ON THE PROBABILITY OF ERROR FOR TRESHOLD BINARY-INPUT TERNARY-OUTPUT DISCRETE MEMORYLESS CHANNELS

Daniel Gebremicheal[1] Paul Cotae[2]
danyzyada@gmail.com pcotae@udc.edu

[1,2]

Electrical and Computer Engineering Department,
School of Engineering and Applied Sciences,
University of the District of Columbia,
Washington DC, 20008

***Abstract*—In this paper we derive the probability of error for binary-input ternary-output discrete memoryless channels (2,3 DMC). We analyse how the performance of the threshold device is determined by the choice of decision points and signal amplitude in the presence of Additive Gaussian White Noise (AWGN). We investigated the trade-off between the probability of false alarm and probability of correct detection by considering symmetric decision threshold around the zero decision thresholds. We considered the underwater acoustic wireless sensor networks where sensor nodes are limited in power, computational capacities and memory. We focused on the (2, 3) DMC by maximizing the detection and estimation of received signal by exploiting stochastic resonance effect.**

KEY WORDS

Probability of error, Binary-input Ternary–output discrete memoryless channels (2, 3) DMC, Threshold devices, Probability of false alarm, Stochastic Resonance (SR), Signal to noise ratio (SNR).

I. INTRODUCTION

In Binary-Input Ternary-Output Discrete Memoryless Channels, the optimum detector compares the input signal with a set of three arbitrarily defined thresholds in the presence of Additive White Gaussian Noise [1]. The placing of the thresholds as shown in Fig.(1) is in such a way that minimizes the probability of error and maximize the probability of correct detection.

Threshold devices can enhance the efficiency of a distributed wireless sensor networks and reduce the cost of target detection by minimizing the probability of false alarm under noisy and realistic conditions. In underwater acoustic wireless sensor networks where sensor nodes are limited in power, computational capacities and memory [2], (2, 3) DMC plays an important role in maximizing channel capacity [3]-[8] and minimizing probability of error [5], [6], [8], [9].

The threshold systems have been the main topic in the study and existence of stochastic resonance (SR) as in [3], [4], [10]. Selecting the right threshold level and noise power at the detector can lead into a better target location detection and direction finding in underwater acoustic wireless sensor networks [2].

An appropriate measure of output performance depends on the task at hand, and the form of input signal. For example, for periodic signals and broadband noise the SNR parameter is often used [11]. When the signal is random and aperiodic, SR can be observed by calculating the mutual information and capacity of the channel [3]-[4]. Not only that but also the noise can be beneficial in bit count increase, a decrease in probability of error, or an increase in detection probability as it analysed in [5], [6]-[7].

In this paper we will consider probability of error as a performance measure and study how the probability of error varies with the decision threshold in the presence of AWGN. We will focus on the same binary input and ternary output system model as they are discussed discussed in [3].

After deriving the analytic relationship for the probability of error in (2, 3) DMC, we will examine how the probability of error varies with respect to the arbitrarily defined threshold and noise standard deviation. Moreover, we will investigate the probability of correct detection versus false alarm for different noise intensity and decision threshold. Eventually we will examine stochastic resonance effect by generating three dimensional plot that relates the probability of correct detection, threshold, theta and noise standard deviation (sigma).

II. SYSTEM MODEL

The input to the threshold communication channel is the signal that takes the binary values ±A as in [3], [4] with probability p_0 and p_1. The physical model is represented in Fig. (2). The (2, 3) DMC detector transforms the observation into a value which is finally compared to a threshold ϴ to make a decision. The DMC is completely characterized by the transition probabilities of the output conditioned by the input probabilities.

Adapting the model of [3], we derived the analytic relationship for the probability of error in the (2, 3) DMC. We will investigate the effect of AWGN on the probability of error of the (2, 3) DMC for different value of decision threshold theta and noise power sigma.

III. THE (2, 3) DMC AND THE PROBABILITY OF ERROR

The (2, 3) DMC is characterized by a binary input random variable X, a ternary output random variable Y and a transition probability matrix, M [3]. We know that in (2, 2) DMC the optimum threshold for antipodal signaling is zero independent of noise when the prior probabilities are equal. However if the prior probabilities can assume different values, then the optimum threshold becomes as a function of signal amplitude and noise variance beside the prior probabilities [12]. Following [4], we defined conditional probabilities as shown in Fig.(1) for an optimum system performance. We assumed antipodal signaling and a new symmetric decision threshold is defined around the optimum threshold in the (2 ,2) DMC which gives rise into three distinct regions at the output as shown in Fig.(2).We defined probability of miss as the conditional probability that the received signal is less than the threshold theta given it is greater than the zero optimum threshold in the (2, 2) DMC. Similarly the probability of false alarm is the conditional probability that the received signal is greater than the symmetric negative threshold theta given that the received signal is less than the optimum threshold in the (2, 2) DMC. We have the following relations:

$$\mathrm{P}(\frac{e}{S_1}) = \mathrm{P}(\frac{B}{A}) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(A)} \quad (1)$$

where B is the region for which the conditional probability $f(\frac{y}{s_1})$ is less than the threshold theta (Θ) as shown in Fig.(1). Also A is the region for which the conditional probability $f(\frac{y}{s_1})$ is greaer than the zero decision threshold theta as shown in Fig.(1).

$$P(A \cap B) = \int_0^{\Theta} f(y)\, dy \quad (2)$$

where $\mathrm{f}(y)$ is the probability density function of the normally distributed Gaussian random variable with mean μ and variance δ^2

$$\mathrm{f}(y) = \frac{1}{\sqrt{2\pi\delta^2}} e^{\frac{-(y-A)^2}{2\delta^2}} \quad (3)$$

$$P(A \cap B) = \int_0^{\theta} \frac{1}{\sqrt{2\pi\delta^2}} * e^{\frac{-(y-A)^2}{2\delta^2}} dy \quad (4)$$

For a Gaussian random variable, $\mathrm{N}(\mu, \delta^2)$, a simple change of variable in the integral in order to compute $p_{r(Y>y)}$, results in

$$P_r(Y > y) = Q\left(\frac{(y-\mu)}{\delta}\right) \quad (5)$$

where $Q(x)$is the error function defined below:

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{\frac{-x^2}{2}} dx \quad (6)$$

$$Q(-x) = 1 - Q(x) \quad (7)$$

$$Q(x_1) - Q(x_2) = \frac{1}{\sqrt{2\pi}} \int_{x_1}^{x_2} e^{\frac{-x^2}{2}} dx \quad (8)$$

Using (5), (6) and (8) we obtain

$$P(A \cap B) = 1 - Q\left(\frac{A}{\delta}\right) - Q\left(\frac{(\theta-A)}{\delta}\right) \quad (9)$$

$$P(A) = \int_0^{\infty} f(y) dy \quad (10)$$

By using (3) we will obtain:

$$P(A) = \int_0^{\infty} \frac{1}{\sqrt{2\pi\delta^2}} * e^{\frac{-(y-A)^2}{2\delta^2}} dy \quad (11)$$

Now, using (5) and (6) one gets

$$P(A) = Q\left(\frac{-A}{\delta}\right) \quad (12)$$

In accordance with (7) we obtain

$$P(A) = 1 - Q(\frac{A}{\delta}) \quad (13)$$

Combining (9) and (13) we will get

$$P(\frac{e}{S_1}) = \frac{1-Q(\frac{(A)}{\delta}) - Q(\frac{(\theta-A)}{\delta})}{1-Q(\frac{(A)}{\delta})} \quad (14)$$

Similarly we have:

$$P(\frac{e}{S_0}) = P(\frac{B}{A}) = \frac{P(A\cap B)}{P(A)} \quad (15)$$

where B is the region for which the conditional probability $f(\frac{y}{S_0})$ is greater than the –Θ threshold as shown in Fig.(1). Also A is the region for which the conditional probability $f(\frac{y}{S_0})$ is less than the zero decision threshold theta as shown in Fig.(1).

$$P(A\cap B) = \int_{-\theta}^{0} f(y)\, dy \quad (16)$$

Using (3), (5) and (8) we will obtain

$$P(A\cap B) = Q\left(\frac{(A-\theta)}{\delta}\right) - Q\left(\frac{(A)}{\delta}\right) \quad (17)$$

$$P(A) = \int_{-\infty}^{0} f(y) dy \quad (18)$$

Using (3) and (7) we will obtain

$$P(A) = 1 - Q(\frac{(A)}{\delta}) \quad (19)$$

Combining (17) and (19)

$$P(\frac{e}{S_0}) = \frac{Q\left(\frac{(A-\theta)}{\delta}\right) - Q(\frac{(A)}{\delta})}{1-Q(\frac{(A)}{\delta})} \quad (20)$$

The average probability of error for the (2, 3) DMC can be calculated as follows

$$P\left(\frac{A}{\theta}\right) = p_0 P\left(\frac{e}{S_0}\right) + p_1 P(\frac{e}{S_0}), \quad [12] \quad (21)$$

$$P(\frac{e}{\theta}) = p_0 \frac{Q\left(\frac{(A-\theta)}{\delta}\right) - Q(\frac{A}{\delta})}{1-Q(\frac{A}{\delta})} + p_1 \frac{1-Q(\frac{A}{\delta}) - Q(\frac{(\theta-A)}{\delta})}{1-Q(\frac{A}{\delta})} \quad (22)$$

Letting $\frac{A}{\delta}=\sqrt{SNR}$, (22) can be written as

$$P\left(\frac{e}{\theta}\right) = p_0 \frac{Q(\sqrt{SNR}(1-\frac{\Theta}{A})) - Q(\sqrt{SNR})}{1-Q(\sqrt{SNR})} + p_1 \frac{1-Q(\sqrt{SNR}\ (\frac{\theta}{A}-1)) - Q(\sqrt{SNR})}{1-Q(\sqrt{SNR})} \quad (23)$$

IV. RESULTS AND CONCLUSIONS

Equation (22) clearly shows the average probability of error is a function of signal amplitude, noise intensity and decision threshold theta. Using this analytic relationship we studied how the probability of error varies as a function of signal to noise ratio (SNR) by either varying the signal amplitude and fixing the threshold or vice versa. We examined how the amplitude of the signal is related to the decision threshold by exploiting the probability of error curves as a function of signal to noise ratio as shown in Fig (3)-(4).

A three dimensional surface that relates the three parameters namely the probability of correct detection, threshold (theta) and noise intensity (sigma) is generated in Matlab in order to investigate the effect of noise intensity in the detection and estimation of received signal in (2,3) DMC. These channels play an important role in underwater acoustic wireless sensor networks where sensor nodes are subject to operational constraints such as power and bandwidth.

As it is shown in the Fig.(3)-(4) we can achieve the same probability of error by either fixing the threshold or the signal amplitude needs to increase in order to compensate the decrease in the Euclidian distance between the threshold and signal amplitude. Alternatively for the same power the probability of error increases as the threshold approaches the amplitude of the signal. However, if we increase the Euclidian distance between the signal amplitude and the threshold beyond a certain limit, the probability of error will increase. Further increase will have a negligible effect on the reduction of the probability of error only increasing our operating costs as shown in Fig.(5). Thus to achieve a minimum probability of error the system need to be sub threshold; the threshold theta should be somewhere between the zero optimum threshold of the (2, 3) DMC and the signal amplitude, $0<\alpha<A$.

As it is depicted in Fig.(6)-(7), for a fixed value of noise standard deviation, the probability of error increase with increase in the decision threshold theta. When the threshold theta is much greater than our signal amplitude we end up of making a total error and there is no reliable communication between the device communicating devices.

The (2,3)DMC exhibits a remarkable boost in the detection probability of the received signal for an optimum value of the noise intensity and the decision threshold theta as shown in Fig.(8). There is a region where the probability of correct detection of the received signal increases with increase of the noise intensity. However beyond a certain limit, the effect of adding noise becomes useless and the detection performance of the detector deteriorates and we end up of making intolerable errors.

Acknowledgement

This work was supported by the Department of Defense (DoD) Grant W911NE-11-1-0144

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FIGURES

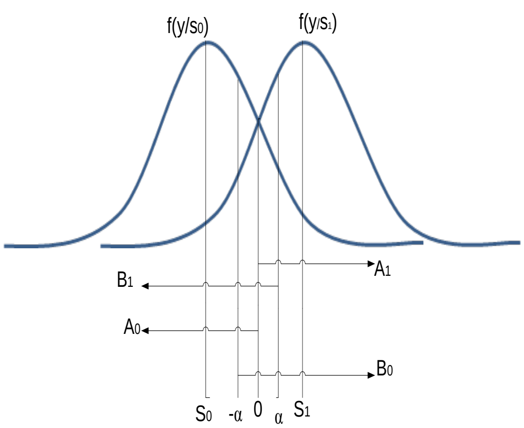


Fig.(1). Represents the Gaussian PDF $\mathbf{f}(\frac{y}{s_0})$ and $\mathbf{f}(\frac{y}{s_1})$ with the symmetric decision threshold

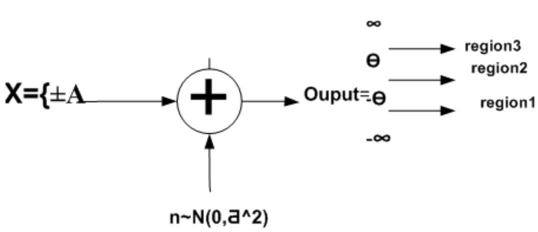


Fig.(2). Physical model of binary-input ternary-output channel with zero mean Gaussian noise

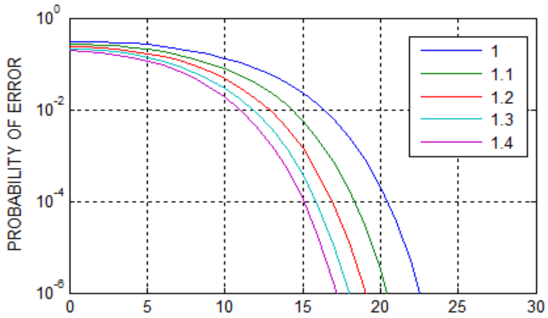
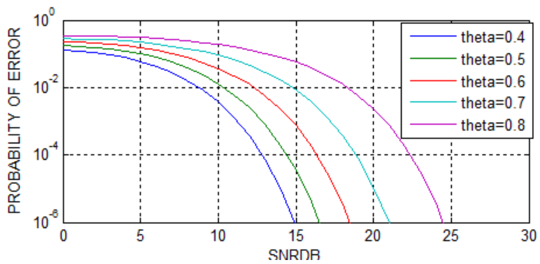
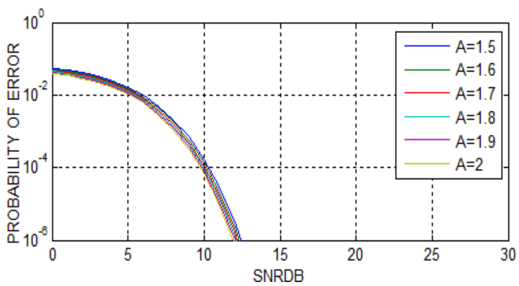


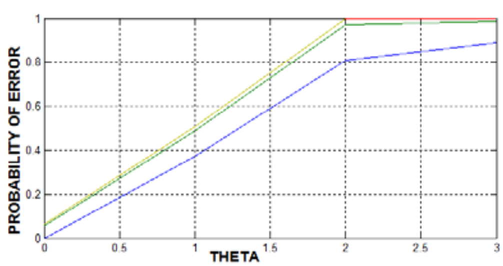
Fig.(3). Probability of error for the (2, 3) DMC as a function of SNRDB for fixed value of threshold, theta=0.75



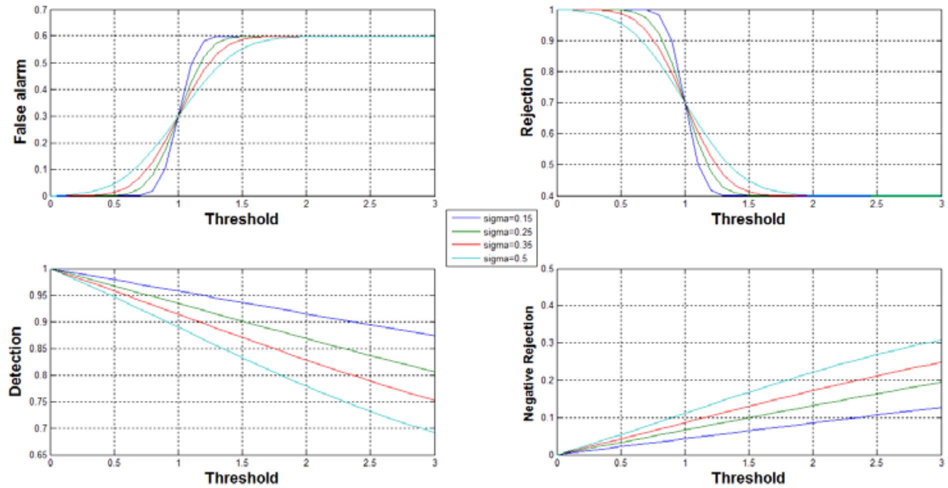
Fig(4). Probability of error for the (2,3) DMC as a function of SNRDB for fixed value of signal amplitude ,A=1



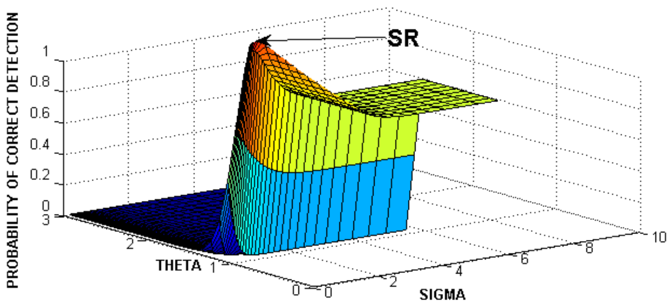
Fig(5). Probability of error for the (2,3) DMC as a function of SNRDB for fixed value of threshold theta=0.5



Fig(6). Probability of error for the (2,3) DMC as a function of theta for fixed values of noise standard deviation sigma.



Fig(7). (2,3) DMC probabilities of error as function of decision threshold theta.



Fig(8). Probability of correct detection as a function of threshold theta and noise standard deviation sigma.