highest level of attainment. This tells us that we were not able to help the students to read mathematics in such a way that, without external sources support, allows them to relate the mathematical concepts with the algorithmic tools proposed to solve specific problems. They are able to solve a problem but not to explain why an algorithm works and where it ends. This obstacle is inherent to most mathematical courses, especially if previous work has been focused on algorithmic processes rather than argumentative or critical reflection on the subjects of study. It is now part of our future research work to search for better strategies to address this issue, not only in linear algebra but in all the courses of our curricula.

In conclusion, we consider that the research presented in this paper contributes to the development of a new roadmap in the investigation of novel strategies to teach linear algebra for our current generation of students. On the aspects to improve, in future redesigns of the linear algebra course, work could be done to i) strengthen the dynamics of the activities so that more discussion processes are generated between the instructor and the students, ii) encourage students to exchange ideas with others colleagues besides their usual partners; and iii) include work sessions, as far as possible, where mathematical concepts are studied in the context of real life project applications.

References

[1] Gueudet-Chartier, G. (2004). Should we teach linear algebra through geometry? Linear algebra and its applications, 379, 491-501.

[2] Dorier, J. L. (Ed.). (2000). On the teaching of linear algebra (Vol. 23). Springer Science & Business Media.

[3] Cowen, C. C. (1997, January). On the centrality of linear algebra in the curriculum. In the Deborah and Franklin Teper Haimo Award for Distinguished College or University Teaching at the Joint Meetings, Jan (Vol. 10).

[4] Hamdan*, M. (2005). Nonlinear learning of linear algebra: Active learning through journal writing. International Journal of Mathematical Education in Science and Technology, 36(6), 607-615

[5] Talbert, R. (2014). Inverting the linear algebra classroom. Primus, 24(5), 361-374.

[6] Muñoz, M., Martínez, C., Cárdenas, C., & Cepeda, M. (2013). Active learning in first-year engineering courses at Universidad Católica de la Santísima Concepción, Chile. Australasian Journal of Engineering Education, 19(1), 27-38.

[7] Carney, D., Ormes, N., & Swanson, R. (2015). Partially flipped linear algebra: A team–based approach. Primus, 25(8), 641-654.

[8] Dogan-Dunlap, H. (2010). Linear algebra students' modes of reasoning: Geometric representations. Linear algebra and its applications, 432(8), 2141-2159.

[9] Roberts, A. J. (2018). Linear Algebra for 21st C Application.

[10] Teixeira, K. C., Silva, T. E., Mota, J. C., Barroso, N. C., & Teixeira, E. V. (2015, October). Peer instruction methodology for linear algebra subject: a case study in an engineering course. In Frontiers in Education Conference (FIE), 2015 IEEE (pp. 1-7). IEEE.

[11] J. Dunne, Back to the rough ground: 'Phronesis' and 'techne' in modern philosophy and Aristotle, Notre Dame, MA: University of Notre Dame Pres, 1993.

[12] ASOCOLME, 15 encuentro colombiano de matemática educativa (ECME 15). Boletín Asociación de Matemática Educativa ASOCOLME, pp. 4. Available online: http://asocolme.org/publicaciones-asocolme/memorias-ecme [Accessed January, 2019]

[13] Bachelard, G. (1993). La formation de l'esprit scientifique: contribution à une psychanalyse de la connaissance. Vrin.

[14] OECD, PISA 2009 Assessment Framework – Key competencies in reading, mathematics and science. Available in www.oecd.org/dataoecd/11/40/44455820.pdf [Accessed January, 2019]

[15] MEN Ministerio de educación Colombiano. (2006). Estándares Básicos de Competencias Matemáticas

[16] L. Lawrence-Wilkes and L. Ashmore, The Reflective Practitioner in Professional Education, MA: Macmillan Publishers Limited, 2014

[17] N. Kellam, A. Babcock, and D. Gattie, "The Engineering Learning Environment: A Proposed Model", ASEE Annual Conference & Exposition, Conference Proceedings, Pittsburgh, Pennsylvania, USA, Junio 22- 25, 2008, Available in https://peer.asee.org/3779

[18] Betty Love, Angie Hodge, Neal Grandgenett & Andrew W. Swift (2014) Student learning and perceptions in a flipped linear algebra course, International Journal of Mathematical Education in Science and Technology, 45:3, 317 – 324, DOI: 10.1080/0020739X.2013.822582

[19] Strang, G. (2019). Linear algebra and learning from data. Wellesley-Cambridge Press.

[20] Boyd, S., & Vandenberghe, L. (2018). Introduction to applied linear algebra: vectors, matrices, and least squares. Cambridge university press.

[21] Cihan, K., Sabri, İ. & Ahmet, I. (2003). "On the Teaching Linear Algebra at the University Level: The Role of Visualization in the Teaching Vector Spaces". Journal of the Korea Society of Mathematical Education Series D: Research in Mathematical Education. 7(1): 59–6

[22] Mason, J. (1996). Expressing generality and roots of algebra. In Approaches to algebra (pp. 65-86). Springer, Dordrecht.

[23] Lay, D. C. (2016). Linear Algebra and its applications 5th edition.