

Solving for the Internal Rate of Return Directly

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Abstract

The trial-and-error and graphical methods for computing the internal rate of return (IRR) are traditionally taught in engineering economics and financial management courses. While developing an understanding for direct computation of the IRR may enhance the learning process, little attention is devoted to direct solution. This paper explores the direct solution of the IRR using the roots of polynomials. Direct computation is feasible in three cases: two cash flows during an N -year study period, three cash flows during a two-year period, and four cash flows during a three-year period with equal revenues in years 1-3. As the degree of the polynomial increases, directly computing the IRR becomes laborious. Sharing these points with engineering economics and management students may improve the understanding of both the IRR's definition and its computational methods.

Introduction

The rate of return is a percentage figure that indicates the relative yield of different uses of capital. Rate of return methods are used to compare investment alternatives. Since the rate of return is an interest rate, the interpretation is straightforward. The internal rate of return (IRR) is one of several that are either commonly used or have been proposed in the literature; the others include the minimum attractive rate of return (MARR) and the external rate of return (ERR),¹⁵ the accounting rate of return (ARR),¹⁴ the overall rate of return,^{3,4} the modified internal rate of return (MIRR),⁹ the effective rate of return,² and the adjusted modified internal rate of return (ADMIRR).¹³ Some of the less frequently used rates of return have similar bases.⁸ The MARR is often used as a benchmark to which another rate of return is compared.¹⁵

One definition of the IRR is "the rate that yields a present worth of zero by assuming that all cash flows are reinvested at the IRR¹¹." Another definition is "the interest rate earned on the unrecovered project balance of investment such that, when the project terminates, the unrecovered project balance will be zero¹⁰." The two definitions are equivalent in that the IRR equates the present worth of project's cash expenses with its cash receipts. For a balance between expenses and income to occur, all income is presumed to be reinvested at an interest rate equivalent to the IRR. That is, all returns on funds remain internally invested in the project, such that there is no unrecovered balance at the project's termination. The IRR is the interest rate at which:

$$EW(\text{receipts}) = EW(\text{expenses}), \quad (1)$$

where EW is the equivalent worth of a project as represented by either the present worth, annual worth, or future worth. The IRR is usually discussed in terms of the present worth, so the remainder of the paper uses this perspective. The IRR can then be expressed as:

$$\sum_{k=1}^N R_k(P/F, \text{IRR}, k) = \sum_{k=1}^N E_k(P/F, \text{IRR}, k) \quad (2)$$

where R_k represents the receipts in year k , k is the year of cash flow R or E , with $0 \leq k \leq N$, N is the service life of the project (or the length of the study period), and E_k represents the absolute value of the expenses in year k . The $(P/F, \text{IRR}, k)$ expression is the discount factor, at the IRR, for the present or year zero equivalent of a future, year k value. Equation (2) can be rewritten as a difference, with the discount factors developed, as follows:

$$\sum_{k=1}^N R_k(1 + \text{IRR})^{-k} - \sum_{k=1}^N E_k(1 + \text{IRR})^{-k} = 0 \quad (3)$$

In both equations (2) and (3), the IRR is the unknown value for which the analyst has to solve.

Problem Statement

Two methods for finding the IRR are typically offered in engineering economics textbooks: trial-and-error and graphical. In the trial-and-error method, an estimate of the IRR is substituted into equation (3). If the result is positive, the interest rate is increased; if the result is negative, the rate is decreased. The process continues until a solution is found. In the graphical method, the value of equation (3) is plotted versus the interest rate. Each point at which the curve crosses the x -axis is i' . Each point of intersection is then determined by selecting two interest rates, one on each side of the abscissa, and connecting them with a straight line. Linear interpolation is used to find i' . If only one i' is greater than zero, then i' is the IRR. If more than one i' is greater than zero, then another project evaluation method is recommended.

Both the trial-and-error and graphical methods are recognized as being tedious and inefficient.¹⁰ Financial calculators, spreadsheet packages with IRR functions, and financial analysis software are substitutes for laborious manual calculations. The user need enter only the cash flows and, if necessary, the time period of analysis. Nonetheless, engineering economics students are taught the trial-and-error and graphical methods as a means of developing an understanding of the concept of the IRR. Further, calculators and computers with the desired functions are not always readily available to the analyst, so it is useful to know how to manually compute the IRR.

This paper examines direct solution methods for finding the IRR. The need for such methods is evident in recognition of the monotony of trial-and-error and graphical approaches. Further, the students in the author's engineering economics course have been, at least initially, confused and dismayed at the lack of a direct solution method for the IRR. The usefulness of direct solution through calculation of the roots of polynomials is examined. It is determined that, for general cash flows, the quadratic

equation -- a second degree polynomial -- and, for constant revenues, the cubic equation -- a third degree polynomial -- are the highest-degree polynomials for which direct solution is practical.

Two Cash Flows

Park¹⁰ discusses direct solution methods for the IRR for two cases: a transaction consisting of two cash flows -- an investment and a single, future payment -- and a project with a service life of two years with up to three cash flows. In the first case, the investment occurs in year zero and the future payment occurs in year N . The setup for this case, using equation (3), is as follows:

$$R_N/(1 + i')^N - E_0 = 0,$$

where i' is a root of the expression. Solving this equation for $(1 + i')^N$ produces the following:

$$(1 + i')^N = R_N/E_0$$

Finding the natural logarithm of both sides of this equation produces:

$$\begin{aligned} N[\ln(1 + i')] &= \ln(R_N/E_0), \text{ so} \\ \ln(1 + i') &= [\ln(R_N/E_0)]/N \end{aligned}$$

Finally, taking the exponential of both sides of the latter expression yields:

$$\begin{aligned} 1 + i' &= \exp\{[\ln(R_N/E_0)]/N\}, \text{ so} \\ i' &= \exp\{[\ln(R_N/E_0)]/N\} - 1 \end{aligned} \tag{4}$$

The IRR is equal to i' if i' is greater than 0, as follows:

$$i' > 0 \text{ if } \exp\{[\ln(R_N/E_0)]/N\} > 1 \tag{5}$$

Three Cash Flow, Two-Year Study Period

Park¹⁰ also discusses a direct solution method for a second case, in which three cash flows occur, one at time zero, then at the end of years 1 and 2, respectively. The following discussion is an expanded examination of this case. Since the concern of rate of return methods is with investment alternatives, an investment E_0 occurs in year zero. It is presumed that the future cash flows are known. Either an expense E_1 or income of R_1 occurs at the end of year 1. Similarly, either an expense E_2 or income R_2 occurs at the end of year 2. The three possible scenarios are: (E_0, R_1, R_2) , (E_0, E_1, R_2) , and (E_0, R_1, E_2) . The equation (3) setups for these cash flows are as follows:

$$\begin{aligned} R_2(1+i')^{-2} + R_1(1+i')^{-1} - E_0 &= 0, \\ R_2(1+i')^{-2} - E_1(1+i')^{-1} - E_0 &= 0, \text{ and} \\ -E_2(1+i')^{-2} + R_1(1+i')^{-1} - E_0 &= 0. \end{aligned}$$

Multiplying each term by $(1+i')^2$ and shifting all terms to the right side of the equal sign yields:

$$\begin{aligned} E_0(1+i')^2 - R_1(1+i') - R_2 &= 0, & (6) \\ E_0(1+i')^2 + E_1(1+i') - R_2 &= 0, \text{ and} & (7) \\ E_0(1+i')^2 - R_1(1+i') + E_2 &= 0. & (8) \end{aligned}$$

Equations (6)-(8) have the quadratic form $ax^2 + bx + c = 0$, where a is (the absolute value of) E_0 , b is $-R_1$ in equations (6) and (8) and E_1 in equation (7), c is $-R_2$ in equations (6) and (7) and E_2 in equation (8), and x is $(1+i')$. The roots of a quadratic equation are $x = [-b \pm (b^2 - 4ac)^{0.5}] / 2a$, so for equation (6):

$$\begin{aligned} (1+i') &= [R_1 \pm (R_1^2 + 4E_0R_2)^{0.5}] / 2E_0, \text{ so} \\ i' &= \{[R_1 \pm (R_1^2 + 4E_0R_2)^{0.5}] / 2E_0\} - 1. \end{aligned} \quad (9)$$

For equation (7),

$$\begin{aligned} (1+i') &= [-E_1 \pm (E_1^2 + 4E_0R_2)^{0.5}] / 2E_0, \text{ so} \\ i' &= \{-E_1 \pm (E_1^2 + 4E_0R_2)^{0.5}\} / 2E_0 - 1. \end{aligned} \quad (10)$$

For equation (8),

$$\begin{aligned} (1+i') &= [R_1 \pm (R_1^2 - 4E_0E_2)^{0.5}] / 2E_0, \text{ so} \\ i' &= \{[R_1 \pm (R_1^2 - 4E_0E_2)^{0.5}] / 2E_0\} - 1 \end{aligned} \quad (11)$$

In both equations (9) and (10), the term under the radical, either $(R_1^2 + 4E_0R_2)$ or $(E_1^2 + 4E_0R_2)$, is always positive. Further, in equation (9), $(R_1^2 + 4E_0R_2)$ is greater than R_1 , and, in equation (10), $(E_1^2 + 4E_0R_2)$ is greater than $|E_1|$. Hence, in both quadratic equations, there are two roots, with one positive and one negative. There is no need to be concerned with the negative root, which is obtained when the minus sign is the first arithmetic operator. For the positive root, a further condition is required for the value of i' to be positive. In equation (9), the condition is:

$$i' > 0 \text{ if } [R_1 + (R_1^2 + 4E_0R_2)^{0.5}] > 2E_0 \quad (12)$$

In equation (10), the condition is:

$$i' > 0 \text{ if } [-E_1 + (E_1^2 + 4E_0R_2)^{0.5}] > 2E_0 \quad (13)$$

If these conditions are satisfied, then i' is the IRR.

Three Cash Flow, Two-Year Study Period, Two Positive Roots

A complication is presented in equation (11) in that the term under the radical can be negative if $4E_0E_2$ is greater than R_1^2 . Hence, a requirement is that this not happen; i.e., $R_1^2 \geq 4E_0E_2$. Given that this condition is satisfied, then the term under the radical is always less than R_1 . Both roots of the quadratic, therefore, will be positive. Then, for the value of i' to be positive, the following conditions must be met:

$$i' > 0 \text{ if } [R_1 \pm (R_1^2 - 4E_0E_2)^{0.5}] > 2E_0 \text{ and } R_1^2 \geq 4E_0E_2 \quad (14)$$

There can be two values of i' which are both positive and which meet the required conditions. If only one of the i' values is greater than zero, then i' is the IRR. If both values of i' are greater than zero, then, without any knowledge of external reinvestment rates, it is not necessarily clear which i' to use as the IRR. The two positive roots problem is inherent in the fact that the sign of the cash flows changes more than once during the study period. Further investigation reveals that, in assessing any investment alternative with a multiyear service life in which the sign of the cash flows changes two or more times, by Descartes' Rule of Signs,⁶ the number of positive roots either equals the number of sign changes, or is less than the number of sign changes by an even integer. When there is more than one positive root, it is a challenge to determine which, if any, of the positive i' values are the IRR.

An extensive amount of research has been conducted on the "multiple IRR" problem. Arrow and Levhari¹ argue that the life of a project can be selected to ensure that its present worth continuously declines as a function of i' . If this condition exists, then a positive i' root will be unique. This approach presumes that the project life is a variable, and that it can be chosen before the project is undertaken. Robinson and Cook¹² offer the perspective of "incentive consistency," in which a project is terminated at the point at which its present worth begins to decrease. To apply these techniques to the somewhat trivial three cash flow case, the sequence expressed in equation (8) might be terminated at the end of year two to avoid the third year's expenses.

Given two positive i' values, to evaluate the ending balance of a project, Park¹⁰ describes a net investment test. In it, the analyst tracks the investment and its return, using each i' , until the end of the project. If the ending balance is less than or equal to zero, then i' is not the IRR. The method is readily applied to a three cash flow – two year study period problem. As the number of cash flows and the length of the study period increase, the test becomes labor intensive.

To avoid the ambiguity of multiple i' values, it is best to use an alternative project evaluation procedure, such as an equivalent worth method or the external rate of return. Both Arrow and Levhari¹ and Robinson and Cook¹² use the present worth method in interim evaluations before selecting an IRR. The external rate of return is particularly useful when an interest rate is desired as a measure, but when i' is potentially greater than the maximum available external investment rate.⁵

Four Cash Flow, Three-Year Study Period

The preceding discussion has established two limitations on the types of problems considered in this paper. First, each project is an investment alternative. Second, it is presumed that the future cash flows are known. To continue the analysis, a third limitation is required: the investment is “simple” in that there is only one change of sign in the cash flows – an investment, possibly followed by expenses during the first few years of the project, is followed by some income, with no further expenses.

Given these conditions, the development of direct-solution methods for the IRR can continue. First, it will be helpful to rewrite equation (3) as a polynomial, in a form similar to equations (6)-(8):

$$E_0(1+i')^N + E_1(1+i')^{N-1} + \dots + E_k(1+i')^{N-k} + R_{k+1}(1+i')^{N-k-1} + R_{k+2}(1+i')^{N-k-2} + \dots + R_{N-1}(1+i') + R_N = 0 \quad (15)$$

Two cases have been discussed: two terms, $E_0(1+i')^N$ and R_N , and three terms, with $N = 2$ and $k = 0$ or 1 . An additional, “nonsimple” investment case was considered in the discussion of the quadratic expression. The next case to consider is a year zero investment followed by receipts in years 1, 2 and 3. Equation (16) expresses this case:

$$E_0(1+i')^3 - R_1(1+i')^2 - R_2(1+i') - R_3 = 0, \text{ or} \\ (1+i')^3 - (R_1/E_0)(1+i')^2 - (R_2/E_0)(1+i') - (R_3/E_0) \quad (16)$$

Equation (16) has the form $y^3 + py^2 + qy + r = 0$, where $(1+i')$ is y , $-(R_1/E_0)$ is p , $-(R_2/E_0)$ is q , and $-(R_3/E_0)$ is r . The absolute value of E_0 is used. To facilitate a solution, a more convenient form is:⁷

$$x^3 + ax + b = 0, \text{ where} \quad (17) \\ y = x - (p/3) = x + (R_2/3E_0), \\ a = (1/3)(3q - p^2) = [-3(R_2/E_0) - (R_1/E_0)^2]/3, \text{ and} \\ b = (1/27)(2p^3 - 9pq + 27r) = [-2(R_1/E_0)^3 - 9(R_1/E_0)(R_2/E_0) - 27(R_3/E_0)]/27$$

The solutions to equation (17) are given by:

$$x = A + B, -0.5(A + B) + 0.5(A - B)(-3)^{0.5}, \text{ and } -0.5(A + B) - 0.5(A - B)(-3)^{0.5}, \text{ where} \\ A = [-0.5b + ((0.25b^2 + (a^3/27))^{0.5})^{(1/3)}], \text{ and} \\ B = [-0.5b - ((0.25b^2 + (a^3/27))^{0.5})^{(1/3)}]$$

If $A = B$; that is, if $0.25b^2 = -a^3/27$, then, there are three real roots; otherwise, two of the roots are imaginary, and the only real root is $x = A + B$:

$$x = \{[(R_1/3E_0)^3 + (R_1R_2/6E_0^2) + (R_3/2E_0)] + \{[0.25[-2(R_1/3E_0)^3 - (R_1R_2/3E_0^2) - (R_3/E_0)]^2 + (1/27)[- (R_2/E_0) - (1/3)(R_1/E_0)^2]^3]^{0.5}\}^{(1/3)}\} + \{[(R_1/3E_0)^3 + (R_1R_2/6E_0^2) + (R_3/2E_0)] - \{[0.25[-2(R_1/3E_0)^3 - (R_1R_2/3E_0^2) - (R_3/E_0)]^2 + (1/27)[- (R_2/E_0) - (1/3)(R_1/E_0)^2]^3]^{0.5}\}^{(1/3)}\}$$

Finally, i' , which is $y - 1$ in the original cubic equation, can be found from x , since $y = x - (p/3)$:

$$i' = \frac{\{[(R_1/3E_0)^3 + (R_1R_2/6E_0^2) + (R_3/2E_0)] + \{[0.25[-2(R_1/3E_0)^3 - (R_1R_2/3E_0^2) - (R_3/E_0)]^2 + (1/27)[- (R_2/E_0) - (1/3)(R_1/E_0)^2]^3\}^{0.5}\}^{(1/3)} + \{[(R_1/3E_0)^3 + (R_1R_2/6E_0^2) + (R_3/2E_0)] - \{[0.25[-2(R_1/3E_0)^3 - (R_1R_2/3E_0^2) - (R_3/E_0)]^2 + (1/27)[- (R_2/E_0) - (1/3)(R_1/E_0)^2]^3\}^{0.5}\}^{(1/3)} + (R_2/3E_0) - 1}{(18)}$$

While solvable, equation (18) is quite tedious. Because there are 17 terms, the manual application of equation (18) would offer no greater efficiency than a trial-and-error or graphical method. An alternative approach would be to enter the formula into a spreadsheet, enter the cash flow data, and allow the software to perform the computations. This approach, though, would merely duplicate the IRR function that many spreadsheets and financial calculators already feature. Spreadsheets typically use an iterative, trial-and-error method to find the IRR. The user must input the cash flows in the proper sequence; in some cases, a starting value and a time period of analysis must be provided.

Another consideration is that the number of real roots of the polynomial varies as a function of the discriminant. For equation (17), the discriminant D is determined as follows:

$$D = -4a^3 - 27b^2, \tag{19}$$

where a and b are as defined for the cubic equation. For $D \leq 0$, there is only one real root, although, if $a = b$, there are three real roots, with two equal. For $D > 0$, there are three distinct real roots. Given that the expression for the roots of a polynomial equation gets longer as the degree of the equation increases, and that the conditions governing the number of real roots become increasingly complex, it is evident that direct solution for general cash flows is practical only for the two and three cash flow cases.

Four Cash Flows, Three Year Study Period, Equal Revenues

A special case exists when the revenues in years 1-3 are the same. This scenario includes a year zero investment of E_0 , followed by a receipt of R in each of years 1, 2 and 3. Equation (20) expresses this case:

$$E_0(1 + i')^3 - R(1 + i')^2 - R(1 + i') - R = 0, \text{ or} \\ (1 + i')^3 - (R/E_0)(1 + i')^2 - (R/E_0)(1 + i') - (R/E_0) = 0 \tag{20}$$

Equation (20) can be reexpressed in the form $x^3 + ax + b = 0$, with a and b as defined for equation (17), as follows:

$$x^3 - [(1/3)(R/E_0)^2 + (R/E_0)]x - [(2/27)(R/E_0)^3 + (1/3)(R/E_0)^2 + (R/E_0)] = 0 \tag{21}$$

The discriminant of equation (21), using equation (19) for D and the development of a and b that follows equation (17), is:

$$D = -3(R/E_0)^4 - 14(R/E_0)^3 - 27(R/E_0)^2$$

Within the constraints of this case – an investment at time zero and revenues in years 1-3 – D will always be negative. Thus, equation (21) has one real root, as follows:

$$x = \{[(1/27)(R/E_0)^3 + (1/6)(R/E_0)^2 + 0.5(R/E_0)] + \{[0.25[-(2/27)(R/E_0)^3 - (1/3)(R/E_0)^2 - (R/E_0)]^2 + (1/27)[- (R/E_0) - (1/3)(R/E_0)^2]^3\}^{0.5}\}^{(1/3)} + \{[(1/27)(R/E_0)^3 + (1/6)(R/E_0)^2 + 0.5(R/E_0)] - \{[0.25[-(2/27)(R/E_0)^3 - (1/3)(R/E_0)^2 - (R/E_0)]^2 + (1/27)[- (R/E_0) - (1/3)(R/E_0)^2]^3\}^{0.5}\}^{(1/3)}\}$$

The equation for x can be simplified by completing the products and combining similar terms. Then i' , which is $y - 1$ in the original cubic equation, can be found from x , since $y = x - (p/3)$:

$$i' = \{(R/E_0)[(1/27)(R/E_0)^2 + (1/6)(R/E_0) + 0.5 + [(1/36)(R/E_0)^2 + (7/54)(R/E_0) + 0.25]^{0.5}]\}^{(1/3)} + \{(R/E_0)[(1/27)(R/E_0)^2 + (1/6)(R/E_0) + 0.5 - [(1/36)(R/E_0)^2 + (7/54)(R/E_0) + 0.25]^{0.5}]\}^{(1/3)} + (1/3)(R/E_0) - 1 \tag{22}$$

To ensure that i' is positive, an approximate criterion is that the sum of the three receipts exceeds the investment. Equation (22), with 13 terms, is less tedious than equation (18). The equation actually features seven different terms, because six of the terms are repetitive. In the special case in which the revenues in years 1-3 are equal, therefore, the direct solution of the IRR is feasible.

Other Forms

A direct-solution procedure is offered by Xepapadeas,¹⁶ who uses a transfer function model to find the IRR. The method is based on the premise that future cash flows are often difficult to obtain. For example, in equation (6), E_0 may be known, but there may be no useful estimates of R_1 and R_2 . The model establishes a transfer function between the investment, or expenses, and the outputs or receipts. The function is based on the notion that a relationship exists between future receipts and past expenses. The function is presumed to be linear and is the ratio of two polynomials. Xepapadeas¹⁶ proceeds to apply the model to several manufacturing industries. Each application requires the estimation of a linear filter equation that enables the transfer function to link the input, or expenses, with the output, or receipts. The equation can be developed empirically. This mathematically rigorous procedure is appropriate when the future cash flows are not known. A lack of knowledge of future cash flows probably represents reality. In contrast, the current paper only considers situations in which the future cash flows are known. Such is the condition in the problems encountered in introductory engineering economics courses. Also, in many situations, the engineering economist may reasonably be able to estimate an investment’s future receipts.

Summary and Conclusion

This paper offers direct solution methods for the IRR for three cases: two cash flows during an N -year study period, three cash flows during a two-year study period, and four cash flows during a three-year study period with equal revenues in years 1-3. Table 1 summarizes the methods. The absolute values of the expenses are used in all of the equations. The equation for the IRR is most useful in four cash flow cases: (E_0, R_N) ,

(E_0, R_1, R_2) , (E_0, E_1, R_2) , and (E_0, R, R, R) . Lines 1-3 and 5 in Table 1 provide the solutions for these. In the expense-receipt-expense case (E_0, R_1, E_2) , it is possible to obtain two positive roots from the equation in line 4. If only one root is positive, then i' is the IRR. If two i' values are positive, it is a challenge to determine which i' , if any, is the IRR. The ambiguity can be avoided by using an alternative project evaluation method, such as an equivalent worth (EW) method or the ERR. A direct solution procedure was developed for the (E_0, R_1, R_2, R_3) case. The computation of 17 terms in the equation for i' is considered to be impractical, however, given the availability of trial-and-error methods, graphical methods, IRR spreadsheet functions, and financial calculators. Nonetheless, when there are two cash flows during an N -year period, three cash flows during a two-year period, or four cash flows during a three-year period with equal revenues in years 1-3, the direct solution of the IRR is an option. The equations listed in Table 1 can be used as both analytical and instructional tools. Equation (18) for the four cash flows, three-year study period case can be used in the learning process to demonstrate the complexity of the problem, potentially enhance the understanding of the IRR, and encourage the usage of more efficient computational tools.

A suggested approach for incorporating the results of this paper into an engineering economics or financial management course is as follows:

- Introduce the IRR and provide a definition, as in paragraph 2 under “Introduction.”
- Present the equation for the IRR – equations (3) and (15) offer two forms.
- Inform the students that the IRR is directly solvable in four cases. Present the equations as listed in Table 1. Alternatively, students could be assigned the task of developing the equations for the (E_0, R_N) and (E_0, R_1, R_2) cases.
- Apply one or more of the equations in Table 1 to example problems, such as those presented in the Appendix.
- Present the 4-cash flows, 3-year period case, and present equation (18). Emphasize that the equation, while solvable, is tedious. Accentuate the discussion with the third sentence following equation (19), which concludes “...it is evident that direct solution for general cash flows is practical only for the two and three cash flow cases.”
- Describe and demonstrate both the trial-and-error and graphical methods for solving for the IRR, emphasizing cases with four or more cash flows and three or more year study periods. Further, discuss the IRR function that is available in spreadsheet software.
- List this paper as recommended (but not necessarily required) reading.

Table 1. Direct Solution Methods for the IRR

Number of Cash Flows	Study Period	Sequence of Cash Flows	IRR	Conditions
2	N	E_0, R_N	$\exp\{\ln(R_N/E_0)/N\} - 1$	$\exp\{\ln(R_N/E_0)/N\} > 1$
3	2	E_0, R_1, R_2	$\{[R_1 + (R_1^2 + 4E_0R_2)^{0.5}]/2E_0\} - 1$	$[R_1 + (R_1^2 + 4E_0R_2)^{0.5}] > 2E_0$
3	2	E_0, E_1, R_2	$\{[-E_1 + (E_1^2 + 4E_0R_2)^{0.5}]/2E_0\} - 1$	$[-E_1 + (E_1^2 + 4E_0R_2)^{0.5}] > 2E_0$
3	2	E_0, R_1, E_2	$\{[R_1 \pm (R_1^2 - 4E_0E_2)^{0.5}]/2E_0\} - 1$	$R_1^2 \geq 4E_0E_2$ and <i>If one i' is positive, then $i' = IRR$. If two roots are positive, then use EW or ERR method.</i>
4	3	E_0, R, R, R	see equation (a) below	ensure that $3R > E_0$

$$(a) IRR = \left\{ \frac{(R/E_0)[(1/27)(R/E_0)^2 + (1/6)(R/E_0) + 0.5 + [(1/36)(R/E_0)^2 + (7/54)(R/E_0) + 0.25]^{0.5}]^{(1/3)}}{(R/E_0)[(1/27)(R/E_0)^2 + (1/6)(R/E_0) + 0.5 - [(1/36)(R/E_0)^2 + (7/54)(R/E_0) + 0.25]^{0.5}]^{(1/3)} + (1/3)(R/E_0) - 1 \right\}$$

Appendix: Example Applications of Direct Solution Methods to the IRR

Two Cash Flows: Equation (4)

Year 0: -\$35,000, Year 8: \$100,000

$$IRR = \exp\{\ln(100,000/35,000)/8\} - 1 = \exp(0.13122) - 1 = 0.1402 \text{ or } 14.02\%$$

Three Cash Flows (E_0, R_1, R_2): Equation (9)

Year 0: -\$7,000, Year 1: \$4,000, Year 2: \$4,500

$$\begin{aligned} IRR &= \{[4,000 + [(4,000)^2 + 4(7,000)(4,500)]^{0.5}]/2(7,000)\} - 1 \\ &= [(4,000 + 11,916.375)/14,000] - 1 \\ &= 0.1369 \text{ or } 13.69\% \end{aligned}$$

Three Cash Flows (E_0, E_1, R_2): Equation (10)

Year 0: -\$600, Year 1: \$-200, Year 2: \$1,200

$$\begin{aligned} IRR &= \{[-200 + [(200)^2 + 4(600)(1,200)]^{0.5}]/2(600)\} - 1 \\ &= [(-200 + 1,708.8007)/1,200] - 1 \\ &= 0.2573 \text{ or } 25.73\% \end{aligned}$$

Four Cash Flows (E_0, R, R, R): Equation (22)

Year 0: -\$3,000, Year 1: \$1,400, Year 2: \$1,400, Year 3: \$1,400

$$\begin{aligned} IRR &= \left\{ \frac{(1,400/3,000)[(1/27)(1,400/3,000)^2 + (1/6)(1,400/3,000) + 0.5 + [(1/36)(1,400/3,000)^2 + (7/54)(1,400/3,000) + 0.25]^{0.5}]^{(1/3)}}{(1,400/3,000)[(1/27)(1,400/3,000)^2 + (1/6)(1,400/3,000) + 0.5 - [(1/36)(1,400/3,000)^2 + (7/54)(1,400/3,000) + 0.25]^{0.5}]^{(1/3)} + (1/3)(1,400/3,000) - 1} \right\} \\ &= \{ (0.46667)[0.008066 + 0.077778 + 0.5 + (0.006049 + 0.060494 + 0.25)^{0.5}]^{(1/3)} + \\ &\quad \{ (0.46667)[0.008066 + 0.077778 + 0.5 - (0.006049 + 0.060494 + 0.25)^{0.5}]^{(1/3)} + 0.15556 - 1 \} \\ &= [(0.466667)(0.585844 + 0.562622)]^{(1/3)} + [(0.466667)(0.585844 - 0.562622)]^{(1/3)} - 0.844444 \\ &= 0.812285 + 0.221293 - 0.844444 \\ &= 0.1891 \text{ or } 18.91\% \end{aligned}$$

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