Session 3553

# Significant Digit Accountability for Exponentiation, Trigonometric, and Logarithmic Operations

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Abstract

Freshman engineering students are typically exposed to a number of elementary subjects during their first semester or quarter. One topic introduced to new engineering students is associated with the number of significant digits to be used for the final answer of a typical engineering problem. Elementary engineering texts often cover the topic as it relates to addition, subtraction, multiplication, and division. However, rules associated with exponentiation, trigonometric function use, and the use of logarithms are not included in these introductory texts, or in any other texts. The application of significant digit practices is proposed as it pertains to trigonometric function use as well as logarithmic and exponentiation operations.

## I. Introduction

Each fall thousands of freshman engineering students are introduced to the engineering profession by taking courses that include a potpourri of topics, including the proper accountability of significant digits. Texts[1,2,3,4,5] used for introductory courses often include a section on the procedures to follow for reading measuring devices and for displaying the appropriate number of significant digits in the final answer of an engineering problem. This paper does not argue the importance of reading scaling devices appropriately, but does question the necessity of teaching the rules given to account for significant digits in the elementary arithmetic operations of addition, subtraction, multiplication, and division.

The purpose of introducing students to the concept of significant digit accountability when solving an engineering problem is to ensure that they display a meaningful final answer when using a hand-held calculator or other computing device for solving a problem. Often new engineering students will write the eight or ten digit calculator or computer display when indicating the answer to an assigned problem. Instructors thus feel obligated to cover the topic of significant digit accountability.

The rules often presented to address significant digit accountability are directed toward the elementary mathematical operations of addition, subtraction, multiplication, and division. Eide states for multiplication and division:

> "The product or quotient should contain the same number of significant digits as are contained in the number with the fewest significant digits."[3]

For addition and subtraction, Eide also states:

"The answer should show significant digits only as far to the right as is seen in the least precise number in the calculation."

For combined operations, Eide continues with:

"If products or quotients are to be added or subtracted, perform the multiplication or division first, establish the correct number of significant figures in the subanswer, perform the addition or subtraction, and round to the proper significant figures. Note, however, that in calculator or computer applications it is not practical to perform intermediate rounding. It is normal practice to perform the entire calculation and then report a reasonable number of significant figures.

"If results from additions or subtractions are to be multiplied or divided, an intermediate determination of significant figures can be made when the calculations are performed manually. Use the suggestion above for calculator or computer answers."[3]

However, these rules are often called into question in special circumstances. Thus, the author might write, as does Eide, that:

"Commonsense application of the rules is necessary to avoid problems"[3].

II. Trigonometric, Exponential, and Logarithmic Function Usage

Two areas in Eide's statements lead to questions. In his rule concerning multiplication and division, Eide states that when using a calculator or computer, one should report a reasonable number of significant digits. He does not, however, define the word "reasonable". In the last quoted statement, Eide does not define the phrase "commonsense application". Thus, it appears, as many intelligent students will argue, that the rules of significant digit accountability may be ignored or applied in any manner desired.

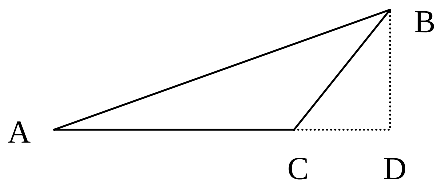
In addition, a bright student, when learning of the rules for addition, subtraction, multiplication, and division, might request the procedure to follow in order to account for significant digits when calculating the trigonometric value associated with an angle, the logarithm of a value, or the value resulting from an exponential operation. Texts for freshman engineering students do not address these mathematical operations and thus often lead both students and instructors to simply ignore the rules stated above.

Is ignoring the rules for the basic arithmetic operations appropriate? Do the rules even need to be addressed at all? These questions are the crux of the argument that follows.

When considering trigonometric, logarithmic, and exponential operations, additional questions related to significant digit accountability arise. Should the value generated by the calculating device be considered exact to the number of digits displayed by the calculating device? Does the calculating device follow the rules associated with the elementary arithmetic operations when presenting the value? Several examples illustrate the problems associated with significant digit accountability in these situations.

Example 1:

Calculate the area of an obtuse triangle ABC shown in the figure below.




Not to scale

AB = 127.3”
AC = 110.1”
Angle BAC = 9.25°

From basic geometry,

Area of a triangle = ½ (length of base) (length of altitude)

= ½ (AC) (BD)

= ½ (AC) (AB) (sin BAC)    since BD = (AB)(sinBAC)

= ½ (110.1) (127.3) (sin 9.25°)

When considering significant digit accountability, the question becomes:

Is the value of the sin 9.25° a number with three significant digits?

Using a typical calculator gives the value of sin 9.25° as 0.160742566. Should the value for the sine of the angle be assumed to have three significant digits, more than three significant digits, or less than three significant digits? Does the fact that a typical protractor shows subdivisions as small as one-half degree (thus allowing one to read to the nearest one-quarter degree) have any effect on this problem?

In the context of a typical problem one might ask if calculator or computer displays for the sine of any angle be considered exact, should one assume that the last digit displayed is doubtful, or should one round the calculator display to n significant digits for use in calculating the final answer for the problem? Thus, one is stymied as to the application of the basic rules.

One may even question if the calculator display followed the standard rules of significant digit accountability. One trigonometric series gives the sine of an angle as:

$\sin x = x - x^3/3! + x^5/5! - x^7/7! + \ldots$ (for real values of x where x is in radians)[6]

For angles near 90°, it may be shown that following the rules for addition, subtraction, multiplication, and division can yield the same value for the sine of several angles. Thus one is inclined to follow Eide’s statement to use common sense when unusual situations arise. Since the use of trigonometric functions is quite common in engineering work, one is lead to the conclusion that following the rules for addition, subtraction, multiplication, and division is not useful. The question then arises – why teach the rules associated with elementary arithmetic operations at all when some common sense guidelines can be followed for all situations?

Example 2:

Determine the equation of the line passing through the data points (6,98), (17,23), (32,3.5), and (42,1.33). This data plots as a straight line on a semilogrithmic grid.

The equation of a straight line on a semilogarithmic grid is of the form:

$$y = be^{mx}$$

which may be rewritten as:

$$\log_e y = \log_e b + \log_e e^{mx} \qquad \text{or}$$

$$\log_e y = \log_e b + mx$$

To solve this problem, it is necessary to select two points on the line and substitute the x and y coordinates of each point into the last equation written above. This results in a pair of simultaneous equations in m and b. Solving the two equations for m and b will then allow the equation of the line to be stated.

In the process of solving the two equations, it will be necessary to determine the logarithm and the antilogarithm of several numbers. If the number has n significant digits, the question is whether the logarithm has n significant digits, the number of significant digits shown in the calculator display, or less than n significant digits. In addition, if a logarithm has n significant digits, does the antilog have n significant digits? An argument similar to that used in Example 1 for the series representation of the sine of an angle may also be used for the series definition of the natural logarithm of a value.

One series for a natural logarithm is:

$$\log_e x = 2\{[(x-1)/(x+1) = 1/3[(x-1)/(x+1)]^3 + 1/5[(x-1)/(x+1)]^5 + \ldots \qquad (\text{for } x > 0)$$ [6]

Just as for the sine trigonometric function, it is possible to follow the rules for addition, subtraction, multiplication, and division and obtain the same answer for several different values of x.

Example 3:

A force of 50 pounds is required to raise a load using a rope that is wrapped around a fixed shaft for one and one-quarter turns. The coefficient of friction between the rope and the fixed shaft is 0.20. What maximum weight can be raised with the 50-pound force?

This problem solution uses the equation:

$$T_2 = T_1 e^{\mu\beta}$$

The value of $T_2$ is 50 pounds, the value of $\mu$ is 0.20, and the value of $\beta$ is $5\pi/2$ radians. The solution requires that e, the base of the natural logarithm system, be raised to the $\mu\beta$ power. The question that arises when considering significant digits concerns the result of the exponential operation. The base of the natural logarithms, e, as well as the value of $\pi$ may be assumed to have many significant digits (at least seven using the hand-held calculator). Should the result of $e^{\mu\beta}$ thus have only two significant digits because of the value of $\mu$ being 0.20?

The series expansion for raising e to a power is:

$$e^x = 1 + x + x^2/2! + x^3/3! = x^4/4! + \ldots \qquad \text{(for all real values of x)}$$ [6]

As in the preceding two examples, it is possible to follow the significant digit rules and obtain the same value for $e^{\mu\beta}$ for different values of $\mu$.

These types of examples cause most instructors to simply make a statement such as, “Your final answer should show three significant digits”, even after having covered in minute detail the rules for significant digit accountability when adding, subtracting, multiplying, and dividing. Thus, it seems appropriate to state and advocate the use of generic guidelines with respect to significant digit accountability.

## III Conclusion

From the author’s experience, it appears that most instructors follow the guidelines stated below.

> When determining the number of significant digits to display in the final answer for an engineering problem, look at the given data. Consider integer values given in the statement of the problem to be exact (having an infinite number of significant digits).
>
> a. If only integer values are used in the statement of the problem, the integer value having the least number of significant digits controls the number of significant digits in the final answer of the problem.
>
> b. If the problem statement includes only values that include a decimal fraction, the value with the least number of significant digits controls the number of significant digits that should appear in the final answer.
>
> c. If the problem statement contains both integer values and decimal values, ignore the integer values (considered exact) and follow paragraph b above.

These generic guidelines, or something similar, take control of the topic of significant digit accountability from the moment they are introduced. Typically, no additional attention is paid to the specific rules for significant digits when adding, subtracting, multiplying, or dividing. The question thus becomes:

> Should any time be spent on significant-digit instruction in the context of engineering problem solving when the generic guidelines given above are usually followed?

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