Richardson Extrapolation Applied to the Numerical Solution of Boundary Integral Equations for a Laplace's Equation Dirichlet Problem

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Abstract

Richardson extrapolation is applied to improve the accuracy of the numerical solution of boundary integral equations. The boundary integral equations arise from a direct boundary integral method for solving a Laplace's equation interior Dirichlet problem. Specifically, the Richardson extrapolation is used to improve the accuracy of collocation. Numerical justification is provided to support the expectation of improved accuracy. The order of the dominant collocation error term is numerically estimated and numerical results are obtained for a simple model problem.

Introduction and statement of problem

The problem of interest involves the numerical solution of an interior Dirichlet problem for Laplace's equation on a rectangular domain. The method of interest is a direct boundary integral method. The boundary integral equations are discretized using collocation, and numerically solved for the unknown outward normal boundary flux (normal derivative of the primary unknown). The discretized boundary integral equations, which are Fredholm integral equations of the first kind, are the boundary element equations. In particular, the emphasis is on efficiently improving the numerical solution of the boundary element equations. That is, we seek a more accurate numerical approximation for the outward normal boundary flux. Then, a more accurate numerical solution for the primary unknown in the domain interior can be computed using this more accurate result for the outward normal boundary flux.

There are papers presenting collocation convergence and error estimation for Fredholm integral equations of the first kind, for example [2, 10]. However, new material that we present in this paper is an enhancement of pointwise approximations along the boundary of the domain. Richardson extrapolation is applied to improve the accuracy of the numerical solution of the boundary integral equations. Specifically, Richardson extrapolation is used to improve the accuracy of the collocation results. The use of a numerical approximation of the rate of convergence, i.e., the order of the dominant error term of the normal boundary flux approximation, (as the boundary grid is refined) in order to justify application of Richardson extrapolation and the manner in which Richardson extrapolation is applied are contributions of this paper. A *Mathematica* notebook implementing the boundary element method was written by the author.

Model problem

The model problem is to solve the Dirichlet problem for Laplace's equation on a square:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \text{in domain } \Omega = \{(x, y) \mid 0 < x < 1, \quad 0 < y < 1\},$$

$$u = u_0, \qquad \text{on boundary } \partial\Omega = \{(x, y) \mid 0 \le x \le 1,\ y = 0 \text{ and } y = 1; \quad 0 \le y \le 1,\ x = 0 \text{ and } x = 1\},$$

where u_0 is a given sufficiently smooth function. We will use the boundary element method.

Richardson extrapolation

The object of Richardson extrapolation is to find a computationally inexpensive way to combine previously computed lower-order (less accurate) numerical results in a way that produces formulas with higher-order (more accurate) numerical results[1]. It is stated that the method is extremely useful when there is a reliable estimate of the form of the discretization error as a function of the grid length[9]. However, it will be proved in this paper that, even if such information is not available, under quite general conditions Richardson extrapolation will improve the accuracy of the numerical result (or, at the very least, maintain the accuracy).

The following material is a brief description of Richardson extrapolation. Let q denote an unknown exact quantity that is desired. Let q_1 and q_2 denote numerical approximations to q that are computed using the same formula (and at the same grid point) but with different, sufficiently small positive grid spacings, h_1 and h_2, respectively. If the dominant term in the discretization error is proportional to h^p, for some positive number p, then we obtain

$$q - q_1 \approx A h_1^p + \text{ higher-order terms} \qquad \text{and} \qquad q - q_2 \approx A h_2^p + \text{ higher-order terms,}$$

where A denotes the constant of proportionality. Taking a linear combination of these two expressions so as to eliminate the dominant error term and solving for q yields

$$q \approx \frac{h_2^p\ q_1 - h_1^p\ q_2}{h_2^p - h_1^p} \quad + \text{ higher-order terms,}$$

where all the quantities on the right-hand side are known. The expression on the right-hand side is the Richardson approximation, denoted $\tilde{q}$

$$q \approx \tilde{q} \equiv \frac{h_2^p\ q_1 - h_1^p\ q_2}{h_2^p - h_1^p}\ .$$

The error associated with the Richardson approximation is of higher-order, since the lowest-order (dominant) error term from the original formula has been eliminated. Therefore, the Richardson approximation is a more accurate result.

Since the construction of this more accurate approximation requires only a weighted average of previously computed results (i.e., another application of the original approximation method is **not** required), Richardson extrapolation can produce more accurate approximations with minimal computational cost.

For situations in which p, the order of the dominant error term in the original approximation formula, is unknown, as will be the case in this paper, we can, instead, use three approximate solutions, q_1, q_2, and q_3. These three approximations are computed using three different positive grid spacings, $h_1 > h_2 > h_3$, respectively.

$$q - q_1 \approx A h_1^p, \qquad q - q_2 \approx A h_2^p, \qquad \text{and} \qquad q - q_3 \approx A h_3^p.$$

Let the grid spacings be chosen so that

$$\frac{h_1}{h_2} = \frac{h_2}{h_3} = c .$$

for some constant c, such that $1 < c$. Then, after some algebra, we obtain for the Richardson extrapolation formula

$$q \approx \tilde{q} \equiv \frac{q_1\, q_3 - q_2^{\;2}}{q_1 - 2\,q_2 + q_3} .$$

Further, the value of p can be numerically approximated using the three previous approximations to obtain

$$p \approx \frac{\ln\left(\frac{q_2 - q_1}{q_3 - q_2}\right)}{\ln(c)} .$$

In the example that is investigated in this paper (see **Numerical results**), we do not have an explicit form for the error terms, and thus the two preceding formulas will be used.

Rationale for the application of Richardson extrapolation

The following question arises: If the form of the error terms is lacking, can we be guaranteed that the Richardson extrapolation result is an improvement?

We can answer this question in the affirmative by developing the connection between Richardson extrapolation and Aitken's delta-squared method for accelerating convergence of iterative methods[1]. We will now show that, under some quite general conditions, Richardson extrapolation is guaranteed to improve the accuracy of the numerical result.

We will use the following theorem[1].

Theorem 1: Suppose that the sequence of approximations $\{q_n\}_{n=1}^{\infty}$ converges to the limit q such that

$$0 \le \lim_{n\to\infty} \frac{q_{n+1}-q}{q_n-q} \equiv \lambda < 1,$$

for some constant λ, $0 \le \lambda < 1$. Then the associated sequence of iterates $\{\tilde{q}_n\}_{n=1}^{\infty}$, where

$$\tilde{q}_n = \frac{q_n\, q_{n+2} - q_{n+1}^2}{q_n - 2q_{n+1} + q_{n+2}}$$

converges to q faster than $\{q_n\}_{n=1}^{\infty}$ in the sense that

$$\lim_{n\to\infty} \frac{\tilde{q}_n-q}{q_n-q} = 0.$$

The proof follows almost directly from[1] (page 87, problem 16), and is most easily done using a computer algebra system, for example, *Mathematica*.

Proof: It is helpful to define the quantities

$$\delta_n \equiv \frac{q_{n+1}-q}{q_n-q} - \lambda \qquad \text{and} \qquad \delta_{n+1} \equiv \frac{q_{n+2}-q}{q_{n+1}-q} - \lambda .$$

Note that by hypothesis, we have

$$\lim_{n\to\infty} \delta_n = 0.$$

We also have

$$\tilde{q}_n \equiv \frac{q_n\, q_{n+2} - q_{n+1}^2}{q_n - 2q_{n+1} + q_{n+2}}.$$

Rearrange the equation defining δ_n so as to solve for q_n. Similarly, solve the equation defining δ_{n+1} for q_{n+2}. This gives

$$q_n = \frac{-q + q\lambda + q_{n+1} + q\delta_n}{\lambda + \delta_n} \qquad \text{and} \qquad q_{n+2} = q - q\lambda + \lambda q_{n+1} - q\delta_{n+1} + q_{n+1}\delta_{n+1}.$$

Substitute these two results into the definition of $\tilde{q}_n$. Then substitute that result and the above expression for q_n into the expression

$$\frac{\tilde{q}_n - q}{q_n - q},$$

and simplify to get

$$\frac{\tilde{q}_n - q}{q_n - q} = \frac{(\lambda + \delta_n)(\delta_n - \delta_{n+1})}{(\lambda - 1)^2 + \lambda\delta_{n+1} + \delta_n(\lambda + \delta_{n+1} - 2)}.$$

Finally, take the limit as $n \to \infty$. The desired result follows. □

Now consider approximating an unknown quantity, for example, the normal boundary flux at a specific boundary point, q, using the three successive grid spacings, $h_1 > h_2 > h_3$, chosen sufficiently small and so that $\frac{q_2 - q}{q_1 - q} \approx \frac{q_3 - q}{q_2 - q}$, where the approximations q_1, q_2, and q_3, are computed using h_1, h_2, and h_3, respectively. Further assume that the errors $q_i - q$, $i = 1, 2, 3$, have the same sign. Theorem 1 implies that if q_1, q_2, and q_3 are converging to q with convergence as described (also see definition[1]), then Richardson extrapolation (which is, in fact, given by the formula for $\tilde{q}_n$) computed using q_1, q_2, and q_3, will give a more accurate approximation.

Boundary element method

The following material gives a summary of the boundary element method[5, 7]. The boundary element method will be described in the restricted context of this paper. We will be interested in solving an interior Dirichlet problem for Laplace's equation on a rectangular domain in the plane. A direct boundary element method will be used, and this involves solving a Fredholm integral equation of the first kind. The relevant equations involve both single and double layer potential integrals[3].

The boundary integral method transforms the given partial differential equation into an equivalent set of integral equations over the boundary of the solution domain. The main advantage is that the numerical solution of the boundary integral equations only requires subdividing the boundary curve of the solution domain. This is in contrast to other standard numerical methods such as finite difference methods or the finite element method, in which the entire solution domain must be discretized. A Green's identity and the fundamental solution of the partial differential equation are used to transform the original partial differential equation problem into a problem which involves only boundary integrals. After discretization, the boundary integral equations are referred to as boundary element equations. Two of the most common discretization methods for numerically solving the boundary element equations are collocation and the Galerkin finite element method. In this paper, we focus on the the use of collocation. We then enhance the collocation method result via Richardson extrapolation (see **Numerical results**).

It turns out that, in the classical direct boundary element method, we use the following boundary integral equation,

$$\int_{\partial\Omega} q(\vec{\xi})\, u^*(\vec{x},\vec{\xi})\, ds_{\vec{\xi}} = -\tfrac{1}{2} u_0(\vec{x}) + \int_{\partial\Omega} u_0(\vec{\xi})\; q^*(\vec{x},\vec{\xi})\, ds_{\vec{\xi}} \qquad \vec{x} \in \partial\Omega,$$

where $\vec{x} = (x, y)$ and $\vec{\xi} = (\xi, \eta)$. The integral equation has both a single layer potential integral and a double layer potential integral. The symbols u and q denote the primary unknown and its outward normal flux on the boundary, respectively. Similarly, the symbols u^* and q^* denote the known fundamental solution and its known outward normal flux on the boundary, respectively. This integral equation is discretized and the resulting linear system of algebraic equations is solved to obtain a numerical approximation for the only unknown quantity appearing in the equation, q, the outward normal flux on the boundary.

Note: This integral equation is a Fredholm integral equation of the first kind. Problems involving Fredholm integral equations of the first kind are not always well-posed. However, it can be shown that for a large class of functions u_0 (the given Dirichlet boundary function), this boundary integral equation has a unique solution[6].

Then, once we have an approximation for q on the boundary, that approximation is substituted into

$$u(\vec{x}) = \int_{\partial\Omega} u_0(\vec{\xi})\; q^*(\vec{x},\vec{\xi})\, ds_{\vec{\xi}} - \int_{\partial\Omega} q(\vec{\xi})\, u^*(\vec{x},\vec{\xi})\, ds_{\vec{\xi}} \qquad \vec{x} \in \Omega.$$

Then, u (as well as q, if desired) can be approximated in the interior of Ω by simply numerically evaluating a set of integral equations in which all integrand quantities are now known, and the problem is solved.

Numerical results

The numerical example for the model problem (see **Model problem**) has the exact solution[4]

$$u(x,y) = \pi\, e^{y} \cos(x - \tfrac{\pi}{7}) \;+\; e^{(1-\pi x)} \;\cos(\pi\, y - \tfrac{\pi}{2}) + \tfrac{1}{100}\pi\, e^{5x} \cos(5\, y - \tfrac{\pi}{2}) \;\; .$$

Mathematica Version 6.0.1 was used to write a boundary element method notebook implementing collocation in the classical direct boundary element method. Piecewise constant (discontinuous) elements (basis functions) for the primary variable and its normal boundary flux were used. The collocation (evaluation) points were selected as the midpoints of each geometric element (subinterval) on a uniform element grid on the square boundary. Note that the geometry nodes that delineate the end points of each boundary element are different from the collocation nodes, which are at the element midpoints.

Initially, a relatively coarse uniform geometry grid was used on the boundary. The number of geometry *x*-nodes on a horizontal edge of the square domain was set equal to the number of geometry *y*-nodes on a vertical edge, denoted by $\text{nxnodes} = \text{nynodes} = 19$. The boundary grid was then refined in such a way so as to have the collocation nodes in the coarse boundary grid remain as collocation nodes in the two other refined grids that were used to construct the Richardson extrapolation result. The numbers of boundary geometry nodes were taken successively as $\text{nxnodes} = \text{nynodes} = 55$ for the intermediate boundary grid, and as $\text{nxnodes} = \text{nynodes} = 163$ for the refined boundary grid. Each successive value of nxnodes (and nynodes) is computed by the formula

$$\text{current number of nxnodes} = 3\,(\text{previous number of nxnodes - 1}) \;+ 1.$$

This formula ensures that the collocation nodes in the coarse boundary grid will also be collocation nodes in the successively refined boundary grids. Each successive boundary element is uniformly trisected by successive Richardson refinements.

Even though the focus of this paper is on numerically approximating q, the outward normal boundary flux (which is then used to approximate u, the primary unknown in the domain interior), it is instructive to view the overall problem solution. A typical coarse grid solution for the primary unknown, u, in the domain interior is shown in Figure 1. If the interior flux were of interest, then that could also be approximated. The 3-dimensional boundary element solution plot is compared with that of the exact solution in Figure 2. Even for this coarse grid, the boundary element solution looks good, although that is in part due to the simplicity of the underlying problem.

In Data Table 1, the numerical results obtained from executing the *Mathematica* boundary element notebook three times, using the coarse, intermediate, and refined boundary grids, are compared with the Richardson extrapolation result that is computed using these three data sets. These four numerical approximations for the outward normal boundary flux are then compared with the exact solution values at the selected flux boundary nodes. Specifically, the Richardson extrapolation should be compared with the results from the fine grid to see which is closer to the exact values. As shown in Data Table 1, the Richardson extrapolation gives the best numerical results.

Data Table 2 gives some of the same information as that in Data Table 1. Instead of the flux values, though, the error values are displayed.

Figure 1.
***Mathematica* graphic of a boundary element coarse grid solution at selected grid points**

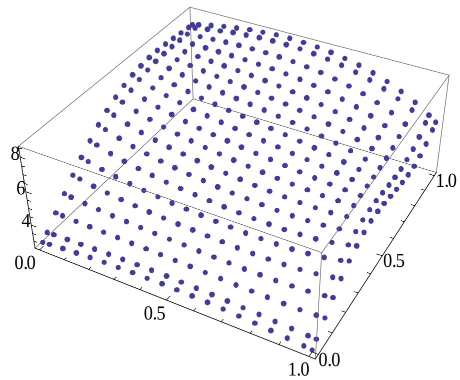
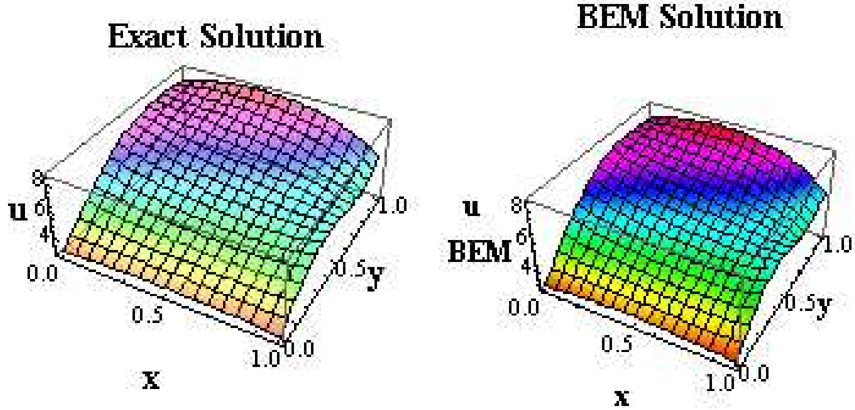


Figure 2.
***Mathematica* graphics of exact versus boundary element solutions**



The numerical justification for application of Richardson extrapolation, with the expectation that this would improve the numerical result, is provided by Data Table 3. The numerical approximations for the order of the dominant error term of our outward normal boundary flux, p, appear in Table 3 (see the approximation formula for p in **Richardson Extrapolation**). The value in Table 3 that appears to be an ‘outlier’ occurs in a region in which the exact solution changes relatively rapidly.

The conclusion from Table 3 is that the approximation (before the Richardson extrapolation is applied) exhibits convergence with, **numerically,** $p \approx 1.8$, for most of the selected points. This is used to justify the application of Richardson extrapolation with the expectation that we will get a more accurate result (see **Rationale for the application of Richardson extrapolation**). For this numerical example, we do obtain more accurate results from the Richardson extrapolation, as supported by the data in Tables 1 and 2. It is observed in this numerical example that the Richardson extrapolation has slightly improved the numerical results for the normal boundary flux.

It should be noted that the numerical approximations of rate of convergence are dependent on the specific grid points (see Table 3). Therefore, these numerical values could be used to perform selective Richardson extrapolation. That is, Richardson extrapolation could be performed only at or near the grid points for which the p values are smaller, with the refined grid results used elsewhere.

Data Table 1. Comparison of numerical normal boundary flux values with the exact value at selected boundary grid points. The column headings specify the rectangular grid point coordinates along the boundary of the square domain. The row headings indicate the numerical method used to compute the normal boundary flux values in that row (or the exact value). The Richardson extrapolation values are computed using a linear combination of the numerical results from the nxnodes=19, 55, and 163 results. All values are rounded.

	(0.25 ,0)	(0.75 ,0)	(1,0.25)	(1,0.75)	(0.75 ,1)	(0.25 ,1)	(0,0.75)	(0,0.25)
nxnodes=19	−7.13323	−5.91798	4.69755	−7.99548	7.94792	4.53466	3.19096	4.25788
nxnodes =55	−7.14568	−5.93317	4.67228	−7.98659	7.94867	4.52827	3.18262	4.24312
nxnodes =163	−7.14751	−5.93527	4.66903	−7.98552	7.94891	4.52754	3.18158	4.24114
Richardson extrapolation	−7.14782	−5.93561	4.66855	−7.98537	7.94903	4.52745	3.18143	4.24083
Exact normal boundary flux	−7.14782	−5.93561	4.66855	−7.98538	7.94897	4.52745	3.18143	4.24082

Data Table 2. Comparison of pointwise normal boundary flux errors at selected boundary grid points. The column headings specify the rectangular grid point coordinates along the boundary of the square domain. The row headings indicate the numerical method used to compute the normal boundary flux in that row. All error values are rounded.

	(0.25,0)	(0.75,0)	(1,0.25)	(1,0.75)	(0.75,1)	(0.25,1)	(0,0.75)	(0,0.25)
nxnodes=19	1.46×10^{-2}	1.76×10^{-2}	2.9×10^{-2}	-1.01×10^{-2}	-1.05×10^{-3}	7.21×10^{-3}	9.53×10^{-3}	1.71×10^{-2}
nxnodes=55	2.14×10^{-3}	2.45×10^{-3}	3.72×10^{-3}	-1.21×10^{-3}	-2.98×10^{-4}	8.18×10^{-4}	1.19×10^{-3}	2.31×10^{-3}
nxnodes=163	3.16×10^{-4}	3.45×10^{-4}	4.75×10^{-4}	-1.37×10^{-4}	-5.51×10^{-5}	9.1×10^{-5}	1.47×10^{-4}	3.19×10^{-4}
Richardson extrapolation	1.85×10^{-6}	7.11×10^{-6}	-3.86×10^{-6}	9.8×10^{-6}	6.11×10^{-5}	-2.23×10^{-6}	-3.13×10^{-6}	9.05×10^{-6}

Data Table 3. Comparison of numerical estimates for the value of p, the order of the dominant error term prior to Richardson extrapolation, at selected boundary grid points. The column headings specify the rectangular grid point coordinates along the boundary of the square domain. All values are rounded.

	(0.25,0)	(0.75,0)	(1,0.25)	(1,0.75)	(0.75,1)	(0.25,1)	(0,0.75)	(0,0.25)
Estimate for p	1.74726	1.79941	1.86807	1.9277	1.02581	1.97894	1.89037	1.82413

Conclusions and extensions

We have presented conditions under which Richardson Extrapolation is guaranteed to improve an approximation. We demonstrated how to approximate the order of the dominant error term, in the case in which the form of the error terms is unknown and how to apply the Richardson extrapolation with respect to collocation. Finally, we verified, for our numerical example, that the Richardson flux approximation was more accurate than the previously obtained flux approximations on which it was based.

An extension of this work is in process, with collocation to be replaced by the Galerkin finite element method. That is, the finite element method will be used to numerically solve the boundary integral equations. Analytic justification that application of Richardson extrapolation improves these numerical approximations will be given using results from Rannacher and Wendland[8]. These results give a formula for the pointwise error bound in which the dominant term is $O(h)$. We will have an analytic foundation which guarantees that Richardson extrapolation will give an accuracy that is better than $O(h)$, where h is the grid spacing.

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