Singularity Functions Revisited: Clarifications and Extensions for the Deflection of Beams of Non-Uniform Flexural Rigidity under Arbitrary Loading

S. Boedo
Department of Mechanical Engineering
Rochester Institute of Technology
Rochester, NY 14623
email: sxbeme@rit.edu

Abstract

The engineering design process involves understanding of the applicability of structural elements associated with a particular application. Beam structural elements are the prototypical example, and it is not surprising that beam stresses and deflections are essential course topics in all undergraduate mechanical and civil engineering degree programs. Singularity functions are a well-known economical and practical solution method for beams subjected to multiple loads and supports. However, the method as presented in most contemporary textbooks is often unclear to the student and instructor alike in the handling of function discontinuities and integration constants. The method also appears to be limited to a small set of concentrated actions and polynomial functional forms, where more complicated loading conditions must be achieved through superposition.

These perceived limitations of the singularity function method were addressed in a recently published paper, where in particular, singularity functions representing general functional forms were re-introduced to construct shear-moment diagrams. The work herein is to extend this paper and show how these general functional forms can be used to determine the deflection of beams of non-uniform flexural rigidity subjected to arbitrary loads.

The solution methods presented here are at a level of mathematical rigor expected in a second-year undergraduate introductory strength of materials course or a subsequent undergraduate machine design course.

1. Introduction

The creative process in engineering design is inevitably constrained by external forces which induce stresses and deflections in the structure. Exoskeleton skyscrapers are a contemporary example, where the objective is to maximize usable space by the elimination of load bearing interior columns. Traditional exoskeleton buildings, such as the John Hancock Center [1] and the Alcoa Building [2] are comprised of an interior core region which takes the majority of the tower's load while a rectilinear exterior beam structure acts as a stabilizing feature. The One Thousand Museum [3] dispenses of the interior core altogether, whereupon the entire load is carried by an exoskeleton structure comprised of undulating, curved beam columns. These columns also contribute as key aesthetic design elements of the interior space. Examples in the arts include kinetic sculptures [4], which are comprised of a variety of structural support elements.

In these examples, beam structural elements are often a critical design component, and it is not surprising that beam stresses and deflections are fundamental course topics in all undergraduate mechanical and civil engineering degree programs. In the Mechanical Engineering Department at RIT, students are introduced to beam bending and deflection in a second-year strength of materials course (MECE 203). The approach is a traditional one, starting with pure beam bending, followed with transverse loading leading to the construction of shear-moment diagrams. Torsion of beams of circular cross-section is also covered. The discussion concludes with the derivation of beam deflection using Euler-Bernoulli beam theory assuming uniform flexural rigidity. Laboratory experiments support the theoretical foundation as well.

When a beam is subjected to a large number of external loads and moments, either in concentrated or distributed form, the process of constructing shear force and bending moment diagrams from repeated sectioning of the beam (as taught in MECE 203) can be a very tedious and time consuming process. Subsequent determination of beam deflection adds another layer of complexity to incorporate additional slope and displacements boundary conditions at singular points on the beam where the functional form of the bending moment is not directly integrable. Complicating the situation are beams which are comprised of different materials or beams which have non-uniform cross-sectional geometry.

Singularity functions for shear-moment diagrams and beam deflections greatly expedite the computational process by eliminating the need to invoke continuity boundary conditions at the singular points. At RIT, singularity functions are introduced in a follow-up upper-division undergraduate elective course (MECE 350), where the method is used to determine stress and deflection of straight and curved beams of non-uniform flexural rigidity. The method is also coupled with Castigliano's theorem and failure theories associated with static and dynamic loading. Singularity functions are reinforced at the graduate level in a mechanics of solids course (MECE 785) for the solution of statically indeterminate beams and determination of structural influence coefficients, and as introductory material for instruction in finite elements (MECE 605).

A sampling of contemporary textbooks [5-9] introduce and discuss the subject of singularity functions in a manner that can be confusing to both instructor and student. In addition to inconsistent sign conventions and inconsistent treatment of singular points among different textbooks, the essential perceived limitation is that the method can only be used for external loading represented by a limited predefined set of polynomial-based functional forms. In a recent paper by Boedo [10], clarification on the use of polynomial-based singularity functions and extension of the method to represent arbitrarily-defined external loads was presented.

The work herein is to extend this paper and show how these general functional forms can be used to determine the deflection of arbitrarily loaded beams of non-uniform flexural rigidity. The level of mathematical rigor employed in this paper is intentionally aimed at a level typically taught in an introductory-level calculus course and typically encountered by a first- or second-year mechanical engineering student.

2. Shear-moment distributions and beam deflections

Much of what follows in this section is taken from Boedo [10] and is presented here for completeness. Figure 1 shows a beam subjected to an external load distribution q(x). The origin of the x,y coordinate frame is attached to the left-most end of the beam, and the y-axis points upward. The load distribution q(x) includes external actions at the supports (concentrated reaction forces and moments) and is constructed using a combination of singularity functions shown graphically in Figure 2. In functional form, this set of singularity functions representing q(x) are given by

$$f_{-2}(x) = \langle x - a \rangle^{-2} \tag{1}$$

$$f_{-1}(x) = \langle x - a \rangle^{-1} \tag{2}$$

$$f_0(x) = \langle x - a \rangle^0 = \begin{cases} 0 & x < a \\ 1 & x > a \end{cases} \tag{3}$$

$$f_n(x) = \langle x - a \rangle^n = \begin{cases} 0 & x \le a \\ (x-a)^n & x \ge a \end{cases} \qquad n = 1, 2, \ldots \tag{4}$$

$$f_s(x) = \langle x - a \rangle^0 \, f(x) = \begin{cases} 0 & x < a \\ f(x) & x > a \end{cases} \tag{5}$$

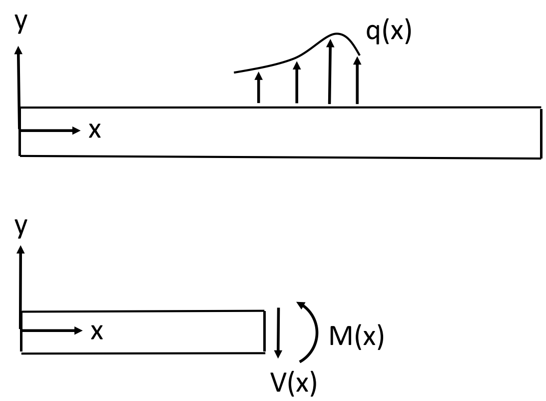

Figure 1. Sign conventions for beam bending

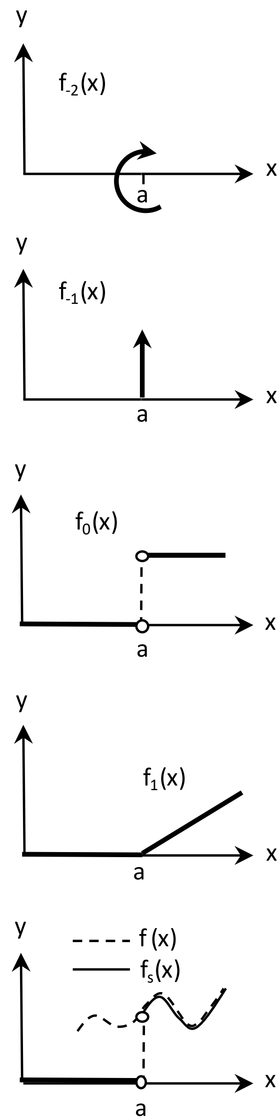

Figure 2. Singularity functions

where $f(x)$ is a smooth function (i.e. continuous, finite, and possessing all derivatives) for $x > a$. In the foregoing, $n \geq 1$ is an integer, and a is an arbitrary real number. Open circles denote the singular points of the function.

Impulse singularity functions $f_{-2}(x)$ and $f_{-1}(x)$ are defined mathematically as a limit process applied to double- and single-pulse distributed loading, respectively [10]. The singularity function $f_0(x)$, also referred to as the step or Heaviside function, is strictly discontinuous at $x = a$ and defined only in the sense of one-sided limits as $x \rightarrow a$. Polynomial-based singularity functions $f_n(x)$ are all continuous for $n \geq 1$, but their nth-order derivatives are discontinuous at the singular point $x = a$.

Contemporary textbook publications do not address the general-form singularity function $f_s(x)$ shown in equation (5), and this function was reintroduced from previous publications by Boedo [10] as a powerful extension to the singularity function method. Neither the continuity of the function $f_s(x)$ nor its derivatives are required at the (singular) point $x = a$.

The integral properties of the singularity functions are given by [10]

$$\int <x - a>^{-2} dx = <x - a>^{-1} + K \quad (6)$$

$$\int <x - a>^{-1} dx = <x - a>^{0} + K \quad (7)$$

$$\int f_n(x)\, dx = <x - a>^{n+1} / (n+1) + K \qquad n = 0, 1, 2, \ldots \quad (8)$$

$$\int f_s(x)\, dx = <x - a>^{0} [g(x) - g(a)] + K \quad (9)$$

where $g(x)$ is the anti-derivative of $f(x)$ defined by $dg/dx = f(x)$.

Concentrated external load of magnitude P_0 and concentrated external moment of magnitude M_0 applied at an arbitrary point $x = a$ on the beam are represented by first- and second-order impulse singularity functions $q(x) = P_0 <x - a>^{-1}$ and $q(x) = M_0 <x - a>^{-2}$, respectively. The sign convention for $q(x)$ represented by point and distributed loads is positive in the direction of the +y-axis. The positive sense of $q(x)$ represented by the concentrated external moment M_0 is a rotation about the –z axis.

Figure 1 shows the sign conventions adopted for the shear force $V(x)$ and bending moment $M(x)$ defined on an arbitrary section cut. Applying force and moment equilibrium to an infinitesimal beam element of width dx, the functions $q(x)$, $V(x)$, and $M(x)$ are related by

$$V(x) = \int q(x)\, dx + K_1 \quad (10)$$

$$M(x) = \int V(x)\, dx + K_2 \quad (11)$$

If q(x) is constructed from singularity functions such that $q(x) \equiv 0$ for $x < 0$, it can be shown [10] that integration constants K_1 and K_2 are always zero.

Assuming Euler-Bernoulli beam theory, the deflection of the beam is given by the second-order differential equation

$$d^2y / dx^2 = M(x) / D(x) \qquad (12)$$

where flexural rigidity D(x) = E(x) I(x), defined by Young's modulus E(x) and area moment of inertia I(x), is allowed to vary along the beam.

3. Sample problems

Three examples are illustrated to clarify the use of singularity functions in the determination of beam deflections where distributed loading and distributed flexural modulus are present. In each case, the external load distribution $q(x) \equiv 0$ for $x < 0$, so that the integration constants K_1 and K_2 in obtaining M(x) are each zero.

3.1 Example I: Point-loaded beam with discontinuous flexural rigidity

Figure 3 shows a simply-supported beam of length L subjected to a concentrated load P at the beam midspan. The half-beam sections to the left and right of P have constant flexural moduli D_0 and αD_0, respectively, where scale factor $\alpha > 0$. The distributed load q(x) is given by

$$q(x) = -(P/2) < x >^{-1} + P < x - (L/2) >^{-1} \qquad (13)$$

Note that in this and succeeding examples, additional singularity functions for $x \geq L$ are not required to "turn off" the external load, as these additional functions do not contribute to the solution. Integrating twice gives M(x), whereupon substitution of M(x) into equation (12) results in

$$d^2y /dx^2 = - P <x>^1 /[2D(x)] + P <x - (L/2)>^1 /[2D(x)] \qquad (14)$$

with boundary conditions y(x = 0) = y(x = L) = 0. The flexural modulus D(x) itself is represented by step singularity functions as

$$D(x) = D_0 [<x>^0 + (\alpha - 1) <x - (L/2)>^0] \qquad (15)$$

Equation (14) with D(x) defined in equation (15) can be rewritten as

$$\begin{aligned} d^2y /dx^2 = & - [P /(2D_0)] <x>^0 x \\ & - [P (1- \alpha)/(2D_0 \alpha)] <x - (L/2) >^0 x \\ & + [P / (D_0 \alpha)] <x - (L/2) >^1 \end{aligned} \qquad (16)$$

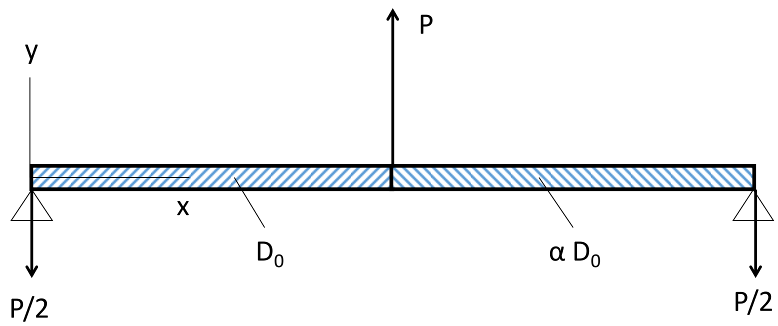

Figure 3. Point loaded beam with discontinuous flexural rigidity (Example I)

Integration yields

$$
\begin{aligned}
dy/dx = & -[P/(2D_0)]\langle x\rangle^0 (x^2/2 - 0) \\
& -[P(1-\alpha)/(2D_0\alpha)]\langle x-(L/2)\rangle^0 (x^2/2 - L^2/8) \\
& +[P/(D_0\alpha)]\langle x-(L/2)\rangle^2/2 + C_1
\end{aligned} \tag{17}
$$

where equation (9) is applied to the integral of general-form singularity functions $\langle x\rangle^0\, x$ and $\langle x-(L/2)\rangle^0\, x$. Integrating again yields the beam deflection

$$
\begin{aligned}
y(x) = & -[P/(12D_0)]\langle x\rangle^0 x^3 \\
& -[P(1-\alpha)/(48D_0\alpha)]\langle x-(L/2)\rangle^0 (4x^3 - 3L^2x + L^3) \\
& +[P/(6D_0\alpha)]\langle x-(L/2)\rangle^3 + C_1x + C_2
\end{aligned} \tag{18}
$$

Boundary conditions $y(x = 0) = y(x = L) = 0$ yield

$$C_1 = PL^2(1 + 2\alpha) / (48D_0\alpha) \tag{19}$$

$$C_2 = 0 \tag{20}$$

For $\alpha = 1$, the beam has uniform flexural rigidity D_0 throughout its span, and the deflection solution simplifies to

$$y(x) = -(P/D_0)[\langle x\rangle^0 x^3/12 - \langle x-(L/2)\rangle^3/6 - L^2x/16] \tag{21}$$

Figure 4 shows beam deflections for a family of scale factors $\alpha \geq 1$. Of particular interest is that the deflection approaches an asymptotic solution as α becomes large. As $\alpha \to \infty$, the right-most beam section becomes a rigid body, while the left-most beam section bends in a manner to maintain moment continuity at the midspan. To find the deflection for α in the range $0 < \alpha \leq 1$, the deflection shape for $1/\alpha$ from Figure 4 is reflected about the midspan, and its corresponding deflection magnitude is scaled by a factor of $1/\alpha$.

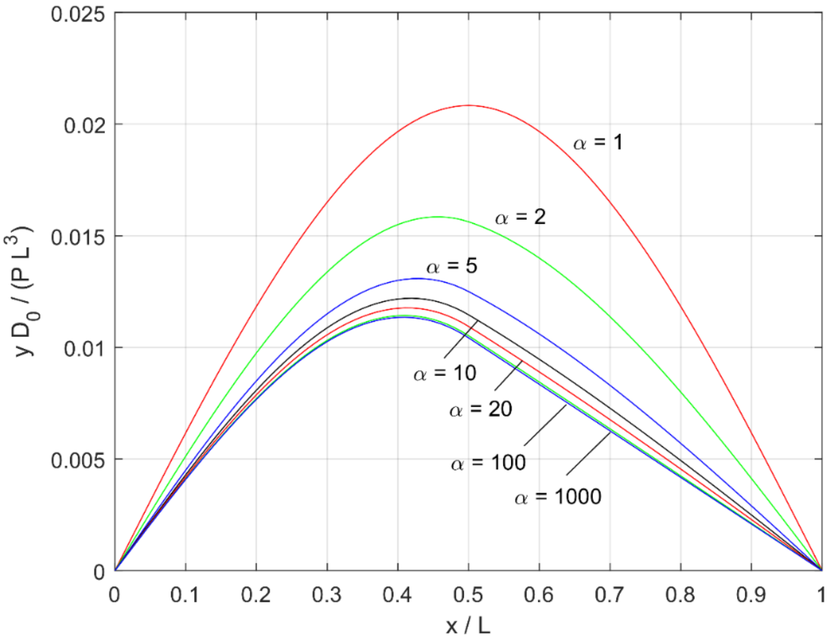

Figure 4. Beam deflections for point loaded beam with discontinuous flexural rigidity (Example I)

3.2 Example II: Cantilever beam with general distributed loading

Figure 5 shows a cantilever beam of length L and uniform flexural rigidity D_0 subjected to a distributed load of exponential form. The external load distribution q(x) for this example is given by

$$q(x) = R_0 <x>^{-1} + M_0<x>^{-2} + q_0 <x - a>^0 e^{(a - x)/L} \tag{22}$$

where reaction force and moment at the support are given by

$$R_0 = q_0 L(e^{a/L - 1} - 1) \tag{23}$$
$$M_0 = q_0 L(a + L) - 2q_0 L^2 e^{a/L - 1} \tag{24}$$

The utility of the general-form singularity function is evident here to represent and integrate the exponential term $<x - a>^0 e^{(a - x)/L}$. Since $q(x) \equiv 0$ for $x < 0$, the constants of integration can be dropped in finding V(x) and M(x).

Solving in the same manner as Example I with boundary conditions y(x = 0) = dy/dx(x = 0) = 0 results in the beam deflection

$$\begin{aligned} D_0 y(x) = \; & R_0 <x>^3/6 + M_0<x>^2/2 + q_0L<x - a>^3/6 \\ & - q_0 L^2 <x - a>^2/2 \\ & + q_0 L^3 <x - a>^1 \\ & - q_0 L^4 <x - a>^0 [1 - e^{(a - x)/L}] \end{aligned} \tag{25}$$

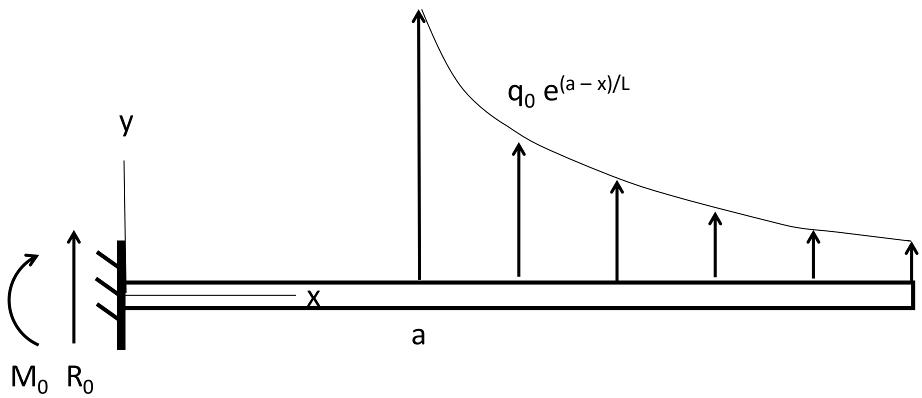

Figure 5. Cantilever beam with distributed load of exponential form (Example II)

The beam deflection δ at the free end (x = L) is evaluated from equation (25) and is given by

$$\delta \; = \; [\, q_0\, L^4 /(6D_0)]\; [\; e^{a/L\,-1} \; - \; (a/L)^3 \;] \qquad (26)$$

Figure 6 shows beam deflections for a family of parameters a/L. Apart from the special case a/L = 0, employing conventional section-cuts to determine either the moment distribution or the deflection curve itself is a very impractical method of solution.

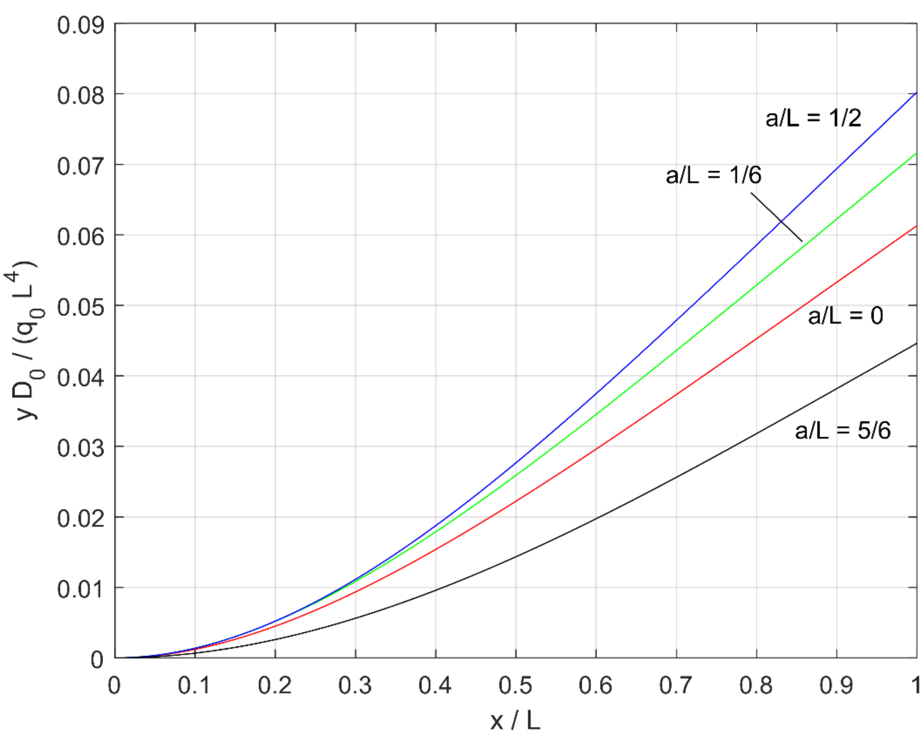

Figure 6. Beam deflections for cantilever beam with distributed load of exponential form (Example II)

3.3 Example III: Point Loaded Beam of Non-Uniform Diameter

Figure 7 shows a simply supported beam of solid circular cross-section and constant Young's modulus E_0 subjected to a concentrated load P at the beam midspan. The beam has a non-uniform diameter given by

$$d(x) = d_0\,(1 + A \sin \pi x/L) \qquad (27)$$

The beam deflection in this example is given by solution of the differential equation

$$d^2y / dx^2 = -\,32P<x>^1 / [E_0\pi d_0^4\,(1 + A\sin \pi x/L)^4\,] + 64P<x - (L/2)>^1 / [E_0\pi d_0^4\,(1 + A\sin \pi x/L)^4\,] \qquad (28)$$

with boundary conditions $y(x = 0) = 0$, $dy/dx\ (x = L/2) = 0$.

Further integration in closed-form is not possible here, so use of numerical integration will be employed. Defining $\xi = x / L$, $Y = E_0\, d_0^4\, y / (PL^3)$, equation (28) in non-dimensional form reads

$$d^2Y / d\xi^2 = -\,32<\xi>^1 / [\pi\,(1 + A\sin \pi\xi)^4\,] + 64<\xi - 1/2>^1 / [\pi\,(1 + A\sin \pi\xi)^4\,] \qquad (29)$$

which can be rewritten as

$$d^2\,Y / d\xi^2 = -\,(32/\pi) < \xi >^0 f_1(\xi) + (64/\pi) < \xi - 1/2 >^0 f_2(\xi) \qquad (30)$$

where

$$f_1(\xi) = \xi / (1 + A \sin \pi\xi)^4 \qquad (31)$$

$$f_2(\xi) = (\xi - 1/2) / (1 + A \sin \pi\xi)^4 \qquad (32)$$

Formal integration of equation (30) gives

$$dY / d\xi = -\,(32/\pi) < \xi >^0 [\, g_1(\xi) - g_1(0)\,] + (64/\pi) < \xi - 1/2 >^0 [\, g_2(\xi) - g_2(1/2)\,] + C_1 \qquad (33)$$

where

$$g_1(\xi) = \int_0^\xi [\, s / (1 + A \sin \pi s)^4\,]\, ds \qquad (34)$$

$$g_2(\xi) = \int_0^\xi [(s - 1/2) / (1 + A \sin \pi s)^4\,]\, ds \qquad (35)$$

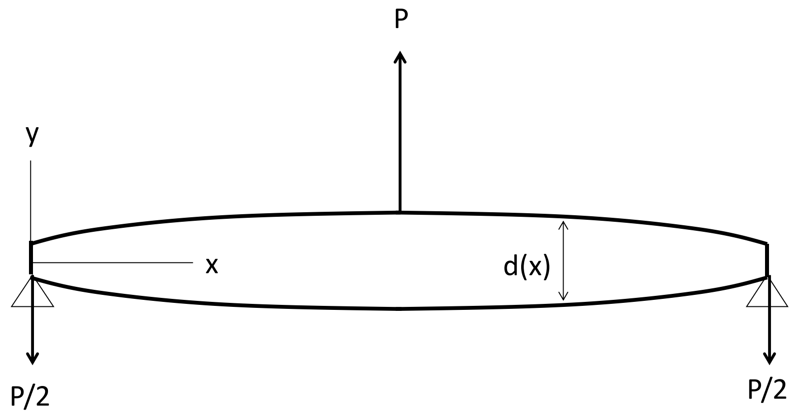

Figure 7. Point loaded beam of non-uniform diameter (Example III)

Employing the boundary condition $dY/d\xi$ ($\xi = 1/2$) = 0 along with the observation that $g_1(0) = 0$ yields $C_1 = (32/\pi)\, g_1(1/2)$, so that equation (33) becomes

$$\begin{aligned} dY/d\xi = \; & -(32/\pi) \langle \xi \rangle^0 g_1(\xi) \\ & + (64/\pi) \langle \xi - 1/2 \rangle^0 [\, g_2(\xi) - g_2(1/2) \,] + (32/\pi)\, g_1(1/2) \end{aligned} \tag{36}$$

Formal integration of equation (36) gives

$$\begin{aligned} Y(\xi) = \; & -(32/\pi) \langle \xi \rangle^0 h_1(\xi) - (64/\pi)\, g_2(1/2) \langle \xi - 1/2 \rangle^1 \\ & + (64/\pi) \langle \xi - 1/2 \rangle^0 [\, h_2(\xi) - h_2(1/2) \,] + (32/\pi)\, g_1(1/2)\, \xi \end{aligned} \tag{37}$$

where

$$h_1(\xi) = \int_0^{\xi} g_1(s)\, ds \tag{38}$$

$$h_2(\xi) = \int_0^{\xi} g_2(s)\, ds \tag{39}$$

by taking into consideration that $h_1(0) = 0$ and by setting integration constant $C_2 = 0$ from boundary condition $Y(\xi = 0) = 0$.

Numerical evaluation of the functions g_1, g_2, h_1, and h_2 can be facilitated using the fundamental theorem of calculus. Given $f(\xi)$ representing one of these functions, if the antiderivative function $F(\xi)$ is defined as

$$F(\xi) = \int_a^{\xi} f(s)\, ds \tag{40}$$

then $F(\xi)$ can be found from solution of the first-order initial value problem

$$dF / d\xi = f(\xi) \tag{41}$$

with

$$F(\xi = a) = 0 \tag{42}$$

which can be solved numerically using well-known extrapolation formulae (Euler, Runge-Kutta, etc.)

Figure 8 shows beam deflections for a family of shape factors A. Positive and negative values of A represent beam barreling and beam tapering, respectively. The g and h functions were evaluated by solving equations (41-42) using Euler's method with 10000 uniformly spaced steps from $\xi = 0$ to $\xi = 1$. Essentially identical answers were obtained with 5000 uniformly spaced steps. Alternative variable step integral methods, such as those found in Matlab, could also be applied here. Apart from the special case A = 0, the general-form singularity functions allow for relative ease of solution which relieves the student of contemplating more sophisticated and unnecessary analysis approaches, such as finite elements.

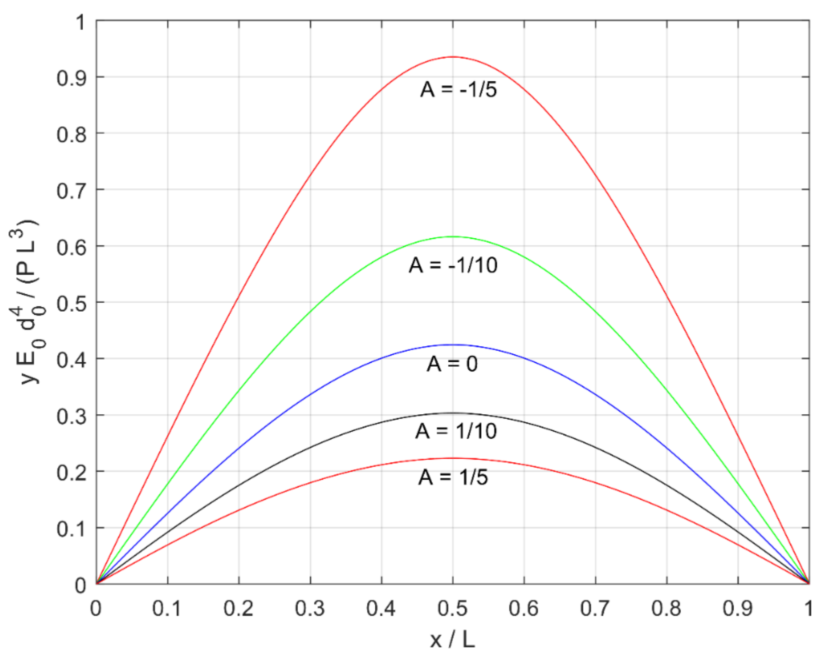

Figure 8. Beam deflections for point loaded beam of non-uniform diameter (Example III)

4. Discussion and Conclusions

This paper has extended previously published work to provide clarification and to offer extension on the use of singularity functions in the determination of beam deflections. Examples have been constructed to provide the student a systematic approach to the solution. The key feature of this paper not emphasized in current textbooks is the ability to apply the method to essentially any specified distributed load function and generally non-uniform flexural rigidity.

Alternatively, bending-induced beam deflection δ at an arbitrary location (x = a) on the beam can be determined from Castigliano's theorem by applying a dummy load Q at the beam location of interest and computing the definite integral

$$\delta \equiv y(x = a) \;=\; \int_0^L \left[M(x)\,(\partial M/\partial Q)\,/\,D(x)\,\right]\Big|_{Q=0}\,dx \qquad (43)$$

where displacement δ is positive in the direction of the dummy load. The midspan deflection for Examples I and III and the end beam deflection for Example II as given by equation (26) were checked in this manner. Castigliano's method has been the traditional solution recourse to solve for beams of non-uniform flexural rigidity. However, it appears that one can readily determine the entire shape y(x) using the integral property of the general-form singularity function with little additional computational effort.

Future work at RIT will incorporate the general-form singularity method into the mechanical engineering course curriculum. As the three examples have attempted to show, the efficacy of general-form singularity functions in a pedagogical sense is self-evident. It is impractical to expect students to solve such problems in a traditional sense using section cuts and additional continuity boundary conditions.

References

1. A.O. Dean, "Trussed tube towering over Chicago," *AIA Journal*, vol. 69, pp. 68-73, 1980.
2. B.H. Holmes, "Alcoa building: Lightweight construction," *Progressive Architecture*, vol. 33, pp. 87-91, 1952.
3. J. Giovannini, "¡Bienvenido a Miami! Zaha Hadid's first skyscraper in the U.S. is a breakthrough in high-end, high rise design," *Architect*, vol. 103, 26 pp., 2014.
4. F.J. Malina, *Kinetic Art: Theory and Practice*. New York, NY, USA: Dover, 1974.
5. R.G. Budynas and J.K. Nisbett, *Shigley's Mechanical Engineering Design*, 10th ed. New York, NY, USA: McGraw-Hill, 2015.
6. F.P. Beer, E.R. Johnston, J.T. DeWolf, and D.F. Mazurek, *Mechanics of Materials*, 7th ed. New York, NY, USA: McGraw-Hill, 2015.
7. T.H.C. Megson, *Aircraft Structures for Engineering Students*, 5th ed. Amsterdam, Netherlands: Elsevier, 2013.
8. B.J. Hamrock, B. Jacobson, and S.R. Schmid, *Fundamentals of Machine Elements*, 3rd ed. Boca Raton, FL, USA: CRC Press, 2014.
9. T.A. Philpot, *Mechanics of Materials: An Integrated Learning System,* 4th ed. Hoboken, NJ, USA: Wiley, 2017.
10. S. Boedo, "Singularity functions revisited: Clarifications and extensions for construction of shear-moment diagrams in beams," *International Journal of Mechanical Engineering Education*, to be published.