# Provably Optimal Economic Decision-Making 

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#### Abstract

Current methods of economic decision-making use multiple criteria that often result in conflicting indications of the best alternatives, none of which are provably optimal. As a consequence, economic decision-making differs between and within organizations. The proof presented in this paper provides a single criterion for selecting engineering and financial alternatives that maximize the net present-value of an organization subject to a capital constraint ${ }^{1}$. Major differences from current practice include forecasting costs of borrowing money for discounting cash flows and measuring capital costs. It is assumed each alternative has accurate input and output cash flow forecasts that incorporate engineering and marketing risks. The proof of optimal economic decision-making can then be validated mathematically and verified with financial accounting statements. The single criterion for selecting alternatives that is proposed here promises to be the best practical guide for optimal economic decision-making not only in industrial firms, but also in financial institutions, government agencies and nonprofit organizations.


## I. Introduction

It is commonly thought that the best way of doing each project would be to select the alternative whose cash flows have the largest net present-value. But net present-values are not defined until discount rates are specified. If low discount rates are specified, alternatives with the largest net present-values could have output cash flows that are received in the distant future. Specifying high discount rates would reduce the net present-values of distant cash flows. How high the discount rate should be is an open question in both theory and practice. Discount rates in current use often include engineering and marketing risks as well as investor opportunity costs. This results in discount rates such as weighted average -costs-of-capital (WACC) ${ }^{3}$ and minimum $\underline{a}^{\text {attractive-rates-of-return (MARR) }}{ }^{4}$ which can be much greater than costs of borrowing money.

However, high discount rates distort the decision-making process. All economic decision-making works within capital constraints derived from loans and investments. When available cash and retained earnings of an organization are insufficient for its investments, money may be borrowed to fulfill investment requirements. This suggests that competing uses of capital funds should be compared by discounting their cash flows with the organization's costs of borrowing money.

If cash flows are discounted instead with WACC or MARR interest rates, the results have very different meanings. For example, suppose a $15 \%$ rate-of-return investment opportunity requires an input of $\$ 100$ that returns an output of $\$ 115$ one-year later. Assuming a $20 \%$ per year WACC
or MARR interest rate, the net present-value (NPV) and net future-value (NFV) of the investment would be $-\$ 4.17$ and $-\$ 5.00$ respectively. These negative results are opportunity costs of investors undertaking the $\$ 100$ investment and foregoing other investments of equal cost that have at least a $20 \%$ rate-of-return. As a result, the $\$ 100$ investment would be rejected. However, if borrowing money costs $8 \%$ per year, the NPV and NFV of the $\$ 100$ investment would be $\$ 6.48$ and $\$ 7.00$ respectively. These positive results are opportunity costs of borrowing $\$ 100$ at $8 \%$ interest per year in order to undertake the $\$ 100$ investment.

If cash flow forecasts of alternatives are accurate, their present values should be discounted with costs of borrowing money just as their future values would be recorded in financial statements. Present-value criteria used before the fact need to be reconcilable with financial statements after the fact. Replacing WACC or MARR discount rates by costs of borrowing money could benefit both financial accounting and economic decision-making as explained below.

Financial accounting statements deal with the outcomes of past decisions. But no data are given of project alternatives that were rejected or not considered. Output revenues and input expenses of funded projects are recorded "as is" without evaluating the time value of money ${ }^{2}$. Input and output cash flows are partitioned into accounting periods where net cash flows are reported as profits or losses. Each accounting period contains mixtures of input and output cash flows, costs of borrowed money and income taxes that are not causally connected. Financial accounting is always done in the aggregate where constraints of debt and equity capital are determined.

Economic decision-making is done on a project level where knowledgeable people closest to available data forecast input and output cash flows of each alternative as accurately as possible. The cash flows of each project alternative must be forecast as time streams of causally connected inputs and outputs over a number of accounting periods. The best way of doing each project is based on the alternative whose cash flows have either the largest net present-value discounted at WACC or MARR interest rates, or least payback time, or largest rate of return on equity. However, multiple criteria often give conflicting answers that are not provably optimal. In contrast, the single criterion based on discounting with costs of borrowing money that is used here is provably optimal when cash flow forecasts of project alternatives are accurate.

In the private sector of the economy, capital with a limited life exceeding one year must first be capitalized and then expensed through depreciation allowances over its estimated life for income tax purposes. Consequently, input cash outflows are smaller after taxes than before taxes. Since interest expenses on debt are deductible from taxable incomes, costs of borrowing money are also smaller after taxes than before taxes. When discounting by costs of borrowing money after taxes, input and output cash flows have the same present values regardless of different relative amounts of debt and equity capital that are being used as shown in the example of Section III.

## II. Convex-envelope proof of optimal economic decision-making

Ricardo's marginal principle defines the necessary and sufficient conditions for optimal capital budgeting as follows: Any accepted alternatives must have greater present-value output revenues for their present-value input costs than any other alternatives with the same present-value input costs that could not be funded because of the capital constraint. The practical problems of applying Ricardo's marginal principle come from using multiple criteria to evaluate alternatives of each project under the capital constraint of the organization. The convex-envelope proof resolves these problems by showing a single criterion is sufficient for selecting alternatives that maximize the net present-value objective of an economic organization under a capital constraint.

The convex-envelope proof is based on an abstract set of simultaneous alternatives. However, the single criterion for selecting alternatives that is derived from the proof is applicable to every alternative regardless of project, timing and type of private or public economic organization. For purposes of the proof, an economic organization is subdivided into non-overlapping projects that compete for funds from a common constraint of debt and equity capital. Each project may have any number of mutually exclusive and indivisible alternatives that may differ in scale, life times, output quantity and/or quality, and sources of financing. Our decision method determines

1) the best way of doing each project,
2) the best projects to do, and
3) the projects that should be funded.

Each way of doing a project is represented by a two-dimensional vector ( $\Delta \mathrm{C}, \Delta \mathrm{R}$ ) where presentvalues of input costs, $\Delta \mathrm{C}$, and output revenues, $\Delta \mathrm{R}$, are discounted at costs of borrowing money. Vectors representing mutually exclusive ways of doing a project are formed into a bundle with a common initial point and distinct terminal points as shown for projects $\mathrm{W}, \mathrm{X}, \mathrm{Y}$ and Z below.


The difference, $\Delta \mathrm{R}-\Delta \mathrm{C}=\Delta \mathrm{NPV}$, is defined as Net Present-Value or absolute profitability which is positive so that $\Delta R$ is greater than $\Delta C$. If $\Delta R$ and $\Delta C$ are changed by the same amount, the resulting vectors have the same $\triangle \mathrm{NPV}$ and their terminal points lie on a $45^{\circ}$ line representing the "null" alternatives whose inputs have equal outputs discounted at the costs of borrowing money. Since all vectors that terminate on a $45^{\circ}$ line have the same absolute profitability, $\triangle \mathrm{NPV}$ cannot be the sole criterion for determining the best alternative despite the objective of the organization to maximize its $\triangle \mathrm{NPV}$ subject to its capital constraint.

If vectors have the same $\triangle \mathrm{NPV}$, the best alternative is defined by the vector with the steepest slope or ratio, $\Delta \mathrm{R} / \Delta \mathrm{C}=\emptyset$, which is called the capital-efficiency criterion or relative profitability of producing $\Delta \mathrm{R}$ from $\Delta \mathrm{C}$. Thus, alternatives $\mathrm{W}_{1}(5,8)$ and $\mathrm{W}_{2}(3,6)$ of project W have the same $\Delta N P V=8-5=6-3=3$ and their terminal points lie on a $45^{\circ}$ line. Since $W_{2}$ has a greater $\Delta R / \Delta C$ ratio and a smaller $\Delta \mathrm{C}$ than $\mathrm{W}_{1}$, alternative $\mathrm{W}_{2}$ is defined to be the best way of doing project W .

Project $X$ has three mutually exclusive alternatives $X_{1}(5,9), X_{2}(4,9)$ and $X_{3}(2,9)$ with different inputs that produce the same output $\Delta \mathrm{R}=9$. Alternative $X_{3}$ with the smallest input $\Delta \mathrm{C}$ is best because its profitability is largest, both absolutely and relatively, for the given output $\Delta \mathrm{R}=9$.

Project $Y$ has three mutually exclusive alternatives $Y_{1}(5,9), Y_{2}(5,11)$ and $Y_{3}(5,13)$ with different outputs produced from the same input $\Delta \mathrm{C}=5$. Alternative $\mathrm{Y}_{3}$ with the largest output $\Delta \mathrm{R}$ is best because its profitability is largest, both absolutely and relatively, for the given input $\Delta \mathrm{C}=5$.

Project $Z$ has two mutually exclusive alternatives $Z_{1}(8,14)$ and $Z_{2}(5,10)$, neither of which have both absolute and relative profitability advantages. Therefore, the best way of doing project Z cannot be based on minimizing $\Delta \mathrm{C}$ for a given $\Delta \mathrm{R}$ or maximizing $\Delta \mathrm{R}$ for a given $\Delta \mathrm{C}$. Ignoring all other projects, $\mathrm{Z}_{1}$ appears to be the best alternative if absolute profitability is used as the sole criterion because $\Delta \operatorname{NPV}\left\{\mathrm{Z}_{1}(8,14)\right\}=14-8=6$ is greater than $\Delta \operatorname{NPV}\left\{\mathrm{Z}_{2}(5,10)\right\}=10-5=5$.

However, $Z_{2}$ has a greater capital-efficiency $\emptyset\left\{Z_{2}(5,10)\right\}=2$ compared to $\emptyset\left\{Z_{1}(8,14)\right\}=1.75$ as well as a smaller input cost $\Delta \mathrm{C}\left\{\mathrm{Z}_{2}\right\}=5$ compared to $\Delta \mathrm{C}\left\{\mathrm{Z}_{1}\right\}=8$. It is possible to utilize the advantages of $Z_{2}$ with alternatives of other projects in order to maximize $\Delta N P V$ for a given $\Delta C$. For example, suppose another project V has an alternative $\mathrm{V}_{1}(3,5)$ where $\Delta \operatorname{NPV}\left\{\mathrm{V}_{1}(3,5)\right\}=2$ and $\emptyset\left\{\mathrm{V}_{1}(3,5)\right\}=1.67$ which are smaller, both absolutely and relatively, than those of $\mathrm{Z}_{1}$ or $\mathrm{Z}_{2}$. Since vector representation is useful not only for mutually exclusive alternatives, but also for combined alternatives of different projects, we find the vector sum of $Z_{2}$ and $V_{1}$ is:

$$
\mathrm{Z}_{2}(5,10)+\mathrm{V}_{1}(3,5)=\left\{\mathrm{Z}_{2}+\mathrm{V}_{1}\right\}(5+3,10+5)=\left\{\mathrm{Z}_{2}+\mathrm{V}_{1}\right\}(8,15)
$$

Since $\Delta N P V\left\{Z_{2}+V_{1}\right\}(8,15)=7$ and $\emptyset\left\{Z_{2}+V_{1}\right\}(8,15)=1.875$ are greater than $\Delta N P V\left\{Z_{1}\right\}(8,14)=6$ and $\emptyset\left\{Z_{1}\right\}(8,14)=1.75$, it follows that $\left\{Z_{2}+V_{1}\right\}$ is more profitable than $Z_{1}$, both absolutely and relatively, for the same input $\Delta \mathrm{C}=8$. Therefore, the best way of doing project Z may not be $\mathrm{Z}_{1}$, the one with greatest $\triangle \mathrm{NPV}$, because its input cost is greater and its capital efficiency is smaller than $Z_{2}$. Indeed, $Z_{2}$ and $V_{1}$ combined has a greater $\Delta N P V$ for the same input $\Delta C=8$.

Decision-making for multiple projects needs coordination of financial accounting and project management. Financial accounting is done in the aggregate where capital constraints are determined. Economic decision-making is done on a project level where input and output cash flows of each alternative can be forecast as accurately as possible. Financial accounting and project management use multiple criteria to select alternatives which provide optimal returns from the capital constraints, but none of the results are provably optimal. In contrast, the single criterion that is used here guarantees optimal results under conditions of economic certainty.

The convex-envelope proof of optimal capital budgeting rapidly scans the best way of doing each project and the best projects to do within a planned range of capital constraints without exhaustively evaluating every possible combination. Vector bundles of each project are first ranked in descending order of their steepest-slope vectors which are added geometrically to form an initial convex envelope as shown in the figure below. In the figure below, four separate mutually exclusive projects (A-D) are shown. Project A contains a vector with the highest efficiency, and therefore occupies the base position at the origin, with two alternatives originating at the same point. Project B has an alternative with the next greatest capital efficiency, and therefore all of its alternatives are positioned with their base at the tip of the best project A alternative. This is then repeated for Projects C and D . It is possible for vectors from bundles with steeper-slope vectors to intersect vectors of the initial convex envelope.


The "outside" vectors with black-filled arrowheads, representing the highest capital efficiency projects, form an initial convex-envelope. While this envelope has these highest capital efficiency projects, we are looking for the highest overall $\triangle$ NPV for any possible combination (vector sum) of projects.

Vectors of the initial convex envelope are the most capital-efficient alternatives of each project that could be replaced by vectors of the same or other project bundles. At the capital-constraint line, $\Sigma \Delta \mathrm{C}$, the convex envelope has its marginal slope, $\emptyset_{\mathrm{m}}$, which must be greater than $45^{\circ}$ so that $\Delta \mathrm{R}$ is greater than $\Delta \mathrm{C}$. Put another way - because all cash flows are discounted at costs of borrowing money, a marginal slope of $45^{\circ}$ would mean money is both borrowed and returned at the cost of borrowing money. Consequently, requiring the marginal slope to be greater than $45^{\circ}$ forces economic reasonableness on all alternatives. The marginal slope, $\emptyset_{\mathrm{m}}$, is translated parallel to itself near the terminal side of each vector bundle until it first encounters

1) the steepest-slope vector that was already in the convex envelope, or
2) another vector with a larger $\Delta C$ and a smaller $\Delta R / \Delta C$ capital-efficiency slope, or
3) two or more vectors with larger $\Delta C$ and smaller $\Delta R / \Delta C$ capital-efficiency slopes.

In cases 1) and 2), vectors first touched by the slope, $\emptyset_{m}$, are ranked in descending order of their slopes and then added geometrically to form a new convex envelope of vectors. Vectors from case 3) need to be merged with vectors from cases 1) and 2) to obtain the greatest $\Sigma \Delta \mathrm{NPV}$.

Hence, if $\emptyset_{\mathrm{m}}$ is rotated continuously in the region $90^{\circ}>\emptyset_{\mathrm{m}}>45^{\circ}$, the convex envelopes that maximize $\Sigma \Delta \mathrm{NPV}$ for a given capital constraint $\Sigma \Delta \mathrm{C}$ are altered. Since the range of $\Sigma \Delta \mathrm{C}$ has finite upper and lower bounds, only a finite number of marginal slopes are needed to exhaust the optimal capital budgets of all project alternatives.

To prove that the marginal capital-efficiency criterion maximizes $\Sigma \Delta \mathrm{NPV}$ for capital constraint $\Sigma \Delta \mathrm{C}$, the marginal comparison slope, $\emptyset_{\mathrm{m}}$, is drawn as a dashed line in the figure below. Three types of changes in the solution are possible:

1) The removal of the convex-envelope vectors which appear in the solution.
2) The vector differences between vectors that are and are not in the solution.
3) The introduction of vectors from bundles which are not in the solution.


From the geometry of the proposed solution, the three possible types of change in the solution would be represented by dashed vectors pointed in the direction of the plane below the dashed line of $\emptyset_{\mathrm{m}}$. Therefore, the resultant of these possible changes cannot be in the portion of the plane above the dashed line of $\emptyset_{\mathrm{m}}$. Hence, no change of the proposed solution could increase $\Sigma \Delta \mathrm{NPV}$ under the capital constraint $\Sigma \Delta \mathrm{C}$.

## III. Comparison of purchasing versus leasing alternatives

A firm plans to use a $\$ 45,000$-machine with operating costs of $\$ 10,000 /$ year in order realize $\$ 25,000 /$ year savings in maintenance costs for the next five years. Assume straight-line depreciation with no salvage value at the end of a five-year useful life. The firm is in a $40 \%$ -income-tax bracket. Since the firm can borrow money at $9 \%$ per year before taxes, the after-tax cost of borrowing money is $(1-0.40) \bullet 9 \%=5.4 \%$ per year. Discounting by WACC/MARR interest rates is at $20 \%$ and $12 \% /$ year, before and after taxes respectively. Debt financing, X , could range between zero and $\$ 45,000$ and could be paid back at the end of five years. The remaining $\$ 45,000-\mathrm{X}$ of the purchase price would be equity financed. The machine could also be leased for five years with $\$ 11,000$ end-of-year rental payments. Assume the firm's marginal output/input ratio is 1.15 after taxes. Should the machine be acquired, and how?

The following notations are used for cash flow descriptions in the spreadsheets:
(1) $\mathbf{E O Y}=$ End-of-year accounting periods.
(2) NCF, ICF or OCF = $\underline{\text { Net, }} \underline{I n p u t}$ or Output $\underline{\text { Cash }} \underline{\text { Flows excluding interest on debt. }}$
(2') $\mathbf{C a p} \mathbf{T r}=\underline{\text { Capital transactions }- \text { capital cash flows for purchasing the machine. }}$
(3) Int $=$ Interest expense on outstanding debt in column (7).
(4) $\mathbf{B T C F}=\underline{B}$ efore-tax cash $\underline{\text { flow. }}$. 4 ) $=(2)+\left(2^{\prime}\right)+(3)$ for each EOY accounting period.

(6) IncTax = Income taxes on (2), (3) and (5). (6) $=-\operatorname{te}\{(2)+(3)+(5)\}$ where te $=40 \%$.
(7) $\mathbf{L n C F}=\underline{\text { Loan }}$ cash flow (nontaxable). Principal (+) and amortization (-) of debt financing.
(8) $\mathbf{A T C F}=\underline{\text { After-tax }}$ cash $\underline{\text { flow. }}(8)=(4)+(6)+(7)$ for each EOY accounting period.

Separate spreadsheets are required for input and output cash flows of each alternative.
Spreadsheet I for the input cash flow of purchasing has variable X which represents the debt financing of the $\$ 45,000$ purchase price. The remaining purchase price is equity financed. Spreadsheet II for the output cash flow is the same for purchasing and leasing. Spreadsheet III for the net cash flow of purchasing is the sum of Spreadsheets I and II. Spreadsheet IV represents the input cash flow of leasing. The spreadsheet for the net cash flow of leasing (not shown) is the sum of Spreadsheets II and IV.

Spreadsheet I. Input Cash Flow - \$10,000/year operating costs of \$45,000-machine for 5 years.

| (1) | (2) | (2') | (3) | (4) | (5) | (6) | (7) | (8) |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| EOY | ICF | CapTr | Int | BTCF | Depr | IncTax | LnCF | ATCF |
| 0 | 0 | -45000 | 0 | -45000 | 0 | 0 | X | -45000+X |
| 1 | -10000 | 0 | -.09X | -10000-.09X | -9000 | 7600+.036X | 0 | -2400-.054X |
| 2 | -10000 | 0 | -.09X | -10000-.09X | -9000 | 7600+.036X | 0 | -2400-.054X |
| 3 | -10000 | 0 | -.09X | -10000-.09X | -9000 | 7600+.036X | 0 | -2400-.054X |
| 4 | -10000 | 0 | -.09X | -10000-.09X | -9000 | 7600+.036X | 0 | -2400-.054X |
| 5 | -10000 | 0 | -.09X | -10000-.09X | -9000 | 7600+.036X | -X | -2400-1.054X |
| $\Sigma$ | -50000 | -45000 | -.45X | -95000-.45X | -45000 | 38000+.18X | 0 | -57000-.27X |

Spreadsheet II. Output Cash Flow - \$25,000/year savings with \$45,000-machine for 5 years.

| $(1)$ | $(2)$ | $\left(2^{\prime}\right)$ | $(3)$ | $(4)$ | $(5)$ | $(6)$ | $(7)$ | $(8)$ |
| :---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| EOY | OCF | CapTr | Int | BTCF | Depr | IncTax | LnCF | ATCF |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 25000 | 0 | 0 | 25000 | 0 | -10000 | 0 | 15000 |
| 2 | 25000 | 0 | 0 | 25000 | 0 | -10000 | 0 | 15000 |
| 3 | 25000 | 0 | 0 | 25000 | 0 | -10000 | 0 | 15000 |
| 4 | 25000 | 0 | 0 | 25000 | 0 | -10000 | 0 | 15000 |
| 5 | 25000 | 0 | 0 | 25000 | 0 | -10000 | 0 | 15000 |
| $\Sigma$ | $\mathbf{1 2 5 0 0 0}$ | $\mathbf{0}$ | $\mathbf{0}$ | $\mathbf{1 2 5 0 0 0}$ | $\mathbf{0}$ | $\mathbf{- 5 0 0 0 0}$ | $\mathbf{0}$ | $\mathbf{7 5 0 0 0}$ |

Spreadsheet III. Net Cash Flow - Purchasing the $\$ 45,000-$ machine for five years of operation.

| (1) | (2) | (2') | (3) | (4) | (5) | (6) | (7) | (8) |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| EOY | NCF | CapTr | Int | BTCF | Depr | IncTax | LnCF | ATCF |
| 0 | 0 | -45000 | 0 | -45000 | 0 | 0 | X | -45000+X |
| 1 | 15000 | 0 | -.09X | 15000-.09X | -9000 | -2400+.036X | 0 | 12600-.054X |
| 2 | 15000 | 0 | -.09X | 15000-.09X | -9000 | -2400+.036X | 0 | 12600-.054X |
| 3 | 15000 | 0 | -.09X | 15000-.09X | -9000 | -2400+.036X | 0 | 12600-.054X |
| 4 | 15000 | 0 | -.09X | 15000-.09X | -9000 | -2400+.036X | 0 | 12600-.054X |
| 5 | 15000 | 0 | -.09X | 15000-.09X | -9000 | -2400+.036X | -X | 12600-1.054X |
| $\Sigma$ | 75000 | -45000 | -.45X | 30000-.45X | -45000 | -12000+.18X | 0 | 18000-.27X |

Spreadsheet IV. Input Cash Flow - $\$ 21,000 /$ year leasing and operating costs for five years.

| $(1)$ | $(2)$ | $\left(2^{\prime}\right)$ | $(3)$ | $(4)$ | $(5)$ | $(6)$ | $(7)$ | $(8)$ |
| :---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| EOY | Input | CapTr | Int | BTCF | Depr | IncTax | LnCF | ATCF |
| 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | -21000 | 0 | 0 | -21000 | 0 | 8400 | 0 | -12600 |
| 2 | -21000 | 0 | 0 | -21000 | 0 | 8400 | 0 | -12600 |
| 3 | -21000 | 0 | 0 | -21000 | 0 | 8400 | 0 | -12600 |
| 4 | -21000 | 0 | 0 | -21000 | 0 | 8400 | 0 | -12600 |
| 5 | -21000 | 0 | 0 | -21000 | 0 | 8400 | 0 | -12600 |
| $\Sigma$ | $\mathbf{- 1 0 5 0 0 0}$ | $\mathbf{0}$ | $\mathbf{0}$ | $\mathbf{- 1 0 5 0 0 0}$ | $\mathbf{0}$ | $\mathbf{4 2 0 0 0}$ | $\mathbf{0}$ | $\mathbf{- 6 3 0 0 0}$ |

Tables IA, IB and IC below analyze present-value input costs of purchasing at various discount rates for different amounts of debt and equity financing. Table IV analyzes the present-value input costs for leasing at the same discount rates. In each present-value analysis, the equation $\Delta \mathrm{NPV}(\mathrm{BTCF})+\Delta \mathrm{NPV}(\operatorname{IncTax})+\Delta \mathrm{NPV}(\operatorname{LnCF})=\Delta \mathrm{NPV}(\mathrm{ATCF})$ is valid at all discount rates.

When discounting at the $5.4 \%$ after-tax cost of borrowing money, the present-value input costs of purchasing remain exactly the same $(\$ 55,276)$ in Tables IA, IB and IC despite increases in debt financing from zero to $\$ 45,000$.

When discounting at the $12 \%$ after-tax WACC/MARR interest rate, the present-value input costs of purchasing decrease from $(\$ 53,561)$ in Table IA to $(\$ 42,945)$ in Table IC. This $\$ 10,616$ decrease in present-value input costs occurred as debt financing increased from zero to $\$ 45,000$. The $\$ 10,616$ decrease of present-value input costs is largely due to $\Delta \mathrm{NPV}(\mathrm{LnCF})$ which increased from zero in Table IA to $\$ 19,465$ in Table IC.

The present-value input costs of purchasing is independent of debt or equity financing only when discounting at $5.4 \%$. This is the only discount rate at which the present-value input costs of purchasing can be sensibly compared to those of leasing which does not involve any debt or equity financing. Thus, the present-value input costs of $(\$ 53,953)$ for leasing in Table IV is less than $(\$ 55,276)$ for purchasing in Table IA, IB or IC regardless of debt or equity financing.

Table IA. Debt $\mathrm{X}=0$, Equity $=45000$ : $\Delta \mathrm{NPV}$ analysis of input cash flow for purchasing.

| Discount Rate | $\Delta$ NPV BTCF | $+\Delta$ NPV IncTax | $+\Delta$ NPV LnCF | $=\Delta$ NPV ATCF |
| :--- | :--- | :--- | :---: | :---: |
| "As is" 0\% | -95000 | 38000 | 0 | -57000 |
| After Tax $5.4 \%$ | -87820 | 32543 | 0 | $\mathbf{- 5 5 2 7 6}$ |
| Before Tax $9.0 \%$ | -83896 | 29561 | 0 | -54335 |
| W/M(AT) $12 \%$ | -81047 | 27396 | 0 | -53651 |
| W/M(BT)20\% | -74906 | 22728 | 0 | -52177 |

Table IB. Debt $\mathrm{X}=22500$, Equity=22500: $\Delta \mathrm{NPV}$ analysis of input cash flow for purchasing.

| Discount Rate | $\Delta$ NPV BTCF | $+\Delta$ NPV IncTax | $+\Delta$ NPV LnCF | $=\Delta$ NPV ATCF |
| :--- | :--- | :---: | :---: | :---: |
| "As is" 0\% | -105125 | 42050 | 0 | -63075 |
| After Tax 5.4\% | -96491 | 36011 | 5202 | $\mathbf{- 5 5 2 7 6}$ |
| Before Tax 9.0\% | -91773 | 32711 | 7876 | -51184 |
| W/M(AT)12\% | -88347 | 30316 | 9732 | -48298 |
| W/M(BT)20\% | -80962 | 25151 | 13457 | -42353 |

Table IC. Debt $\mathrm{X}=45000$, Equity $=0$ : $\Delta \mathrm{NPV}$ analysis of input cash flow for purchasing.

| Discount Rate | $\Delta$ NPV BTCF | $+\Delta$ NPV IncTax | $+\Delta$ NPV LnCF | $=\Delta$ NPV ATCF |
| :--- | :--- | :--- | :---: | :---: |
| "As is" 0\% | -115250 | 46100 | 0 | -69150 |
| After Tax 5.4\% | -105162 | 39480 | 10405 | $\mathbf{- 5 5 2 7 6}$ |
| Before Tax 9.0\% | -99649 | 35862 | 15753 | -48033 |
| W/M(AT)12\% | -95647 | 33236 | 19465 | -42945 |
| W/M(BT)20\% | -87018 | 27573 | 26915 | -32529 |

Table IV. $\triangle$ NPV analysis of input cash flow (Spreadsheet IV) for leasing.

| Discount Rate | $\Delta$ NPV BTCF | $+\Delta$ NPV IncTax | $+\Delta$ NPV LnCF | $=\Delta$ NPV ATCF |
| :--- | :--- | :---: | :---: | :---: |
| "As is" $0 \%$ | -105000 | 42000 | 0 | -63000 |
| After Tax $5.4 \%$ | -89922 | 35968 | 0 | $\mathbf{- 5 3 9 5 3}$ |
| Before Tax $9.0 \%$ | -81682 | 32673 | 0 | -49009 |
| W/M(AT) $12 \%$ | -75700 | 30280 | 0 | -45420 |
| W/M(BT)20\% | -62802 | 25121 | 0 | -37681 |

Table V. Profitability comparison of purchasing versus leasing alternatives after taxes.

| Discount Rate <br> Purchase/Lease | $\Delta \mathrm{NPV}=\Delta \mathrm{R}-\Delta \mathrm{C}$ | $\Delta \mathrm{R}$ Output <br> Revenues | $\Delta \mathrm{C}$ Input <br> Costs | $\Delta \mathrm{R} / \Delta \mathrm{C}$ <br> Ratio | Annual Lease <br> Equivalent |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $5.4 \%$ Purchase | 8953 | 64230 | 55276 | 1.162 | 11515 |
| $5.4 \%$ Lease | 10276 | 64230 | 53953 | 1.190 | 11000 |

The present-value comparison of purchasing versus leasing after taxes is given in Table V. The purchasing $\Delta \mathrm{NPV}=\$ 8,953$ equals the sum of $\Delta \mathrm{R}=\Delta \mathrm{NPV}(\mathrm{ATCF})=\$ 64,230$ output revenues and $\Delta \mathrm{C}=\Delta \mathrm{NPV}(\mathrm{ATCF})=-\$ 55,276$ input costs. The leasing $\Delta \mathrm{NPV}=\$ 10,276$ is greater than the purchasing $\triangle \mathrm{NPV}$ by $\$ 1,323$. However, a greater $\triangle \mathrm{NPV}$ is not sufficient reason to prefer leasing to purchasing. The leasing $\Delta \mathrm{R} / \Delta \mathrm{C}=1.190$ needs to be greater than that of purchasing to
establish the preference for leasing. Also, the leasing $\Delta \mathrm{R} / \Delta \mathrm{C}$ must be greater than the 1.15 marginal ratio in order for leasing to be acceptable in the firm's budget.

## IV. Summary and Conclusions

A convex-envelope method of optimal economic decision-making is developed and proven here under conditions of economic certainty which assume accurate forecasts for the input and output cash flows of each alternative. Present values of input and output cash flows are obtained with discount rates based on costs of borrowing money for the organization after taxes in order to reconcile results of economic decision-making with borrowing costs that would be recorded in financial statements. The convex-envelope proof shows that a single criterion is sufficient for selecting alternatives that maximize the net present-value of an organization under given capital constraints. The single criterion is applicable to both private and public economic organizations.

The practice of economic decision-making is often hindered by imperfect communication between financial accounting and economic decision-makers at the project level. Financial accounting is done in the aggregate where capital constraints are determined and success is finally measured. Economic decision-making is done on a project by project basis where managers who are closest to engineering and marketing data forecast the input and output cash flows of each alternative as accurately as possible. Communication between financial accounting and project managers is carried out at present with multiple criteria such as payback times and rates of return on equity investments which cloud the fundamental thrust of decision-making, namely, to make the most money from given constraints of available money. In contrast, the single criterion developed here provides an intuitive language for communicating between financial accounting and project managers that can guarantee the best possible economic returns under conditions of economic certainty.

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