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Spring Connectivity Diagram: An Intuitive Approach to Determining the Equivalent Lumped Stiffness of a System of Springs and Simple Continuous Elements

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Abstract

Lumping parameters for a continuous mechanical system is one of the early topics of discussion in courses like System Dynamics and Vibrations. When the system includes multiple continuous elements such as beams, bars, or rods along with spring elements, determining the equivalent lumped stiffness can be difficult, especially with respect to how each stiffness element is arranged (in parallel or series) with respect to other elements. In this paper, the spring connectivity diagram (SCD) approach is introduced as a simple way to analyze such problems. In this method, the effective stiffness of all continuous elements is first determined, the nodes at the ends of all stiffness elements are identified and the SCD is obtained by connecting each stiffness element to the proper nodes. The stiffnesses in the resulting diagram may then be combined to determine the equivalent lumped stiffness of the entire system. Three examples are provided to show the application of this intuitive approach to different types of problems that involve continuous and spring stiffness elements. Two assessment methods that compare the SCD and infinite stiffness approaches suggest that a significantly larger number of students who are taught the SCD approach are able to correctly complete a similar lumped stiffness problem on an exam.

Introduction and Motivation

Every mechanical system in the real world is continuous, which means that the mass, stiffness, and damping elements are distributed throughout the system. One of the first topics of discussion in a System Dynamics or Vibrations course usually involves lumping elements of continuous mechanical systems such that their total inertia, stiffness, and damping can be represented as mass, spring, and damper elements, respectively. Distributed-parameter models involve partial differential equations (PDEs), which are significantly more difficult to solve than the ordinary differential equations (ODEs) associated with lumped-parameter models. The mathematical complexity of distributed-parameter models, however, allows for the motion of the system to be determined at any location and at any time. If the primary behavior(s) of the system are only of importance, lumped-parameter models usually suffice. At the undergraduate level, lumped-parameter modeling is almost exclusively considered.

The manner in which lumped parameters of a simple mechanical system are estimated is logically straightforward, but may be mathematically difficult for an undergraduate student. The effective or lumped mass of simple continuous structures such as rods, bars, and beams can be determined by equating the kinetic energy of the member when subjected to a displacement to the kinetic energy of a single-degree-of-freedom system with an effective mass, m_{eff}. Similarly, the effective stiffness of such structures can be determined by equating the strain energy contained within the elastically-deformed member to the energy stored in a spring with an effective stiffness, k_{eff}. The effective lumped damping of a mechanical system is determined by equating the energy dissipated over one vibration cycle to that of a viscous damper with an effective coefficient, c_{eff} [1]. When multiple lumped stiffnesses or masses are involved as part of a larger system, it may be possible to combine them together to determine the equivalent or overall stiffness or mass, which will be explored in more depth. For complex, real-world setups, lumped parameters may be estimated from the step[2] or frequency response of the system.

In this paper, the spring connectivity diagram (SCD) approach is introduced as a way to combine the stiffnesses of a system of springs and continuous elements such as beams, rods, and bars. In advanced texts[3], Castigliano's theorem has been used to determine the equivalent stiffness of such a system, but to this author's knowledge, no textbook or publication provides a *simple* approach to this type of problem that can be applied by most undergraduate engineering students.

Spring Connectivity Diagram (SCD) Approach to Determining the Equivalent Lumped Stiffness of a System

The main idea behind the SCD approach is to have students draw an intermediate diagram in order for them to better visualize lumping parameters (like stiffness) of continuous systems and how these lumped parameters may be combined with one another. The series of steps that follow are presented to help any student develop the spring connectivity diagram in order to determine the equivalent lumped stiffness of a system. Some of the steps that will be outlined may seem trivial, but they will be effective in helping students draw the proper SCD, particularly for those who are newly introduced to this topic.

Step 1: Determine the effective stiffness of all non-spring elements At the undergraduate level, students are typically not expected to derive the effective stiffness relations for bars, rods, and beams on their own; oftentimes, a summary of effective stiffness values are provided to students in tabulated form similar to what is shown in Table 1 for various beam elements. A more comprehensive list can be found in reference texts such as Shigley's Mechanical Engineering Design[4].

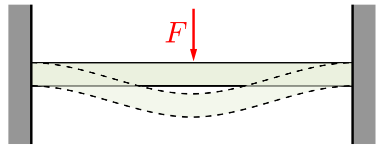
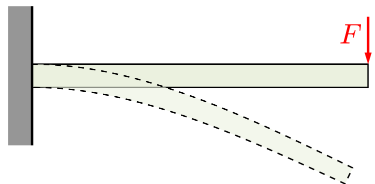
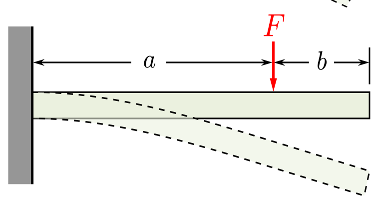
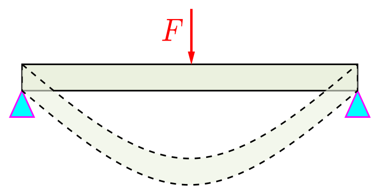
Step 2: Identify the nodes at the ends of each element The displacement at each end of all spring and non-spring elements must be identified. This is particularly helpful in determining whether stiffness elements are in series, parallel, or neither with one another.

Step 3: Draw the spring connectivity diagram (SCD) The spring connectivity diagram is a diagrammatic representation of the relationship between all spring and non-spring elements in the system as springs. In order to draw the SCD, the following steps are needed:

i. Draw the nodes identified in Step 2 separately, including a fixed support. Draw the external force at the node at which it acts along with the displacement of each node.

ii. Connect each spring and non-spring element to its proper pair of nodes.

Table 1 Effective stiffness of uniform beams with Young's modulus E, area moment of inertia I and length L (based on maximum deflection)

Description	Figure	Effective stiffness
Fixed-fixed beam subjected to a point load at its center		$192\frac{EI}{L^3}$
Cantilever beam subjected to a point load at its free end		$3\frac{EI}{L^3}$
Cantilever beam subjected to a point load at a distance *a* from its fixed end		$\frac{6}{\left(\frac{a}{L}\right)^2\left(3-\frac{a}{L}\right)}\frac{EI}{L^3}$
Simply-supported beam subjected to a point load at its center		$48\frac{EI}{L^3}$

iii. Reorganize the diagram so that relationships between springs, *i.e.*, series or parallel, are more recognizable. From Rao[1], if two springs k_1 and k_2 are in series, they are subjected to the same force, resulting in an equivalent stiffness of

$$k_{\text{eq,series}} = \left[\frac{1}{k_1} + \frac{1}{k_2}\right]^{-1} \tag{1}$$

If the same two springs are arranged in parallel, they are subjected to the same displacement, resulting in an equivalent stiffness of

$$k_{\text{eq,parallel}} = k_1 + k_2 \tag{2}$$

Step 4: Combine springs in the proper manner to determine the equivalent stiffness

Spring arrangements that are clearly in series or parallel should be combined. For example, if two springs share the same two pairs of nodes, they are in parallel. If the force applied to two springs is the same and they share a common node, they will be in series. If the spring arrangements in the SCD cannot be combined in series and parallel to yield a single equivalent stiffness, Newton's 2^{nd} law must be applied at each node. It may be possible to combine the resulting equations in a manner that will yield an equivalent stiffness for the setup.

Examples

Example 1: Parallel elements Determine the equivalent lumped stiffness of the system shown in Figure 1, k_{eq}. Assume all elements to be massless.

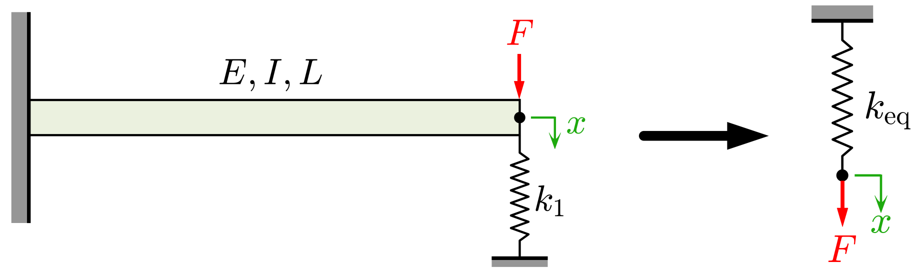


Figure 1 Example 1

Example 1 Solution There is one continuous element in this system, a cantilever beam of

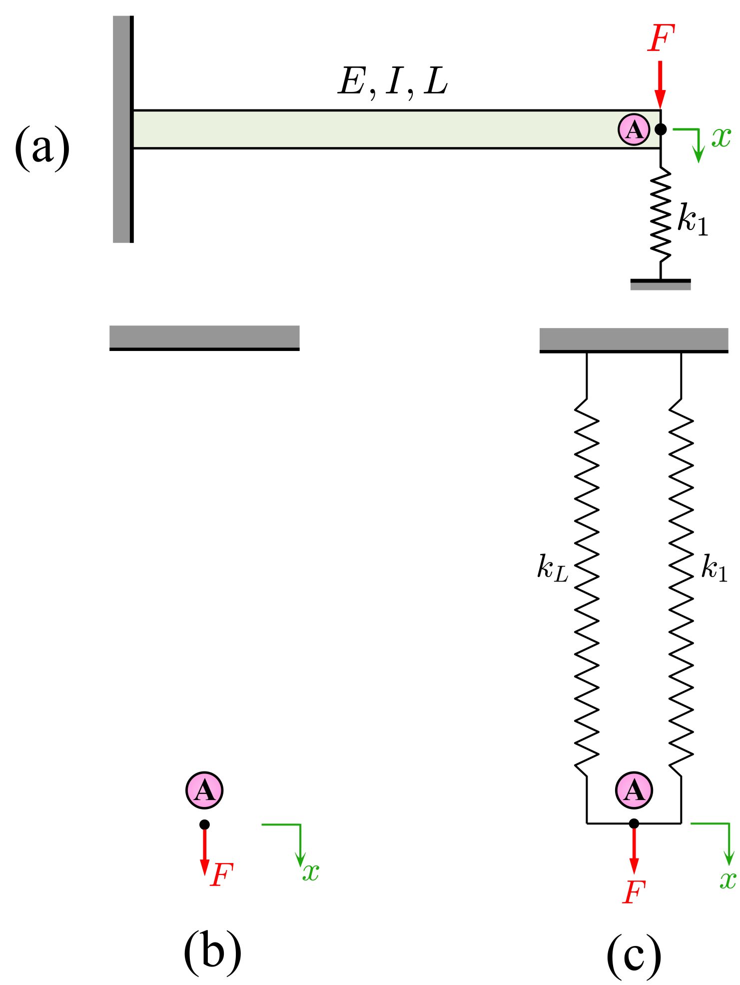


Figure 2 Example 1 (a) Node identification, (b) Nodes of the SCD and (c) completed SCD

length L where the force is applied at the tip. From Table 1, the effective stiffness of this beam is

$$k_L = \frac{3EI}{L^3} \tag{3}$$

There are two stiffness elements present in this system, each of which is connected to a fixed support at one end and a common node, which will be denoted as A, as shown in Figure 2(a). To establish the SCD, the fixed node and common node A are isolated, as shown in Figure 2(b). Both the spring k_1 and the effective beam stiffness k_L share the aforementioned common node and are connected to a fixed end. This simply leads to the SCD in Figure 2(c). As shown, these two springs are clearly in a parallel arrangement. Since the stiffnesses are in parallel, the equivalent stiffness of this system is found to be

$$k_{\text{eq}} = k_1 + k_L = \boxed{k_1 + \frac{3EI}{L^3}} \tag{4}$$

After determining the equivalent system stiffness, a suggested check would be to set one of the stiffnesses to infinity and logically deduce whether the result obtained from the SCD approach makes sense. For this example, if $k_1 \to \infty$ (is rigid), the system is not expected to experience any displacement because the cantilever beam would not be able to bend. Looking at (4), setting $k_1 \to \infty$ leads to $k_{\text{eq}} \to \infty$, implying that the system would be rigid, which is exactly the conclusion that was reached just by visualizing how the system would behave.

Example 2: Elements in series and in parallel Determine the equivalent lumped stiffness of the system shown in Figure 3, k_{eq}. Assume all elements to be massless.

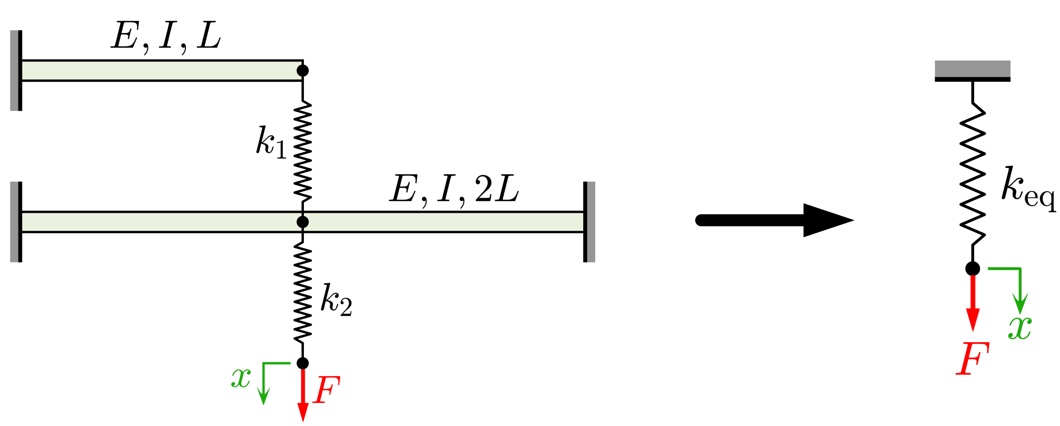


Figure 3 Example 2

Example 2 Solution There are two continuous elements in this system: a cantilever beam of length L where the (spring) force is applied at the tip and a fixed-fixed beam of length $2L$ where the force is applied at its middle. From Table 1, the effective stiffness of these beams are

$$k_L = \frac{3EI}{L^3} \tag{5}$$

$$k_{2L} = \frac{192EI}{(2L)^3} = \frac{24EI}{L^3} \tag{6}$$

There are four stiffness elements in this system, as shown in Figure 4(a):

- The spring associated with the cantilever beam is fixed at one end and has a node at **C** which it shares with k_1.
- The spring associated with the fixed-fixed beam is fixed at one end and has a common node **B** with k_1 and k_2.
- Spring k_1 has nodes at **B** and **C**.
- Spring k_2 has nodes **B** and **A**, at which the external force F is applied.

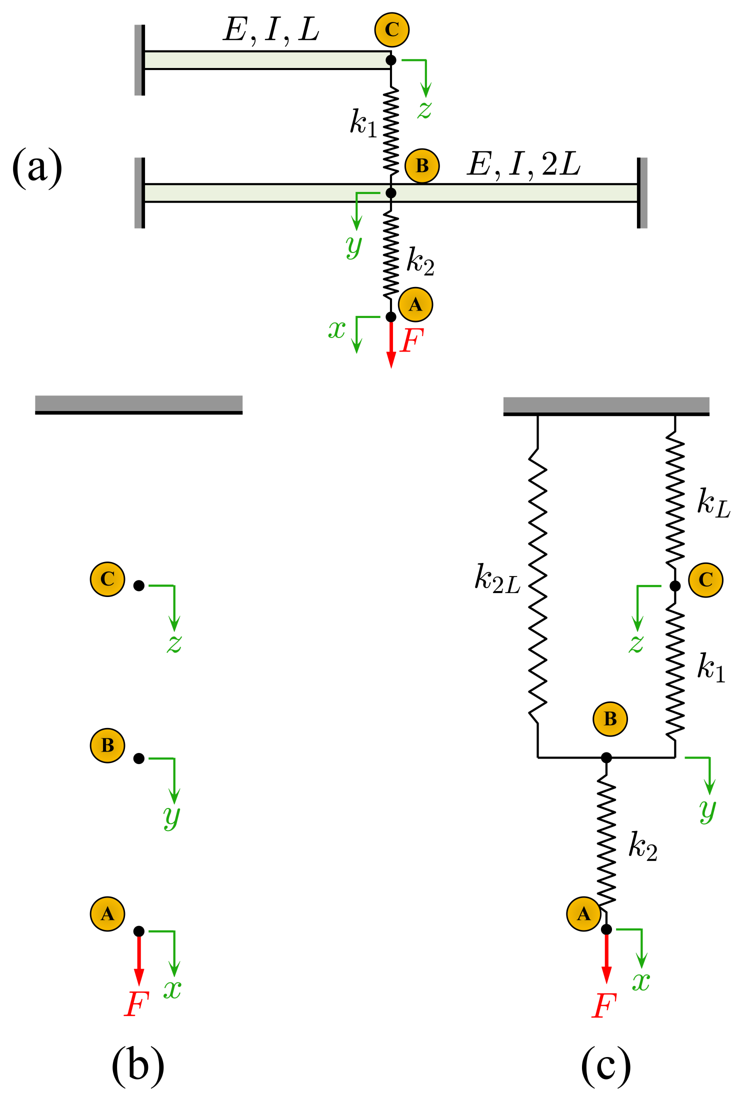


Figure 4 Example 2 (a) Node identification, (b) Nodes of the SCD and (c) completed SCD

To establish the SCD, the fixed node and nodes A, B, and C are isolated, as shown in Figure 4(b). Drawing the springs between the nodes as outlined above leads to the SCD shown in Figure 4(c). As shown, springs k_1 and k_L are in series. The equivalent stiffness of these two springs is in parallel with k_{2L}. The resulting stiffness is in series with k_2. Mathematically, the equivalent stiffness of this system is found to be

$$k_{\mathrm{eq},1} = \left[\frac{1}{k_L} + \frac{1}{k_1}\right]^{-1}$$

$$k_{\mathrm{eq},2} = k_{\mathrm{eq},1} + k_{2L} = \left[\frac{1}{k_L} + \frac{1}{k_1}\right]^{-1} + k_{2L}$$

$$k_{\mathrm{eq}} = \left[\frac{1}{k_{\mathrm{eq},2}} + \frac{1}{k_2}\right]^{-1} = \boxed{\left[\frac{1}{k_2} + \frac{1}{\left[\frac{1}{k_L} + \frac{1}{k_1}\right]^{-1} + k_{2L}}\right]^{-1}} \tag{7}$$

where k_L and k_{2L} have been defined in equations (5) and (6).

This example demonstrates the ease with which the SCD approach can be used to determine the equivalent lumped stiffness of a system containing elements that are arranged both in parallel and in series. Most students are usually comfortable simplifying the SCD by combining elements in series and parallel since it looks similar to combining resistors, but special care should be taken since springs in parallel combine similarly to resistors in series and vice versa.

Example 3: Neither in series nor in parallel Determine the equivalent lumped stiffness of the system shown in Figure 5, k_{eq}. Assume all elements to be massless.

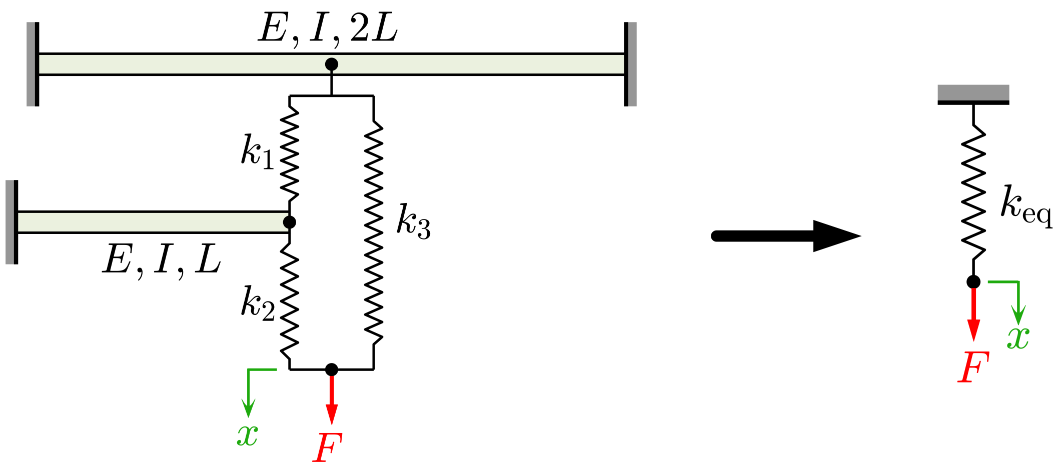


Figure 5 Example 3

Example 3 Solution There are two continuous elements in this system: a cantilever beam of length L where the (spring) force is applied at the tip and a fixed-fixed beam of length $2L$

where the force is applied at its middle. From Table 1, the effective stiffness of these beams are

$$k_L = \frac{3EI}{L^3} \tag{8}$$

$$k_{2L} = \frac{192EI}{(2L)^3} = \frac{24EI}{L^3} \tag{9}$$

There are five stiffness elements in this system, as shown in Figure 6(a):

- The spring associated with the fixed-fixed beam is fixed at one end and has a node at C which it shares with k_1 and k_3.
- The spring associated with the cantilever beam is fixed at one end and has a common node B with k_1 and k_2.

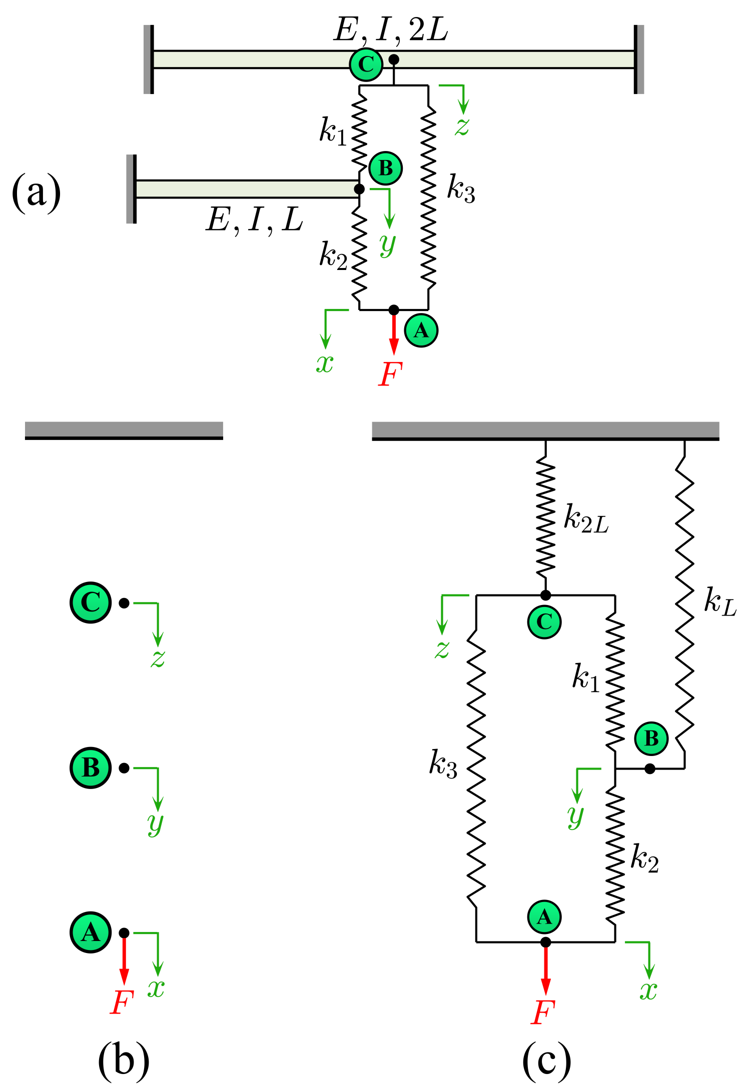


Figure 6 Example 3 (a) Node identification, (b) Nodes of the SCD and (c) completed SCD

- Spring k_1 has nodes at **B** and **C**.
- Spring k_2 has nodes at **B** and **A**, at which the external force F is applied.
- Spring k_3 has nodes at **C** and **A**.

To establish the SCD, the fixed node and nodes **A**, **B** and **C** are isolated, as shown in Figure 6(b). Drawing the springs between the nodes as outlined above leads to the SCD shown in Figure 6(c). The springs in this system cannot be combined in a straightforward manner. Thus, the equations of motion at the nodes should be determined first and subsequently combined to yield the equivalent stiffness of this system.
The kinetic (KD) and free-body diagrams (FBD) of the nodes are shown in Figure 7. Note that the terms in the kinetic diagram are zero at the nodes for this system since every element is massless.

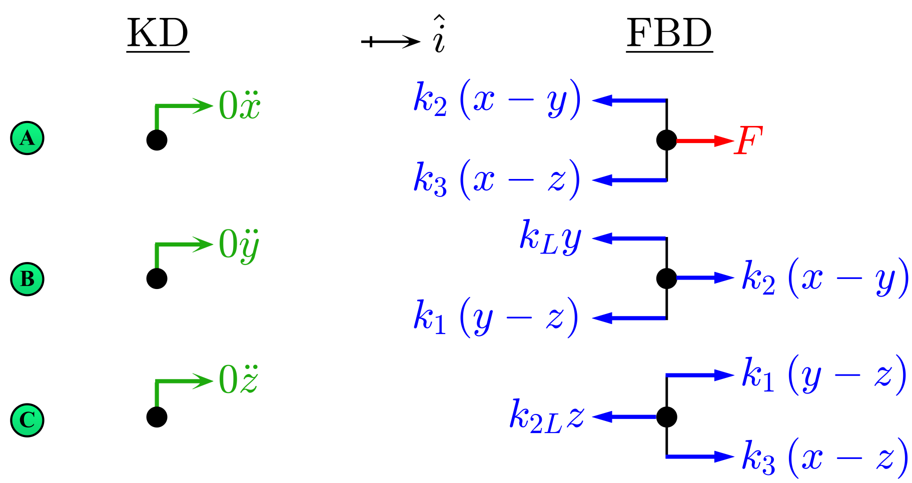


Figure 7 Example 3 kinetic and free-body diagrams of important nodes (rotated 90° here for demonstration purposes)

Newton's 2nd law yields the following system of equations:

$$\begin{cases} 0 = F - k_2(x-y) - k_3(x-z) \\ 0 = k_2(x-y) - k_L y - k_1(y-z) \\ 0 = k_1(y-z) + k_3(x-z) - k_{2L} z \end{cases} \tag{10}$$

where k_L and k_{2L} have been defined in equations (8) and (9).
Simultaneously solving the three equations in (10) leads to

$$F = k_{\text{eq}} x \tag{11}$$

where

$$k_{\text{eq}} = \boxed{\frac{k_1k_2k_{2L} + k_1k_2k_L + k_1k_{2L}k_3 + k_1k_3k_L + k_2k_{2L}k_3 + k_2k_{2L}k_L + k_2k_3k_L + k_{2L}k_3k_L}{k_1k_2 + k_1k_{2L} + k_1k_3 + k_1k_L + k_2k_{2L} + k_2k_3 + k_{2L}k_L + k_3k_L}} \tag{12}$$

This example demonstrates how the SCD can be used to find the equivalent stiffness of the system when determining whether the springs are in series or in parallel is difficult if not impossible. If the mass of either beam is not negligible, it would not be possible to determine the equivalent stiffness of the system in a closed-form fashion, even though the equations of motion from Newton's 2^{nd} law are sufficient to completely find each degree of freedom.

Assessment

Lumping the stiffness of continuous systems, particularly those involving springs, is a rather specialized type of problem. At Rose-Hulman Institute of Technology, it constitutes less than one 50-minute lecture in the Analysis and Design of Engineering Systems course, which is equivalent to a sophomore- or junior-level System Dynamics in most other mechanical engineering programs. Therefore, properly assessing the effectiveness of the SCD approach has been a challenging task. Nonetheless, two evaluation methods have been considered by the author to test whether this technique helps students better understand how to lump stiffnesses compared to the alternative, the infinite stiffness approach (ISA). Admittedly, each assessment method on its own may not provide strong enough evidence of the efficacy of the SCD approach but when taken as a whole, the results are hopefully convincing.

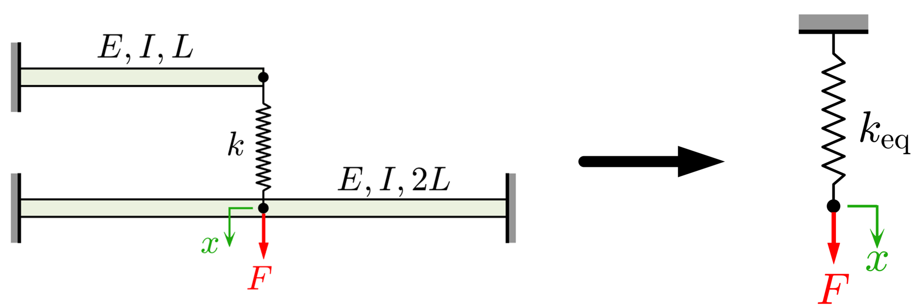
Before talking about each assessment method, it is necessary to briefly introduce the ISA. With this technique, the students identify the pair of stiffnesses furthest from the node at which the force is applied and combine them by assuming one stiffness is infinitely large and considering whether the pair would experience motion when the system is subjected to the force. If the pair of stiffnesses will move, then they will be in series. If not, the stiffnesses are in parallel. Once the first pair of stiffnesses is combined, the resulting stiffness is combined with the next spring or beam and the process continues until the stiffnesses whose node(s) are subjected to a force are combined.

- **Anecdotal evidence**: Experience shows that prior to introducing either the SCD or the ISA, a vast majority of students will say that the stiffness of the beam will be in series with the spring stiffness when first shown Example 1. Similarly, many students tend to take k_1 and k_2 to be in series with one another in Example 2, even when using the ISA (more on this in the next bullet point), without accounting for how the beams will influence the equivalent system stiffness. Without incorporating the SCD and depending solely on the ISA, students cannot solve Example 3 at all since some stiffnesses are arranged neither in parallel with nor in series to one another.
- **Performance on exams**: As a summative assessment, a similar question relating to the lumped stiffness of continuous systems was asked from three different cohorts of students in an exam for the Analysis and Design of Engineering Systems course, which were all taught by the author in an in-person setting over a 10-week period in three different quarters. Cohort 1 consisted of 22 mechanical engineering sophomores and juniors, 1 engineering design junior and 1 biomedical engineering senior who were taught to lump stiffnesses using the ISA. Cohort 2, which consisted of 15 mechanical engineering sophomores and juniors, were also taught the ISA. Cohort 3 was made up of 43 mechanical

engineering sophomores and juniors and 1 biomedical engineering senior and they were taught the SCD approach to lumping stiffnesses. While GPA information was not accessible to the author for each cohort, as a matter of comparison, the average Final Grade for the students in each cohort were in the $80 \pm 3\%$ range. An unpaired two-tailed Student's t-Test yields that, at a 95% confidence interval, there is no statistically significant difference between the Final Grade averages between any combination of the three cohorts.

Cohorts 2 and 3 received an identical question, which is shown in the text box below. The students in Cohort 1 received a very similar question, with the difference being that the force was applied at the upper node of the connecting spring (as opposed to the lower one) and that no multiple-choice options were provided. For Cohorts 2 and 3, students' work was also checked to ensure that the correct answer was not the result of a lucky guess.

Assuming that all the beams and springs are massless and that $EI/L^3 = 1\ \mathrm{N/m}$ and $k = 6\ \mathrm{N/m}$, the equivalent stiffness of the system is closest to which of the following values?



(a) $1\ \mathrm{N/m}$
(b) $1.85\ \mathrm{N/m}$
(c) $3\ \mathrm{N/m}$
(d) $6.55\ \mathrm{N/m}$
(e) $7.8\ \mathrm{N/m}$
(f) $26\ \mathrm{N/m}$
(g) $33\ \mathrm{N/m}$
(h) $194\ \mathrm{N/m}$
(i) $201\ \mathrm{N/m}$

For Cohort 1, who were not given multiple-choice options and were taught the ISA, the average score on the lumped stiffness problem on the exam was 59%. This was the lowest average on any individual problem on any of the exams for this group of students. For Cohort 2, who were also taught the ISA, 2 of 15 students managed to correctly answer the question shown in the text box above. For Cohort 3, however, 20 out of 44 students answered the multiple-choice question correctly using the SCD approach. Furthermore, all but 9 students in this cohort drew the SCD correctly, which is very significant, and those who made mistakes either forgot to take the double length of the fixed-fixed beam into account, treated the fixed-fixed beam as simply supported, or combined springs in series and in parallel similar to how resistors are combined. Since students were instructed to document their work on this problem on the exam (and they all did), an effort was made to compare their performance to that of Cohort 1 by assigning partial credit to their solution based on a grading rubric similar to that of the problem solved by Cohort 1. This effort led to an average problem score of 84% for Cohort 3. Thus, with 95% confidence, the results for Cohort 3, who were taught the SCD approach, were statistically significantly better than those of Cohorts 1 and 2.

To this author's knowledge, teaching options with respect to lumping stiffnesses are limited to the SCD approach and the ISA for sophomore- and junior-level students who will likely struggle with more advanced methods such as Castigliano's theorem. While more extensive assessment would certainly be welcome, taken in conjunction with anecdotal evidence, the results of this assessment seem to at least suggest that the SCD approach is superior to the ISA.

Conclusions

In this work, the spring connectivity diagram (SCD) approach to determining the equivalent lumped stiffness of a system containing springs and simple continuous elements has been presented and its utility is demonstrated with three examples. As the three examples in this paper demonstrate, this approach can be applied to virtually any combination of springs and simple continuous elements (here only beams have been considered, but the process is identical for rods and bars). The primary advantage of the SCD is that it provides a visual framework for the student to build off of and takes guesswork out of the process. Anecdotal evidence and the assessment of student performance on a related exam problem suggest that the SCD approach is, to this author's knowledge, the most helpful technique to teach undergraduate students how to lump stiffness elements. Further assessment of this approach with a true control and experimental group would certainly aid in determining whether the conclusion the author has reached here is the correct one. It is the author's sincere hope that this technique will help make this topic less confusing for students compared to how he learned it when he was in their shoes.

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