Opening the Black Box: The Direct Stiffness Method Uncovered

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Abstract

This paper describes our use of common computer tools to help students unlock the mysteries embedded in structural analysis computer programs that are based on the Direct Stiffness Method. The methodology described in this paper is taught in an Advanced Structural Analysis course in the ABET-accredited civil engineering program at the United States Military Academy. This formulation is based on our strong belief that students must understand the basic assumptions inherent in the Direct Stiffness Method before they can confidently and competently perform computer-based structural analyses. We find that students understand these assumptions best when they have an opportunity to work through each major step in the Direct Stiffness Method by hand—aided by appropriate software to perform computations and matrix manipulations.

I. Introduction

In our Advanced Structural Analysis course at the U. S. Military Academy, students learn and apply the Direct Stiffness Method in three different blocks of instruction—Trusses, Beams, and Frames. In each block, we develop the direct stiffness formulation for the appropriate structural element, then have students work through one or more problems involving the analysis of a relatively simple structure. In every case, the students perform the Direct Stiffness Method manually, but use Excel spreadsheet software to perform matrix manipulations and MathCAD computational software to perform mathematical computations. This presented educational methodology is effective for peering inside any type Black Box tool as long as the key learning steps are clearly delineated and common computer tools are used only to perform the mundane, time consuming tasks.

Specifically, students solve each problem as follows:

- Use MathCAD to define *local* element stiffness matrices.
- Use MathCAD to transform the *local* element stiffness matrices to *global* element stiffness matrices.
- Use Excel to assemble (i.e., stack) the *global* element stiffness matrices into a *global* structure stiffness matrix.
- Use Excel to reorder the rows and columns of the *global* structure stiffness matrix to better solve for unknown values of displacements and external forces.
- Use MathCAD to solve for reactions and unknown nodal displacements.
- Use MathCAD to solve for *local* internal member forces.
- Use a commercial structural analysis software package to analyze the same structure, and compare the displacements and member forces with those obtained through the manual solution.

This paper describes this problem-solving methodology in detail. It provides an example of a typical student homework problem involving a manual solution of the Direct Stiffness Method, to include representative portions of the MathCAD worksheet and the Excel spreadsheet used to obtain the solution. The paper will also present student assessment data demonstrating the effectiveness of the methodology in promoting better understanding of: (1) the Direct Stiffness Method itself; (2) the relationship between the Direct Stiffness Method and classical structural analysis techniques like Slope Deflection and Moment Distribution; and (3) the Finite Element Method.

II. The Direct Stiffness Method

Table 1 outlines a step-by-step procedure for employing the direct stiffness method to analyze engineering structures (1). For simplicity, this table assumes that there are no fixed-end effects.

Table 1: The Direct Stiffness Method

Step	Procedure
1	Establish the global coordinate system.
2	Label all nodes and members.
3	Determine and write $\{F'\} = [k']\{\delta'\}$ for an element with respect to the local coordinate system.
4	Determine $\{F\} = [k]\{\delta\}$ for the element with respect to the global coordinate system using $[k] = [T]^T[k'][T]$
5	Assemble each element's [k] into the structure's force-displacement equation $\{P\}_{nx1} = [K]_{nxn}\{\delta\}_{nx1}$ where n is the total number of structural degrees of freedom (DOF)
6	Substitute known values of P and δ into: $\{P\}_{nx1} = [K]_{nxn}\{\delta\}_{nx1}$
7	Rearrange $\{P\}_{nx1} = [K]_{nxn}\{\delta\}_{nx1}$ such that known values of P are up (by rearranging rows) and known values of δ are down (by rearranging columns).
8	Solve for the displacements at the unrestrained DOF and the forces/moments at the restrained DOF.
9	Solve for the local element end forces using: $\{F'\} = [T][k]\{\delta\}$
10	Draw the Free-Body Diagram and deflected shape.

DOF = degrees of freedom
$\{\delta\}$ = global element displacement vector
$\{\delta'\}$ = local element displacement vector
$\{\delta\}_{nx1}$ = global structure displacement vector
$\{F\}$ = global element force vector
$\{F'\}$ = local element force vector
$[k]$ = global element stiffness matrix
$[k']$ = local element stiffness matrix
$[K]_{nxn}$ = global structure stiffness matrix
n = total number of structural degrees of freedom
$\{P\}_{nx1}$ = global structure force vector
$[T]$ = transformation matrix

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The procedure listed in Table 1 assumes the usual unknowns of a direct stiffness analysis problem: the displacements of unrestrained degrees of freedom (DOF), the reaction forces or moments at the restrained DOF, and the member end forces in the *local* coordinate system. To solve for these unknown quantities using matrix algebra, the analyst must properly identify the known forces and moments at the unrestrained DOF, specify the known displacements at the restrained DOF, and formulate a global structure stiffness matrix.

Understanding how to transform a *local* element stiffness matrix to a *global* element stiffness matrix, how a *global* structure stiffness matrix is generated from assembling or "stacking" *global* element stiffness matrices into it, and how to rearrange the *global* force-displacement equation to simplify the solution for the unknown displacements and reactions is critical to comprehending and employing successfully the Direct Stiffness Method. Unfortunately, performing these steps manually can be very tedious without the use of computer tools.

III. Homework Problem Example

We will use a typical homework problem—the truss structure shown in Figure 1—to describe how the students use the computer tools, MathCAD and Excel, to manually analyze a problem using the Direct Stiffness Method. Initially the students must establish the global coordinate system and label all nodes and members (Steps 1 and 2).

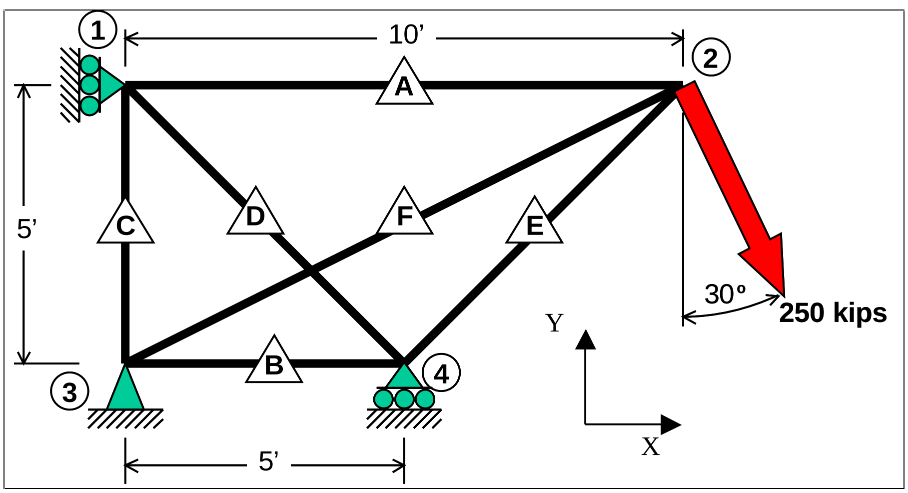

Figure 1. Truss Homework Problem

Figure 1 shows the resulting global coordinate system, the node numbers (1, 2, 3, and 4), and the member labels (A, B, C, D, E, and F). In this homework problem, we provide students with the node numbers and member labels, in order to simplify grading. We make no effort to number the nodes in a manner that would optimize the numerical solution, as we do not discuss this topic until later in the course. We also provide students with the cross-sectional area for each truss member, as listed in Table 2. All members are A572 Grade 50 steel.

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Table 2. Member Data

Members	Area
A, E	10 in^2
B	6 in^2
C, D, F	1 in^2

After accomplishing these preliminaries, the student is prepared to develop the *global* element stiffness matrix for each element using MathCAD. First the student develops the *local* element stiffness matrix, starting with the theoretically derived truss *local* element stiffness matrix (for element E, defined as $k_{E.local}$ in Figure 2). Using the proper orientation of the *global* x-coordinate system to the element's *local* x-coordinate system (i.e., counter-clockwise from global to local, Figure 3), the *local* element stiffness matrix is transformed to the *global* element stiffness matrix using the matrix operation: $[k] = [T]^T[k'][T]$ (Figure 2 for member E). The local x-coordinate system is always defined positively along the longitudinal axis of the member from the near (smaller) node to the far (larger) node. The transformation angle for members A, C, and E is provided in Figure 3.

Enter the required geometric and material properties for Element E

$\mathbf{Area} := 10\cdot\mathbf{in}^2$	$\mathbf{A} := \mathbf{Area}$	**evaluation follows ==>**	$\mathbf{A} = 10\,\mathbf{in}^2$
$\mathbf{Length} := 7.071\cdot\mathbf{ft}$	$\mathbf{L} := \mathbf{Length}$	**evaluation follows ==>**	$\mathbf{L} = 84.852\,\mathbf{in}$
$\mathbf{Modulus} := 29000\cdot\mathbf{ksi}$	$\mathbf{E} := \mathbf{Modulus}$	**evaluation follows ==>**	$\mathbf{E} = 2.9\times 10^4\,\mathbf{ksi}$
$\mathbf{Angle} := 225\cdot\mathbf{deg}$	$\theta := \mathbf{Angle}$	**evaluation follows ==>**	$\theta = 225\,\mathbf{deg}$

Define (& evaluate) the transformation and local element stiffness matrices:

$$\mathbf{T_E} := \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 & 0 \\ -\sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & \cos(\theta) & \sin(\theta) \\ 0 & 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix} \qquad \mathbf{T_E} = \begin{pmatrix} -0.707 & -0.707 & 0.000 & 0.000 \\ 0.707 & -0.707 & 0.000 & 0.000 \\ 0.000 & 0.000 & -0.707 & -0.707 \\ 0.000 & 0.000 & 0.707 & -0.707 \end{pmatrix}$$

$$\mathbf{k_{E.local}} := \begin{pmatrix} \frac{A\cdot E}{L} & 0 & -\frac{A\cdot E}{L} & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{A\cdot E}{L} & 0 & \frac{A\cdot E}{L} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{k_{E.local}} = \begin{pmatrix} 3.42\times 10^3 & 0 & -3.42\times 10^3 & 0 \\ 0 & 0 & 0 & 0 \\ -3.42\times 10^3 & 0 & 3.42\times 10^3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \frac{\mathbf{kips}}{\mathbf{in}}$$

Define (& evaluate) the global element stiffness matrix:

$$\mathbf{k_{E.global}} := \mathbf{T_E}^{\mathbf{T}}\cdot\mathbf{k_{E.local}}\cdot\mathbf{T_E} \qquad \mathbf{k_{E.global}} = \begin{pmatrix} 1708.86 & 1708.86 & -1708.86 & -1708.86 \\ 1708.86 & 1708.86 & -1708.86 & -1708.86 \\ -1708.86 & -1708.86 & 1708.86 & 1708.86 \\ -1708.86 & -1708.86 & 1708.86 & 1708.86 \end{pmatrix} \frac{\mathbf{kips}}{\mathbf{in}} \quad \mathbf{E}$$

Figure 2. Element E Local to Global Stiffness Matrix Transformation

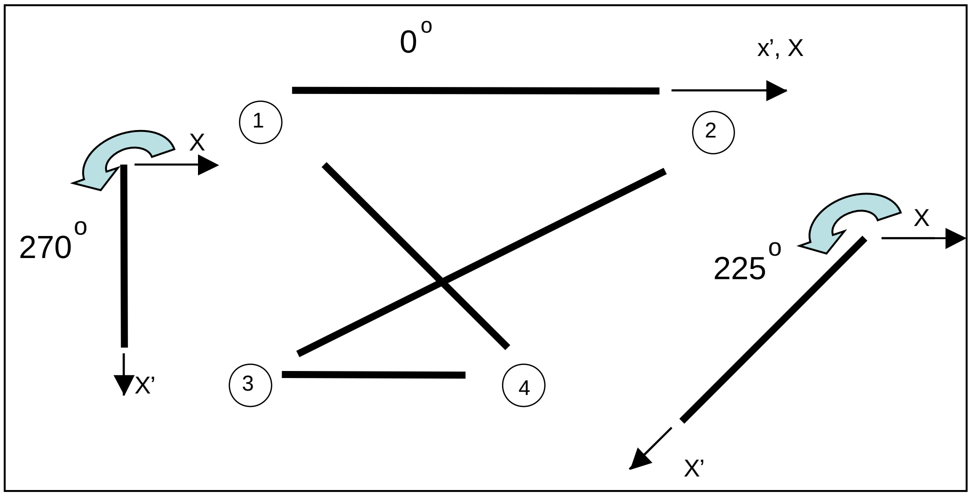

Figure 3. The Disconnected Structure With Transformation Angles for A, C, and E

Once all of the global element stiffness matrices have been determined in MathCAD, it is time to assemble the global structure stiffness matrix (Step 5). Before this can happen, we must size the global structure stiffness matrix. The dimensions of this square matrix are a function of the number of nodes times the number of DOF at each node. For a truss structure, each node has two DOF--translation in the x and y directions. Therefore, the global structure stiffness matrix for this problem is 8x8, determined as follows:

$$(n = 4 \text{ nodes}) \times (2 \text{ DOF}) = 8$$

Since stacking of matrices is best done in a spreadsheet, and the node numbers at the end of members are not consecutive, the global element stiffness matrices are copied into Excel via the Windows Clipboard, and the size of the global structure stiffness matrix is determined (Figure 4).

You will notice in Figure 4 that the *global* force-displacement relationship for each element is provided to express how each element is connected in the structure and to assist the student in properly stacking the *global* element stiffness matrices into the *global* structure stiffness matrix. Elements A, B, and F can be stacked as complete 4x4 matrices since their node numbers are consecutive, while elements C, D, and E must be stacked in 2x2 chunks. The ability in Excel to easily add values to existing cell values in a matrix and later rearrange rows and columns makes performing Steps 5-7 efficient in Excel. While the global structure stiffness matrix is empty, simply copying and pasting is satisfactory. Once there are stiffness values where additional stiffness values are to be pasted, we need to use the command Paste Special to add the new values to the old ones. Adding the stiffness values represents the connecting of the elements at the node. Normally we would stack Elements A through F in order to ensure none are missed, but for the sake of clarity, Figure 5 shows the stacking only of Element E with 2x2 chunks. The upper left cell of the upper left 2x2 chunk of Element E, F2x - u2, needs to be pasted in the P2x – u2 cell of the global structure stiffness matrix. The listing of the global displacements underneath the matrix rather than only along the right side of the matrix greatly assists in highlighting the proper cell for pasting operations.

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CE-491 **Advanced Structural Analysis**

Stack the Global Structure Stiffness Matrix

Import the global element stiffness matrix from MathCAD using Excel's Paste Special "uncoded text"

Element A

F1x	2416.67	0.00	-2416.67	0.00	u1
F1y	0.00	0.00	0.00	0.00	v1
F2x	-2416.67	0.00	2416.67	0.00	u2
F2y	0.00	0.00	0.00	0.00	v2

Element B

F3x	2900.00	0.00	-2900.00	0.00	u3
F3y	0.00	0.00	0.00	0.00	v3
F4x	-2900.00	0.00	2900.00	0.00	u4
F4y	0.00	0.00	0.00	0.00	v4

Element C

F1x	0.00	0.00	0.00	0.00	u1
F1y	0.00	483.33	0.00	-483.33	v1
F3x	0.00	0.00	0.00	0.00	u3
F3y	0.00	-483.33	0.00	483.33	v3

Element D

F1x	170.89	-170.89	-170.89	170.89	u1
F1y	-170.89	170.89	170.89	-170.89	v1
F4x	-170.89	170.89	170.89	-170.89	u4
F4y	170.89	-170.89	-170.89	170.89	v4

Element E

F2x	1708.86	1708.86	-1708.86	-1708.86	u2
F2y	1708.86	1708.86	-1708.86	-1708.86	v2
F4x	-1708.86	-1708.86	1708.86	1708.86	u4
F4y	-1708.86	-1708.86	1708.86	1708.86	v4

Element F

F2x	172.94	86.45	-172.94	-86.45	u2
F2y	86.45	43.22	-86.45	-43.22	v2
F3x	-172.94	-86.45	172.94	86.45	u3
F3y	-86.45	-43.22	86.45	43.22	v3

Step 5. Stack the Global Structure Stiffness Method (using Copy, Paste Special, Values, Add)

P1x									u1
P1y									v1
P2x									u2
P2y									v2
P3x									u3
P3y									v3
P4x									u4
P4y									v4
	u1	v1	u2	v2	u3	v3	u4	v4	

Figure 4. Excel Worksheet Prior to Assembling the Global Structure Stiffness Matrix

Step 5. Stack the Global Structure Stiffness Method (using Copy, Paste Special, Values, Add)

P1x									u1
P1y									v1
P2x			1708.86	1708.86			-1708.86	-1708.86	u2
P2y			1708.86	1708.86			-1708.86	-1708.86	v2
P3x									u3
P3y									v3
P4x			-1708.86	-1708.86			1708.86	1708.86	u4
P4y			-1708.86	-1708.86			1708.86	1708.86	v4
	u1	v1	u2	v2	u3	v3	u4	v4	

Figure 5. Stacking of Element E into the Global Structure Stiffness Matrix.

Figure 6 shows the stacking of Element F, while Figure 7 shows the completed assembly of the global structure stiffness matrix for this problem. Note Element F being stacked as a 4x4 matrix and adding to existing stiffness values from the first 2x2 chunk when pasting Element F.

Step 5. Stack the Global Structure Stiffness Method (using Copy, Paste Special, Values, Add)									
P1x									u1
P1y									v1
P2x			1881.80	1795.31	-172.94	-86.45	-1708.86	-1708.86	u2
P2y			1795.31	1752.08	-86.45	-43.22	-1708.86	-1708.86	v2
P3x			-172.94	-86.45	172.94	86.45			u3
P3y			-86.45	-43.22	86.45	43.22			v3
P4x			-1708.86	-1708.86			1708.86	1708.86	u4
P4y			-1708.86	-1708.86			1708.86	1708.86	v4
	u1	v1	u2	v2	u3	v3	u4	v4	

Figure 6. Next Stacking Element F into the Global Structure Stiffness Matrix

Step 5. Stack the Global Structure Stiffness Method (using Copy, Paste Special, Values, Add)									
P1x	2587.56	-170.89	-2416.67	0.00	0.00	0.00	-170.89	170.89	u1
P1y	-170.89	654.22	0.00	0.00	0.00	-483.33	170.89	-170.89	v1
P2x	-2416.67	0.00	4298.47	1795.31	-172.94	-86.45	-1708.86	-1708.86	u2
P2y	0.00	0.00	1795.31	1752.08	-86.45	-43.22	-1708.86	-1708.86	v2
P3x	0.00	0.00	-172.94	-86.45	3072.94	86.45	-2900.00	0.00	u3
P3y	0.00	-483.33	-86.45	-43.22	86.45	526.55	0.00	0.00	v3
P4x	-170.89	170.89	-1708.86	-1708.86	-2900.00	0.00	4779.75	1537.97	u4
P4y	170.89	-170.89	-1708.86	-1708.86	0.00	0.00	1537.97	1879.75	v4
	u1	v1	u2	v2	u3	v3	u4	v4	

Figure 7. Completely Assembled Global Structure Stiffness Matrix

Step 6 is substituting known values of external forces and displacements. The key learning point here is that for each DOF, we must know either the external force or the displacement. Determining these values assists in determining the final order of rows and columns in Step 7 as we prepare to solve for the unknown reactions and displacements. The known forces are at nodes 1, 2, and 4. The known displacements are at nodes 1, 3, and 4. At node 1 (Figure 1), we know the displacement in the x-direction is zero, since the roller prevents movement in the x-direction and no support movement is specified in the problem. Since we do not know the displacement in the y-direction, we must know the external force. With no y-direction force shown at node 1, the external force P1y must be zero. At node 2, there are no support conditions preventing movement, so we do not know the displacements, but we do know the external forces. The 250 kip force shown in Figure 1 must be broken into global x and y components resulting in P2x = 125 kips and P2y = -216.5 kips. Similarly, at node 4 the displacement is unknown in the x-direction, so the external force must be known. Since no external force is shown, we know that P4x = 0. Displacement in the y-direction is zero, due to the roller support. The displacements in both the x- and y-directions are zero at node 3. Having determined the known forces and displacements, we annotate them on the spreadsheet, as shown in Figure 8.

Step 5. Stack the Global Structure Stiffness Method (using Copy, Paste Special, Values, Add)									
P1x	2587.56	-170.89	-2416.67	0.00	0.00	0.00	-170.89	170.89	u1=0
P1y=0	-170.89	654.22	0.00	0.00	0.00	-483.33	170.89	-170.89	v1
P2x=125	-2416.67	0.00	4298.47	1795.31	-172.94	-86.45	-1708.86	-1708.86	u2
P2y=-216.5	0.00	0.00	1795.31	1752.08	-86.45	-43.22	-1708.86	-1708.86	v2
P3x	0.00	0.00	-172.94	-86.45	3072.94	86.45	-2900.00	0.00	u3=0
P3y	0.00	-483.33	-86.45	-43.22	86.45	526.55	0.00	0.00	v3=0
P4x=0	-170.89	170.89	-1708.86	-1708.86	-2900.00	0.00	4779.75	1537.97	u4
P4y	170.89	-170.89	-1708.86	-1708.86	0.00	0.00	1537.97	1879.75	v4=0
	u1	v1	u2	v2	u3	v3	u4	v4	

Figure 8. The Known External Forces and Displacements Included

Figure 8 is the matrix equation {P} = [K] {δ} , a system of eight linear equations representing the relationship between forces and displacements in the structure shown in Figure 1. The order of the variables u1 through v4 in the equations can be rearranged as long as the individual equations remain unchanged. Such reordering of variables can be achieved by changing the order of rows *and* columns, such that the final matrix is also symmetric, square, and singular, with non-negative values on the diagonal. Note these qualities in the assembled matrix in Figure 8. The substitution of known values for external forces and displacements guides the movement of rows and then columns to ensure we reach the desired goal. Note that we are systematic in how we move rows (and later columns). We move the rows with known external forces *up*, in order. Note that P1y, P2x, P2y and P4x remain in the same order with respect to each other. Similarly P1x, P3x, P3y and P4y also remain in the same order with respect to each other (Figure 9).

Step 7a. Known Values of P up (rearrange rows only)									
P1y=0	-170.89	654.22	0.00	0.00	0.00	-483.33	170.89	-170.89	v1
P2x=125	-2416.67	0.00	4298.47	1795.31	-172.94	-86.45	-1708.86	-1708.86	u2
P2y=-216.5	0.00	0.00	1795.31	1752.08	-86.45	-43.22	-1708.86	-1708.86	v2
P4x=0	-170.89	170.89	-1708.86	-1708.86	-2900.00	0.00	4779.75	1537.97	u4
P1x	2587.56	-170.89	-2416.67	0.00	0.00	0.00	-170.89	170.89	u1=0
P3x	0.00	0.00	-172.94	-86.45	3072.94	86.45	-2900.00	0.00	u3=0
P3y	0.00	-483.33	-86.45	-43.22	86.45	526.55	0.00	0.00	v3=0
P4y	170.89	-170.89	-1708.86	-1708.86	0.00	0.00	1537.97	1879.75	v4=0
	u1=0	v1	u2	v2	u3=0	v3=0	u4	v4=0	

Figure 9. Movement of Rows With Known External Forces Up

At this point, many students often think they have finished rearranging {P} = [K] {δ}, since the far right column now shows the known displacements at the bottom. Upon review of the matrix, however, we notice that it is square, but no longer is it symmetric, nor does it have a non-negative diagonal. The listing of the displacements below the matrix also highlights that the columns still need to be rearranged. Figure 10 shows the movement of the columns with known displacements at the right, resulting in a symmetric, square, singular matrix, with non-negative values on the diagonal. Bold lines have been added to help clarify the next step in the problem solution.

Step 7b. Known Values of Displacements down (rearrange columns only)									
P1y=0	654.22	0.00	0.00	170.89	-170.89	0.00	-483.33	-170.89	v1
P2x=125	0.00	4298.47	1795.31	-1708.86	-2416.67	-172.94	-86.45	-1708.86	u2
P2y=-216.5	0.00	1795.31	1752.08	-1708.86	0.00	-86.45	-43.22	-1708.86	v2
P4x=0	170.89	-1708.86	-1708.86	4779.75	-170.89	-2900.00	0.00	1537.97	u4
P1x	-170.89	-2416.67	0.00	-170.89	2587.56	0.00	0.00	170.89	u1=0
P3x	0.00	-172.94	-86.45	-2900.00	0.00	3072.94	86.45	0.00	u3=0
P3y	-483.33	-86.45	-43.22	0.00	0.00	86.45	526.55	0.00	v3=0
P4y	-170.89	-1708.86	-1708.86	1537.97	170.89	0.00	0.00	1879.75	v4=0
	v1	u2	v2	u4	u1=0	u3=0	v3=0	v4=0	
Import into MathCAD									

Figure 10. Completely Rearranged Global Structure Stiffness Matrix

It is now time to return to MathCAD to solve for the unknown reactions and displacements. The reordering of rows and columns has allowed for the partitioning of the force-displacement equations into the following two equations each representing four equations:

$$\begin{Bmatrix} P_f \\ P_s \end{Bmatrix} = \begin{bmatrix} K_{ff} & K_{fs} \\ K_{sf} & K_{ss} \end{bmatrix} \begin{Bmatrix} \delta_f \\ \delta_s \end{Bmatrix} \qquad (1)$$

A close review of Equation 1, Figure 8, and Figure 10 indicates that the only unknowns are δ_f and P_s ; therefore, the first equation line in Equation 1 has only one unknown, which can be solved directly by manipulating the equation as shown:

$$\{\delta_f\} = [K_{ff}]^{-1}\left(\{P_f\} - [K_{fs}]\{\delta_s\}\right) \qquad (2)$$

Substituting δ_f into the second equation line, P_s can be directly solved for:

$$\{P_s\} = [K_{sf}]\{\delta_f\} + [K_{ss}]\{\delta_s\} \qquad (3)$$

These mathematical computations (Step 8) are completed in the MathCAD worksheet by copying the partitioned matrices K_{ff}, K_{fs}, K_{sf}, and K_{ss}, and the known external forces P_f and displacements δ_s from the spreadsheet back to the MathCAD worksheet (Figure 11). Again, this transfer of data is easily accomplished via the Windows Clipboard. In the worksheet, displacement values are further defined by their symbolic representation to simplify the calculation of *local* element end forces. The calculation of local element end forces (Step 9) is accomplished using available information for each member in the existing MathCAD worksheet and the following equation (Figure 12):

$$\{F'\} = [T][K]\{\delta\} \qquad (4)$$

Import known matrix quantities from Excel.

$$K_{ff} := \begin{pmatrix} 654.22 & 0.00 & 0.00 & 170.89 \\ 0.00 & 4298.47 & 1795.31 & -1708.86 \\ 0.00 & 1795.31 & 1752.08 & -1708.86 \\ 170.89 & -1708.86 & -1708.86 & 4779.75 \end{pmatrix} \cdot \frac{kips}{in}$$

$$K_{fs} := \begin{pmatrix} -170.89 & 0.00 & -483.33 & -170.89 \\ -2416.67 & -172.94 & -86.45 & -1708.86 \\ 0.00 & -86.45 & -43.22 & -1708.86 \\ -170.89 & -2900.00 & 0.00 & 1537.97 \end{pmatrix} \cdot \frac{kips}{in}$$

$$K_{sf} := \begin{pmatrix} -170.89 & -2416.67 & 0.00 & -170.89 \\ 0.00 & -172.94 & -86.45 & -2900.00 \\ -483.33 & -86.45 & -43.22 & 0.00 \\ -170.89 & -1708.86 & -1708.86 & 1537.97 \end{pmatrix} \cdot \frac{kips}{in}$$

$$K_{ss} := \begin{pmatrix} 2587.56 & 0.00 & 0.00 & 170.89 \\ 0.00 & 3072.94 & 86.45 & 0.00 \\ 0.00 & 86.45 & 526.55 & 0.00 \\ 170.89 & 0.00 & 0.00 & 1879.75 \end{pmatrix} \cdot \frac{kips}{in}$$

Input known Forces and Displacements

$$P_f := \begin{pmatrix} 0 \\ 125 \\ -216.5 \\ 0 \end{pmatrix} \cdot kips \qquad \Delta_s := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \cdot in \qquad \begin{pmatrix} u_1 \\ u_3 \\ v_3 \\ v_4 \end{pmatrix} := \Delta_s$$

Calculate unknown quantities. Express unknown forces in kips - - and unknown displacements in inches.

$$\Delta_f := K_{ff}^{-1} \cdot (P_f - K_{fs} \cdot \Delta_s) \qquad \Delta_f = \begin{pmatrix} 0.01849 \\ 0.14227 \\ -0.33838 \\ -0.07077 \end{pmatrix} in$$

$$\begin{pmatrix} v_1 \\ u_2 \\ v_2 \\ u_4 \end{pmatrix} := \Delta_f \qquad P_s := K_{sf} \cdot \Delta_f + K_{ss} \cdot \Delta_s \qquad P_s = \begin{pmatrix} -334.8878 \\ 209.8878 \\ -6.6099 \\ 223.1099 \end{pmatrix} kips$$

Figure 11. Using MathCAD to Solve for Unknown Displacements and Reactions

Students can now complete Step 10 by drawing free-body diagrams of each member, placing the *local* member end forces on them, and answering the question: Does the magnitude and direction of the force in the member make sense, based on how we think the structure should behave under the given loading?

The homework assignment is not complete until the students analyze the same structure with a commercial analysis package and compare the two sets of results. We let them know ahead of time that the answers should be identical. When the answers are *not* identical, the discrepancy is often caused by round-off errors in the manual method. These errors can occur in input parameters--member and material properties, transformation angles, and forces. Round-off error can also occur when copying values between MathCAD and Excel. Due to an apparent software glitch, when numbers are displayed in scientific notation in MathCAD, only the displayed digits are actually transferred to the spreadsheet in a copy-paste operation.

Define the local element displacement vectors and then Evaluate the local element force vectors.

Element A

$$\Delta_A := \begin{pmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{pmatrix} \quad \Delta_A = \begin{pmatrix} 0 \\ 0.018 \\ 0.142 \\ -0.338 \end{pmatrix} \mathbf{in} \quad \mathbf{F'_A := AT \cdot Ak_{global}} \cdot \Delta_A \quad \mathbf{F'_A} = \begin{pmatrix} -343.822 \\ 0.000 \\ 343.822 \\ 0.000 \end{pmatrix} \mathbf{kips}$$

Element B

$$\Delta_B := \begin{pmatrix} u_3 \\ v_3 \\ u_4 \\ v_4 \end{pmatrix} \quad \Delta_B = \begin{pmatrix} 0 \\ 0 \\ -0.071 \\ 0 \end{pmatrix} \mathbf{in} \quad \mathbf{F'_B := BT \cdot Bk_{global}} \cdot \Delta_B \quad \mathbf{F'_B} = \begin{pmatrix} 205.240 \\ 0.000 \\ -205.240 \\ 0.000 \end{pmatrix} \mathbf{kips}$$

Element C

$$\Delta_C := \begin{pmatrix} u_1 \\ v_1 \\ u_3 \\ v_3 \end{pmatrix} \quad \Delta_C = \begin{pmatrix} 0 \\ 0.018 \\ 0 \\ 0 \end{pmatrix} \mathbf{in} \quad \mathbf{F'_C := CT \cdot Ck_{global}} \cdot \Delta_C \quad \mathbf{F'_C} = \begin{pmatrix} -8.935 \\ 0.000 \\ 8.935 \\ 0.000 \end{pmatrix} \mathbf{kips}$$

Element D

$$\Delta_D := \begin{pmatrix} u_1 \\ v_1 \\ u_4 \\ v_4 \end{pmatrix} \quad \Delta_D = \begin{pmatrix} 0 \\ 0.018 \\ -0.071 \\ 0 \end{pmatrix} \mathbf{in} \quad \mathbf{F'_D := DT \cdot Dk_{global}} \cdot \Delta_D \quad \mathbf{F'_D} = \begin{pmatrix} 12.636 \\ 0.000 \\ -12.636 \\ 0.000 \end{pmatrix} \mathbf{kips}$$

Element E

$$\Delta_E := \begin{pmatrix} u_2 \\ v_2 \\ u_4 \\ v_4 \end{pmatrix} \quad \Delta_E = \begin{pmatrix} 0.142 \\ -0.338 \\ -0.071 \\ 0 \end{pmatrix} \mathbf{in} \quad \mathbf{F'_E := T_E \cdot k_{E.global}} \cdot \Delta_E \quad \mathbf{F'_E} = \begin{pmatrix} 302.888 \\ -0.000 \\ -302.888 \\ 0.000 \end{pmatrix} \mathbf{kips}$$

Element F

$$\Delta_F := \begin{pmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{pmatrix} \quad \Delta_F = \begin{pmatrix} 0.142 \\ -0.338 \\ 0 \\ 0 \end{pmatrix} \mathbf{in} \quad \mathbf{F'_F := FT \cdot Fk_{global}} \cdot \Delta_F \quad \mathbf{F'_F} = \begin{pmatrix} 5.197 \\ 0.000 \\ -5.197 \\ 0.000 \end{pmatrix} \mathbf{kips}$$

Figure 12. Local Member End Forces

Knowing that the two sets of answers should agree leads most students to review their work carefully for possible errors. Many times they will discover errors in both their use of the commercial analysis package and manual solution. In the beam and frame analysis blocks, we take the process one step farther, also giving students the opportunity to solve their homework problems with the slope deflection and moment distribution methods. By performing these classical solutions in conjunction with the more modern computer-based methods, students often rediscover concepts that were never fully learned during the introductory structural analysis course.

At the end of the Advanced Structural Analysis course, we introduce Finite Element Methods using the constant strain triangle element. And though the derivation of the stiffness element is obviously different from the Direct Stiffness method, the matrix computations are very similar. Here we use the same technique of solving a small problem manually and with a commercial program. MathCAD and Excel are once again used to perform the computations necessary for determining stresses in a finite element.

IV. Assessment

We have observed many ways in which manually solving the Direct Stiffness Method with common computer tools has helped our students learn about how the standard structural analysis packages work and about structural behavior. We find that most students entering the Advanced Structural Analysis course are not completely comfortable with the manual analysis methods taught in their introductory structural analysis course. Many students do not recognize the links between slope deflection, moment distribution and the direct stiffness method. Most are unsure of the basic assumptions underlying the commercial programs that they are already using so they can analyze/design structures in their steel and concrete design courses.

Manually analyzing small structures with the direct stiffness method helps students understand the importance of each variable they input into a commercial software package and how it affects the solution mathematically. They learn about the importance of accuracy and the hazards of round-off error. By requiring students to compare their manual solutions with those obtained from a commercial analysis program, we give them an opportunity to familiarize with the common errors associated with using any commercial program--inaccurate modeling of support conditions, missing input, and not assessing whether the results make sense. Indeed, in checking their answers, many students discover that they have made errors in *both* solutions. The entire experience clearly accomplishes the goal of ensuring that students understand what is happening within the “black box” structural analysis packages they are using.

We have assessed the effectiveness of this attempt to “open the black box” principally through the use of our institution’s course-end feedback system. This system is administered entirely over the worldwide web and features a small number of USMA-standard survey questions, supplemented by department-specific and course-specific questions of our own choosing. Students respond to these questions using a scale of 1 (strongly disagree) to 5 (strongly agree). For the USMA-standard questions, this system allows us to compare our own students’ survey responses to those of all other students at the institution. More importantly, the inclusion of course-specific questions allows us to survey our students about their achievement of specific course objectives.

On their course-end feedback from the fall semester of 2000, our students were extremely supportive of using manual methods to strengthen their understanding of the Direct Stiffness Method. Relevant data are provided in Table 3. The upper half of the table shows students’ responses to USMA-standard questions that relate specifically to the quality of instruction and student learning; nonetheless, we also believe these particular responses also reflect student satisfaction with the instructional methodology described in this paper. In the lower half of the table, Questions D6, 7, 8, 12, 13, and 14 shows students’ self-assessments of their own ability to achieve the DSM-related objectives of the course. The high average responses—above 4.2 in every case—reflects a high level of confidence in a challenging subject. In response to a “free text” question, one student appropriately summarized the learning experience as follows:

"This course was one of the best courses that I have ever had that primarily focused on theoretical understanding…. Many classes with similar goals have failed to meet this objective. Specifically, this course used a common sense approach to relatively difficult mathematical derivations and then immediately applied them in a simple problem. Based on this, they moved to more complex problems. I believe that this is the type of teaching that I will remember in the long run."

Table 3: Students' Course-End Feedback Responses, Fall 2000

Q.#	**USMA Standard Questions**	**Course Average**	**USMA Average**
A2	This instructor used effective techniques	4.78	4.19
A6	Motivation to learn increased	4.47	3.94
B1	The instructor stimulated my thinking	4.65	4.19
B2	Critical thinking ability increased	4.57	4.00
	Course-Specific Questions	**Course Average**	
D6	I can analyze trusses, beams and frames (prismatic or non-prismatic) using the DSM	4.55	
D7	I can analyze beams (prismatic or non-prismatic) using Moment Distribution	4.27	
D8	I can analyze beams (prismatic or non-prismatic) using Slope Deflection	4.25	
D12	I can calculate the approximate displacements/stresses using the FEM	4.29	
D13	I can solve 2D/plane stress problems using a FEM program/interpret results	4.29	
D14	I can create/analyze 2D/3D models of trusses, beams and frames using commercial packages & interpret results	4.45	
D19	Advanced Structural Analysis spent too much time reviewing material that I mastered in Structural Analysis	2.44	
D20	Advanced Structural Analysis allowed me to finally master material from Structural Analysis - made more sense and fit together	4.30	

V. Conclusion

We have described our successful use of common computer tools to help students unlock the mysteries embedded in structural analysis computer programs that are based on the Direct Stiffness Method. We believe that students must understand the basic assumptions inherent in the method before they can confidently and competently perform computer-based structural analyses. We find that students understand these assumptions best when they have an opportunity to work through each major step in the Direct Stiffness Method by hand—aided by appropriate software like MathCAD and Excel to perform computations and matrix manipulations.

Student assessment data supports our contention that solving simple problems first by hand and then comparing results with commercial structural analysis packages promotes a strong understanding of the cornerstone of structural analysis--the Direct Stiffness Method. An added benefit was the thorough understanding of the techniques and tools necessary to take the initial step toward understanding Finite Element Methods.

Bibliography

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Symbols

DOF	degrees of freedom
$\{\delta\}$	global element displacement vector
$\{\delta'\}$	local element displacement vector
$\{\delta\}_{nx1}$	global structure displacement vector
$\{F\}$	global element force vector
$\{F'\}$	local element force vector
Fx1	member end force in global x-direction at node 1
Fy1	member end force in global y-direction at node 1
[k]	global element stiffness matrix
[k']	local element stiffness matrix
$[K]_{nxn}$	global structure stiffness matrix
n	total number of structural degrees of freedom
$\{P\}_{nx1}$	global structure force vector
Px1	structure force/reaction/moment in global x-direction at node 1
Py1	structure force/reaction/moment in global y-direction at node 1
[T]	transformation matrix
u1	displacement in the global x-direction at node 1

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