Problem Solving in Statics and Dynamics: A Proposal for a Structured Approach

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Abstract

It has been the authors' experience that, even with the most careful presentation, students perceive the solutions to problems in statics, and especially dynamics, to be a "hodgepodge" of techniques and tricks. This is also born out by feedback the author's have received from colleagues and from the approximately 50 expert reviewers of the statics and dynamics books that the authors are currently writing. Interestingly, this state of affairs has changed little in the more than 40 years since the publication of the first editions of Meriam 1952, Shames in 1959, and Beer and Johnston in 1962 changed the way engineering mechanics was taught.

In this paper, we present a formal procedure that we are using in the statics and dynamics texts we are writing. The procedure we are using is not new in that it derives from the approach used in more advanced mechanics courses in which the equations needed to solve problems derive from three areas or places:

1. balance laws (e.g., momentum,* angular momentum, energy, etc.);
2. constitutive equations (e.g., friction laws, drag laws, etc.); and
3. kinematics or constraints.

On the other hand, it is new in the sense that we are applying it in freshman and sophomore-level mechanics courses. We will close with several examples from statics and dynamics for which we use our approach.

Introduction

Engineering courses in mechanics differ from their companion courses offered by physics departments in that, in engineering, there is a strong emphasis on issues concerning engineering standards and design on the one hand and on the acquisition of effective problem solving techniques, on the other. In this paper we focus our attention on how problem solving is treated and fostered in current freshman/sophomore-level mechanics books in statics and dynamics. Specifically, we are interested in investigating the notion of structured problem solving, where, by structured problem solving, we mean an approach to problem solving that can be applied almost universally to mechanics problems and helps the student in avoiding a trial and error approach to the assembling of the equations governing a problem's solution. Our motivation is that, in our experience, students perceive the solutions to problems in statics and, especially, dynamics to be a "hodgepodge" of tricks

*Of course, the balance of momentum as given by Euler's First Law for a particle, i.e., $\vec{F} = \dot{\vec{p}}$, where $\vec{p}$ is the particle's momentum, contains the "equilibrium equations" of statics as a special case.

that are very much problem specific instead of generally applicable principles. This is also born out by feedback we have received from colleagues and from the approximately 50 expert reviewers of the statics and dynamics books that the we are currently writing [1, 2]. Interestingly, it appears that the teaching of problem solving has changed little in the more than 40 years since the publication of the first editions of Meriam 1952, Shames in 1959, and Beer and Johnston in 1962 changed the way engineering mechanics was taught. Furthermore, it appears that indeed most books, while making an effort to develop problem solving skills, do not focus enough on the development of a problem solving framework that can be applied to all problem concerning the statics and kinetics of particles and rigid bodies.

The most successful books currently available on the market [3–12] all have outstanding features and have been responsible for educating many generations of students, including the authors of this paper. However, it has been our experience that these textbooks, with the exception of the recent books by Tongue and Sheppard [11, 12] (more on these books below), do not explicitly present a structured problem solving approach that can guide a student through *any* problem they will encounter in mechanics, not just statics and dynamics. That is, current approaches tend to present mechanics as a host of special cases and leave students wondering where to begin a problem when it does not fit into the framework of one of those cases. In addition, current approaches leave students wondering when they have enough equations to solve for the unknowns in a problem. With this in mind, this paper offers a structured approach to problem solving that we feel will serve students throughout their careers. While we freely admit that this approach does not offer anything new (we aren't presenting a new means of formulating governing equations), it is, we believe, the first time anyone has tried to implement a universally applicable problem solving methodology in sophomore-level mechanics courses.

We should mention that the recent books by Tongue and Sheppard [11, 12] make the development of structured problem solving one of their main objectives. We view their developments in this area as a welcome advance in defining and promoting problem solving skills. To develop structured problem solving skills, they suggest a six-step program (though seven steps are actually listed) and, for the most part, the solution steps are followed consistently throughout the book. Furthermore, Tongue and Sheppard do make an effort to explicitly discuss important modeling assumptions. However, many of the assumptions made in words during their "Assume" step are not always given a corresponding mathematical form and are not verified a posteriori, i.e., once a candidate solution corresponding to those assumptions is available. While we do not want to turn our presentation into a review of their text, we do find that their structured approach falls a bit short of what we feel is one of the desired goals, that is, the goal of not having to "forage" for additional equations if you get to some point and discover that you do not have enough. Unfortunately, there are a number of solved examples in which we find this to be the case.

We will begin by outlining our approach to problem solving and then we will report two examples, one in statics and one in dynamics, of how our approach is practically implemented. The article is then concluded with a brief summary and discussion.

Our Approach to Problem Solving

Models and the Modeling Process

We emphasize the process of modeling of mechanical systems, that is, the process by which one takes a *real system* and, via a number of assumptions, defines a corresponding *mathematically tractable system* whose behavior can be predicted. In every problem solved, we are careful to point out the assumptions used in the solution of that problem and we take every opportunity to remove assumptions as the introduction of material allows. This allows us to compare the response of the same system under two or more different sets of assumptions so that the efficacy of each assumption can be determined. Modeling is also being emphasized by selecting a few problems for *re-analysis* throughout the books, each time using a slightly relaxed set of assumptions so that students can explore and contrast the outcomes of different models.

We have also emphasized that in statics and dynamics there is a close relationship between modeling and problem-solving skills. This relationship has been reinforced by constructing a *modeling-based problem-solving strategy*. This effort was also motivated by our direct observation of students' homework and exam solving practices in which "pattern matching" and "coming up with *any* n equations in n unknowns" seem to be the guiding principles for many students. To turn this lack of organization into effective problem solving, we have created a solution paradigm based on the fact that *any* model of equilibrium and motion (at least within the confines of classical mechanics) is constructed using three basic elements:

(i) the Newton-Euler equations and/or balance laws,

(ii) material or constitutive equations, and

(iii) the kinematic equations,

where by Newton-Euler equations and/or balance laws, we mean Newton's second law for particles, the rigid body rotation equations as developed Euler, and balance laws for energy and momentum that are derived from them. This solution paradigm is universally practiced in graduate courses as well as in real-life engineering modeling. Our approach emphasizes to the students that exhausting each of the three items mentioned above results in a *complete system of independent equations* (i.e., not just any n equations in n unknowns) leading to the solution of the problem. This approach removes some of the mystery as to where to begin to write the equations in dynamics since students often just keep writing equations hoping that they will come up with enough of them. In addition, it gives the teaching of statics and dynamics the same mathematical and conceptual foundation as other mechanics courses that the students encounter (e.g., strength of materials, continuum mechanics, elasticity).

Our Five Steps of Problem Solving

To put these three basic elements of modeling into a structured framework for problem solving, we have created a five-step problem solving process that is used, *without exception*, in every equilibrium and kinetics problem that we solve. The five steps are described below in the order in which they are always used.

Road Map This is a summary of the *given* pieces of information, an extremely concise statement of what needs to be found, and an outline of the overall solution strategy.

Modeling This is a discussion of the assumptions and idealizations necessary to make the problem tractable. For example, are we including or neglecting effects such as friction, air drag, and nonlinearities? Whether or not we are including these effects, we make it very clear how sophomore-level statics and dynamics deals with them and are careful to discuss the fact that our solution is restricted to the particular model system that has been analyzed. The free-body diagram (FBD), a visual sketch of the forces acting on a body, is the central element of the modeling feature and is included here.

Governing Equations The governing equations are *all* the equations needed for the solution of the problem. These equations are organized according to the paradigm discussed earlier, that is, (*i*) Newton-Euler/Balance Equations, (*ii*) Material Models, and (*iii*) Kinematic Equations. In statics, the Newton-Euler/Balance Equations are called Equilibrium Equations. At this point in our approach we encourage students to verify that the number of unknowns they have previously identified equals the number of equations they have written in the Governing Equations section.

Computation The manipulation and solution of the governing equations.

Discussion & Verification A verification of whether the solution is correct and a discussion of the solution's physical meaning with an emphasis on the role played by the assumptions stated under the *Modeling* heading.

In those problems where the writing of governing equations alternates with computations (e.g., static analysis of truss structures), the third and fourth steps may be grouped together as "Governing Equations and Computation". This five-step procedure is presented to the students as a *universal* problem solving procedure to be applied to any problem concerning forces and motion both in undergraduate and graduate courses, as well as in research and development. We feel that this approach to problem solving is quite different from what can actually be found in current textbooks, though we realize that many engineering faculty may already teach problem solving using this structure. This is a recognition that, in our classroom teaching, most of us do deviate, to one extent or another, from the presentation found in textbooks.

Additional Remarks

We also want to mention some other aspects of our modeling pedagogy that we feel are important in developing the skills of students in statics and dynamics.

The figures associated with our problem statements almost never include arrows representing velocities, forces, or other vectorial aspects of the given problem. We expect the student to, with our guidance via example problems, determine the *model* for the system based entirely on the problem description and a simple figure portraying the physical system. In addition, our models do not consist entirely of linear springs, Coulomb friction, and negligible air drag. We present models with nonlinear elastic elements, different models of air drag, and the like. It is important for the student to understand two very important things when modeling physical systems:

1. the real world contains ugly, nasty elements that sometimes can and sometimes cannot be included in a model and choices need to be made as to how complicated a model needs to be;

2. the model they create and solve is just that: a *model*; that is, it is not the *real* system and assessments need to be made as to the accuracy and adequacy of their model.

As part of this, we should also mention that even the choice of component system (e.g., polar, path, Cartesian, etc.) is an essential part of the modeling process for any given problem.

We now present two example of our approach—one from statics and one from dynamics,

Examples of Our Structured Approach

In this section we report some examples to demonstrate the practical implementation of the proposed model-based problem solving strategy. We will discuss an example from statics and one from dynamics. Every example will consists of a problem statement and its corresponding solution.

Example from Statics: Equilibrium of a Rigid Body

Problem Statement The rear door of a minivan is hinged at point A and is supported by two struts; one strut is between points B and C and the second strut is immediately behind this on the opposite side of the door. If the door weighs 350 N with center of gravity at point D and it is desired that a 40 N vertical force applied by a person's hand at point E will begin closing the door, determine the force each of the two struts must support and the reactions at the hinge.

Solution
Road Map Although the problem is really three dimensional, a two dimensional idealization is sufficient and will be used here. We will neglect the weights of the two struts since they are likely very small compared to the weight of the door.

Modeling The FBD is shown in Fig. 2 and is constructed as follows. The door is sketched first and an xy coordinate system is chosen. The person's hand at E applies a 40 N downward

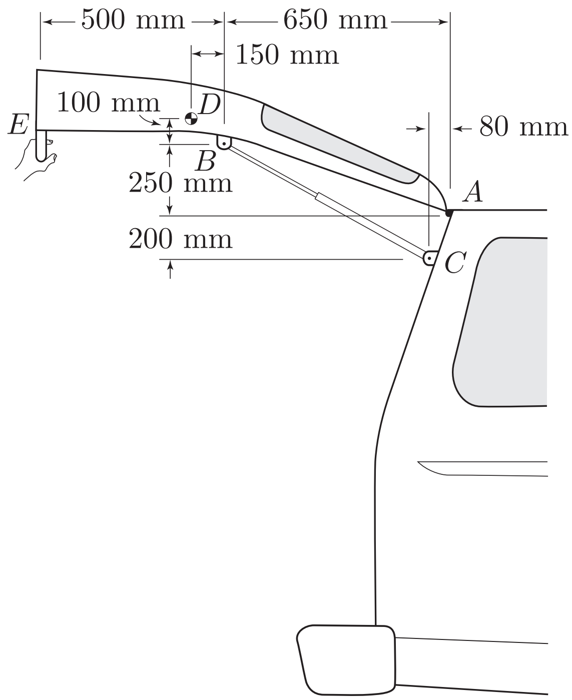

Figure 1. Rear door of a minivan.

vertical force, and the 350 N weight of the door is a vertical force that acts through point D. The hinge (or pin) at A has horizontal and vertical reactions A_x and A_y. F_{BC} represents the force in *one* strut, with a positive value corresponding to compression. Thus, the total force applied by the two struts is $2F_{BC}$. In Fig. 2, the horizontal and vertical components of the strut force are determined using the geometry of the triangles shown.

Governing Equations & Computation Summing moments about point A is convenient because it will produce an equilibrium equation where F_{BC} is the only unknown:

$$\sum M_A = 0: \quad (40\,\mathrm{N})(1.150\,\mathrm{m}) + (350\,\mathrm{N})(0.800\,\mathrm{m}) - 2F_{BC}\frac{450}{726.2}(0.650\,\mathrm{m}) + 2F_{BC}\frac{570}{726.2}(0.250\,\mathrm{m}) = 0 \quad (1)$$
$$\Rightarrow \quad F_{BC} = 789.1\,\mathrm{N}.$$

Thus, the force in one strut is $F_{BC} = 789.1\,\mathrm{N}$.

The reactions at point A are found by writing the remaining two equilibrium equations

$$\sum F_x = 0: \quad -2F_{BC}\frac{570}{726.2} + A_x = 0 \quad (2)$$
$$\Rightarrow A_x = 1239\,\mathrm{N},$$

$$\sum F_y = 0: \quad -40\,\mathrm{N} - 350\,\mathrm{N} + 2F_{BC}\frac{450}{726.2} + A_y = 0 \quad (3)$$
$$\Rightarrow A_y = -588.0\,\mathrm{N}.$$

Discussion & Verification

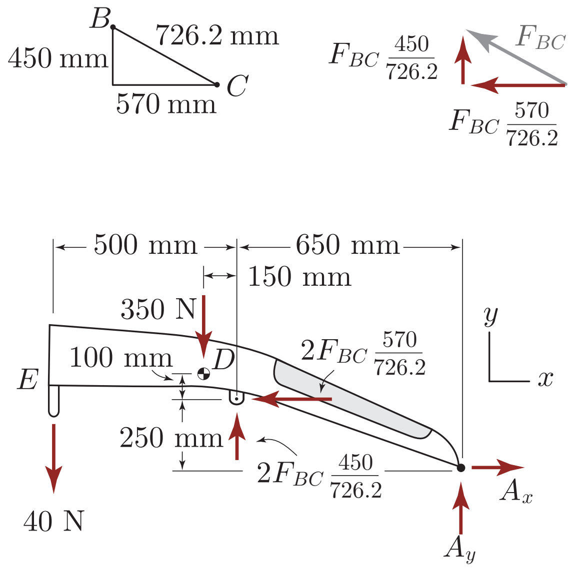

Figure 2. FBD of the rear door.

- Because of the geometry and loading for this problem, we intuitively expect the struts to be in compression. Since the strut force F_{BC} was defined to be positive in compression, we expect the solution to Eq. (1) to give $F_{BC} > 0$, which it does.
- You should verify that the solutions are mathematically correct by substituting F_{BC}, A_x and A_y into *all* equilibrium equations to check that *each* of them is satisfied. However, this check does not verify the accuracy of the equilibrium equations themselves, so it is essential that you draw accurate FBDs and check that your solution is reasonable.

Example from Dynamics: Kinetics of a Two-Particle System

Problem Statement A student throws a pair of stacked books, whose masses are $m_1 = 1.5\,\text{kg}$ and $m_2 = 1\,\text{kg}$, on a table as shown in Fig. 3. The books strike the table with essentially zero vertical speed and their common horizontal speed is $v_0 = 0.75\,\text{m/s}$. The coefficient of kinetic friction between the bottom book and the table is $\mu_{k1} = 0.45$, the coefficient of kinetic friction between the two books is $\mu_{k2} = 0.3$, and the coefficient of static friction between the two books is $\mu_{s2} = 0.4$. If the books strike the table as shown in Fig. 3, determine their final positions.

Solution

Road Map When we throw the two books on to the table, we *know* that the bottom book *must* slide on the table since, if it did not slide, then the bottom book would have to experience infinite acceleration in going from the speed v_0 to zero. On the other hand, we are not assured that the top book must slide relative to the bottom book. Hence, we begin by assuming that m_2 does *not* slip on m_1 and compute the corresponding solution. Once this solution is obtained we will be in a position to check whether or not the starting

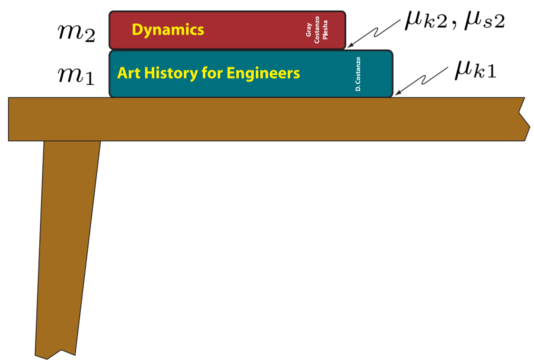

Figure 3. Two books are thrown onto a table and slide over it.

working assumption was correct. If not, we will conclude that the books slide relative to one another and we will have to compute a corresponding new solution.

Modeling Figure 4 shows the unit vectors $\hat{\imath}$ and $\hat{\jmath}$ of a Cartesian coordinate system with

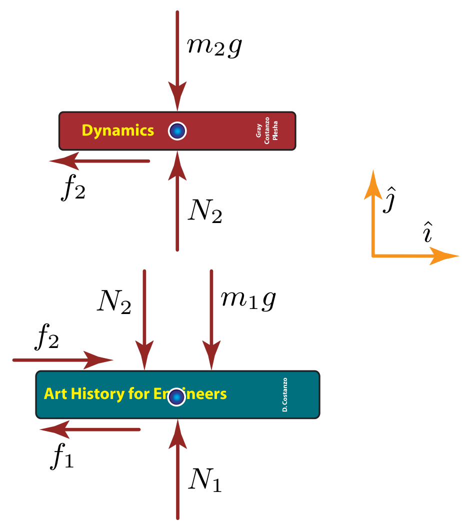

Figure 4. FBDs for the two books modeled as particles.

x and y axes parallel and perpendicular to the table, respectively. This choice of axes is motivated by the horizontal nature of books' motion. The origin of the chosen system is taken to be the point at which the books first impact the table. Next, referring to the FBDs of the two books shown in Fig. 4, it is reasonable to assume that the books' initial velocity is too small for air resistance to have an effect. Hence, the forces we retain in our model are simply the weights and the contact forces with the table and between the books. As far as the modeling of friction is concerned, we will use the standard Coulomb friction model. Finally, to be consistent with the theory seen thus far, we will model the two books as particles.

Governing Equations

Newton-Euler/Balance Equations Referring to the FBDs of the two books shown in

Fig. 4, we can write the Newton-Euler equations for m_2 as

$$\sum F_x = ma_x: \qquad -f_2 = m_2(a_2)_x, \tag{4}$$
$$\sum F_y = ma_y: \quad N_2 - m_2 g = m_2(a_2)_y, \tag{5}$$

and for m_1 as

$$\sum F_x = ma_x: \qquad f_2 - f_1 = m_1(a_1)_x, \tag{6}$$
$$\sum F_y = ma_y: \quad N_1 - N_2 - m_1 g = m_1(a_1)_y. \tag{7}$$

Material Models The material relations to use under the current working assumption consists of the Coulomb friction law describing slip between book 1 and the table along with the inequality consistent with the no slip condition between book 1 and 2. These relations take on the form

$$f_1 = \mu_{k1} N_1 \quad \text{and} \quad |f_2/N_2| < \mu_{s2}. \tag{8}$$

Kinematic Equations The associated kinematic relations are

$$(a_2)_x = a, \quad (a_2)_y = 0, \tag{9}$$
$$(a_1)_x = a, \quad (a_1)_y = 0, \tag{10}$$

in which, due to the no slip assumption between the books, we have called a their common horizontal acceleration.

Computation Now that the governing equation for the problem have been assembled, we observe that Eqs. (4)–(7) along with the first of the relations in Eqs. (8), and Eqs. (9) and (10) form a system of five equations in the five unknowns: f_1, f_2, a, N_1, and N_2. Since our first objective is to verify whether or not our current working assumption is verified, we begin with solving for just f_2 and N_2:

$$f_2 = \mu_{k1} m_1 g = 6.62\,\text{N} \quad \text{and} \quad N_2 = m_2 g = 9.81\,\text{N}. \tag{11}$$

Discussion & Verification To verify our working assumption, we need to check whether or not the inequality in (8) is satisfied, i.e.,

$$|f_2/N_2| = 0.675 \overset{?}{<} \mu_{s2} = 0.4. \tag{12}$$

Clearly, the above inequality is *not* satisfied and we must therefore conclude that our working assumption was incorrect and that, in reality, the top book *does* slip on the bottom one. We now need to start the problem over and solve it for the case of slipping between the books. Fortunately, it isn't as bad as it sounds. The FBDs in Fig. 4 still apply, so the Newton-Euler equations given by Eqs. (4)–(7) also still apply. The only changes concern the material and the kinematic equations as they need to be consistent with the the fact that the books can slip relative to one another.

Material Models The material relations are now as follows:

$$f_1 = \mu_{k1} N_1 \quad \text{and} \quad f_2 = \mu_{k2} N_2. \tag{13}$$

Kinematic Equations The kinematic relations consistent with the books slipping relative to one another are

$$(a_2)_x = a_2, \quad (a_2)_y = 0, \tag{14}$$
$$(a_1)_x = a_1, \quad (a_1)_y = 0. \tag{15}$$

Computation We now notice that Eqs. (4)–(7) along with Eqs. (13), the first of Eqs. (14) and the first of Eqs. (15) provide us with a system of 6 equations in the unknowns a_1, a_2, f_1, f_2, N_1, and N_2. Solving and then plugging in numbers, we obtain

$$a_1 = -\frac{g}{m_1}\left[\mu_{k1}(m_1 + m_2) - \mu_{k2} m_2\right] = -5.396\,\mathrm{m/s^2}, \tag{16}$$
$$a_2 = -\mu_{k2} g = -2.943\,\mathrm{m/s^2}, \tag{17}$$
$$f_1 = \mu_{k1} g(m_1 + m_2) = 11.04\,\mathrm{N}, \tag{18}$$
$$f_2 = \mu_{k2} m_2 g = 2.943\,\mathrm{N}, \tag{19}$$
$$N_1 = g(m_1 + m_2) = 24.52\,\mathrm{N}, \tag{20}$$
$$N_2 = g m_2 = 9.81\,\mathrm{N}. \tag{21}$$

Now that we know the acceleration of each of the two masses, and since we know their common initial velocity, we can figure out how far each of them slides. Applying the constant acceleration kinematic equations to the motion of the masses m_1 and m_2 between the position at which they hit the table to the position at which they stop gives, respectively,

$$0 = v_0^2 + 2a_1 x_1 = v_0^2 - \frac{2g}{m_1}\left[\mu_{k1}(m_1 + m_2) - \mu_{k2} m_2\right], \tag{22}$$
$$0 = v_0^2 + 2a_2 x_2 = v_0^2 - 2\mu_{k2} g. \tag{23}$$

Solving for x_1 and x_2 and plugging in numbers, we obtain

$$x_1 = \frac{m_1 v_0^2}{2g\left[\mu_{k1}(m_1 + m_2) - \mu_{k2} m_2\right]} = 0.052\,13\,\mathrm{m}, \tag{24}$$
$$x_2 = \frac{v_0^2}{2g\mu_{k2}} = 0.095\,57\,\mathrm{m}. \tag{25}$$

So, the bottom book slides 5.2 cm, but where does the top book end up? Referring to Fig. 5, as viewed by an inertial observer, i.e., someone sitting on the table, it slides 9.6 cm, but *relative* to the bottom book, it slides

$$x_{\text{top/bottom}} = x_{2/1} = x_2 - x_1 = 0.043\,\mathrm{m}, \tag{26}$$

or, it slides about 4.3 cm on top of, or relative to, the bottom book.

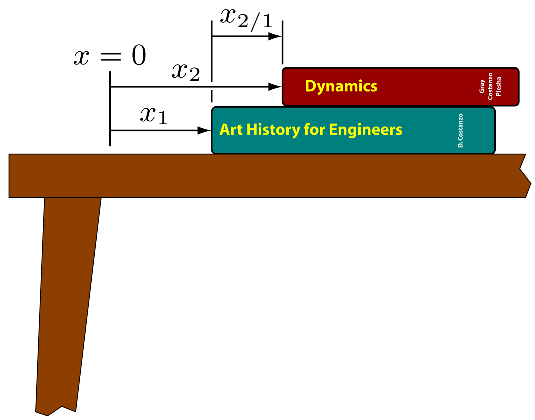

Figure 5. Sliding motion of the books.

Discussion & Verification With the solution completed, we can now think about whether or not the answers given by Eqs. (24)–(26) seem reasonable. The signs on all three equations are what we expect, that is the both book 1 and book 2 both move to the right and book 2 moves to the right relative to book 1. In addition, the magnitudes of the distances slid by the books is on the order of centimeters, which seems reasonable. Had any of the answers been many meters or kilometers, we would have to re-evaluate our solution. Finally, since we are treating the books as particles, the dimensions of the books don't come into play. Therefore, we don't really know if book 2 is still sitting on book 1 at the end of the motion.

Discussion

We hope that the reader can see that these examples have followed the structured approach that we present at the beginning of this paper and we encourage the reader to compare our proposed approach with those found in the best-selling textbooks currently on the market. While we have only presented two examples in this paper, one from statics and one from dynamics, the prospect of *every* problem in a statics and dynamics textbook combination being presented in this way provides an opportunity to instill in the students a sense that mechanics has an underlying set of principles that apply to *every single* problem.

Summary

In this paper, we have proposed a new strategy for structured problem solving in equilibrium and kinetics problems in freshman/sophomore-level mechanics courses. We believe that this approach "stands on the shoulders of giants" in the sense that it builds upon the foundation laid by the most recent generations of statics and dynamics textbooks that began appearing about 50 years ago. We believe that our approach discourages the attitude developed by students according to which all problems in statics, and especially dynamics, are solved using a bag of tricks that appear to be different for every problem and it develops a viewpoint that is applicable in all future courses in mechanics.

Acknowledgments

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References

[1] Plesha, M. E., G. L. Gray, and F. Costanzo (2009) *Engineering Mechanics: Statics*, McGraw-Hill, in preparation.

[2] Gray, G. L., F. Costanzo, and M. E. Plesha (2009) *Engineering Mechanics: Dynamics*, McGraw-Hill, in preparation.

[3] Bedford, A. and W. Fowler (2002) *Engineering Mechanics: Dynamics*, 3rd ed., Prentice Hall.

[4] ——— (2002) *Engineering Mechanics: Statics*, 3rd ed., Prentice Hall.

[5] Beer, F. P., E. R. Johnston, and E. R. Eisenberg (2004) *Vector Mechanics for Engineers: Statics*, 7th ed., McGraw-Hill.

[6] Beer, F. P., E. R. Johnston, and W. E. Clausen (2004) *Vector Mechanics for Engineers: Dynamics*, 7th ed., McGraw-Hill.

[7] Hibbeler, R. C. (2004) *Engineering Mechanics: Statics*, 10th ed., Pearson Prentice Hall, Upper Saddle River, New Jersey.

[8] ——— (2004) *Engineering Mechanics: Dynamics*, 10th ed., Pearson Prentice Hall, Upper Saddle River, New Jersey.

[9] Meriam, J. L. and L. G. Kraige (2002) *Engineering Mechanics: Dynamics*, 5th ed., John Wiley & Sons.

[10] ——— (2002) *Engineering Mechanics: Statics*, 5th ed., John Wiley & Sons.

[11] Sheppard, S. D. and B. H. Tongue (2005) *Statics: Analysis and Design of Systems in Equilibrium*, John Wiley & Sons, Hoboken, NJ.

[12] Tongue, B. H. and S. D. Sheppard (2005) *Dynamics: Analysis and Design of Systems in Motion*, John Wiley & Sons, Hoboken, NJ.

Biographies

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