STEM Majors' Ability to Relate Integral and Area Concepts

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STEM Majors’ Ability to Relate Integral and Area Concepts

Area and integral concepts are interrelated under certain circumstances. In this work, senior undergraduate and graduate mathematics and engineering students’ ability to combine concept image and concept definition based on their integral knowledge is observed. Seventeen participants of this study were either enrolled or completed a Numerical Methods/Analysis course at a large Midwest University during a particular semester. The participants completed a questionnaire and got interviewed to explain their written questionnaire responses. The questionnaire questions covered concepts such as functions, differentiation, function integrals, power series, and programming preferences of the participants. Action-Process-Object-Schema (APOS) theory of Asiala, Brown, DeVries, Dubinsky, Mathews and Thomas (1996) is considered initially for evaluation of the research question, however this theory is determined to be inappropriate for evaluating the research question. The data collected from the written questionnaire and video recorded interview responses are evaluated by using the concept image and concept definition approach of Dreyfus & Vinner (1989). In addition, Triad classification of the participants are determined to obtain the qualitative and quantitative results presented in this work.

Key Words: Riemann integral, area, functions, concept image, concept definition, APOS theory

Introduction

An important application of Riemann integral is determining the area between a single variable continuous function and the input axis. Given a continuous function f on the interval [a, b], the area between the function and the input axis can be calculated by using the formula

$$Area = \left| \int_a^b f(x)dx \right|$$

This definition of area by using integral concept requires a well-developed knowledge of concept image and concept definition of Riemann integrals. The use of absolute value with definite integral is an important aspect of the research question for the area calculations. In this work, the goal is to observe graduate and senior undergraduate mathematics and engineering students’ ability to combine integral and absolute value concepts by evaluating their responses to an integral question.

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Methodology

In pedagogy, researchers needed to observe students' comprehension of the function concept. The definitions in mathematical pedagogy are based on the research on students contrary to the mathematics development of the function concept. A mixture of two methodologies; Triad classification and Concept Image and Concept Definition, will be used for data evaluation in this work. This mixture is expected to yield a better understanding of the data, and therefore a better understanding of the participating students' thinking process from two different research perspectives.

The pedagogical approach to the function concept in the undergraduate curriculum is not explored until the 1970s. The concept image and concept definition of functions in mathematical education are defined by Hershkowitz and Vinner (1980) with a geometric approach and again by Tall & Vinner (1981, pg. 153); however, the most extensive research in the undergraduate curriculum was done by Dreyfus & Vinner (1989) in which they defined the concept image and concept definition of functions based on their research with undergraduate students. In this work, concept image and concept definition of functions' definite integral will be used to understand undergraduate and graduate STEM majors' ability to relate the area between a function and the input axis to the definite integral of functions.

Action-Process-Object-Schema (APOS) theory is used in mathematics and engineering education to measure knowledge of undergraduate students' conceptual classification in topics such as function, limit, derivative, and integral. APOS theory is particularly useful in measuring students' knowledge of a specific concept by determining how much students know about the prerequisite topics of the concept. Students' cognitive knowledge of concepts in APOS theory is determined with in-depth questions. See Arnon, Cottrill, Dubinsky, Oktac, Fuentes, Trigueros, and Weller (2014) for the most recent comprehensive work on APOS theory. In this work, students' conceptual knowledge classification is determined by evaluating their responses to the following question:

Question: What is the connection between definite integral of a function f(x) and the area between the graph of f(x) and the x-axis?

The participants of this study are 17 graduate and senior undergraduate mathematics and engineering students who were either enrolled or recently completed (i.e. 1 week after the completion of the course) a Numerical Methods or Numerical Analysis course at a large Midwest university during a particular semester. The participants completed a questionnaire and each participant is interviewed to explain his/her written responses to the questionnaire questions. Qualitative and quantitative results are displayed in this paper by using the written and video recorded interview responses to the question stated above. The connection between participants' concept image and concept definition knowledge is evaluated in this work. Next section is devoted to the literature review on triad classification and APOS theory.

Relevant Literature

By relying on Piaget`s study of functions in 1977 (Piaget et al. 1977), the Action-Process-Object idea in mathematics education for the undergraduate curriculum was initiated by Breidenbach, Dubinsky, Hawks and Nichols in 1992 who studied students' conceptual view of the function in their research. In 1996, Asiala, Brown, DeVries, Dubinsky, Mathews and Thomas applied Action, Process, Object and Schema theory (called APOS theory) to understand students' function knowledge and explained this theory as the combined knowledge of a student in a specific subject based on Piaget`s philosophy. Dubinsky and McDonald (2001) explained the components of the APOS theory as follows:

An action is a transformation of objects perceived by the individual as essentially external and as requiring, either explicitly or from memory, step-by-step instructions on how to perform the operation...

When an action is repeated and the individual reflects upon it, he or she can make an internal mental construction called a process which the individual can think of as performing the same kind of action, but no longer with the need of external stimuli...

An object is constructed from a process when the individual becomes aware of the process as a totality and realizes that transformations can act on it...

A schema is an ... individuals collection of actions, processes, objects, and other schemas which are linked by some general principles to form a framework in individual's mind...

Baker, Cooley and Trigueros (2000) applied APOS theory to understand undergraduate students' conceptual function knowledge to the data collected for a calculus graphing problem. Cooley, Trigueros and Baker (2007) continued in the line of their previous work from 2000 (Baker et al. 2000) by focusing on the thematization of the schema with the intent to expose those possible structures acquired at the most sophisticated stages of schema development. For a detailed review of APOS theory see Arnon, Cottrill, Dubinsky, Oktac, Fuentes, Trigueros, and Weller (2014). Some of the researchers such as Clark et al. (1997) didn't find APOS theory appropriate for analyzing data in their research.

APOS theory is widely used in several educational research areas in the past decade: It is used by Parraguez and Oktac (2010) to lead the students' towards constructing the vector space concept, Mathews and Clark (2007) to observe successful students' conceptual knowledge of mean, standard deviation, and the central limit theorem who completed an elementary statistics course with a grade of "A", by Trigueros and Martinez-Planell (2009), and Kashefi, Ismail, and Yusof (2010) to observe students' ability to construct and develop two variable functions. Tokgöz and

Gualpa (2015), and Tokgöz (2015) recently worked on understanding undergraduate and graduate students' ability to respond to a variety of calculus questions by using APOS theory. Evaluation of the collected data indicated a variety of APOS classification of the participants' depending on the research question.

In this work APOS theory appeared to be not applicable for observing students' ability to relate the integral concept with the area between a function and the input axis. Arnon et al. (2014, pg. 20) explained application of APOS theory on area calculations as follows:

In Calculus: *Actions are needed to construct an estimate of the definite integral as the area under a curve: for example, in dividing an interval into specific subintervals of a given size, constructing a rectangle under the curve for each subinterval, calculating the area of each rectangle, and calculating the sum of the areas of the rectangles.*

…The area under the curve for a function on a closed interval is the limit of Riemann sums—an Action applied to the Riemann sum Process. In order to determine the existence of this limit and/or to calculate its value, the student needs to encapsulate the Riemann sum Process into an Object.

There isn't an extensive literature on students' ability to determine paper-pencil solution to the Riemann integral of functions. Asiala, Brown, DeVries, Dubinsky, Mathews, and Thomas (1997) pointed out the difficulty of writing a code to find the integral of functions and asked the participating students to write a code to approximate the integral by sampling points. Thompson (1994) states

...We must think of integration as the culmination of a limiting process, but at the same time consider that process, applied over an interval of variable length, as producing a correspondence...

and invites to do research on determining the integral of functions:

...A curricular and instructional emphasis in algebra and pre-calculus on having students' develop images of arithmetic operations in analytically-defined functions as operations on functions would seem to prepare them for a deeper understanding of this aspect of the calculus. At the same time, a conception of operations in expressions as operating on numbers and not on functions would seem to be an obstacle to understanding the derivative and integral as linear operators. These are empirically testable hypothesis; I would welcome research on them...

The way the research question was structured in this work appeared to be not appropriate for applying the APOS theory to evaluate the research question. This is due to the fact that there are not many concepts with not many steps to be taken for answering the research question; therefore understanding in-depth knowledge of the participants did not appear to be appropriate by using the APOS theory. In addition, neither before nor after the interviews the majority of the students attempted to explain the approximation of the area by using rectangles and its integral connection, therefore APOS theory did not appear to be appropriate for evaluating the collected data.

Triad Classification

In APOS theory the development of the individual schemas are accomplished by using "the triad;" a progression of three stages proposed by Piaget and Garcia (1989). Clark, Cordero, Cottrill, Czarnocha, DeVries, St. John, Tolias, and Vidakovic (1997) used the stages of the triad; Intra, Inter, and Trans to investigate how first year calculus students construct the chain rule concept. Their attempt to use the APOS theory resulted in insufficiency by itself therefore they included the schema development idea of Piaget et al. (1989). Clark et al. (1997) classified students in the intra stage if they knew some of the derivative rules and were able to apply the chain rule, however did not know the relationship between these rules. In the same study participants are classified to be in the inter stage if they had the ability to begin to collect all different cases and recognize that they are related, and in the trans stage if they were able to construct and apply the chain rule classified. Similar to Clark et al. (1997) APOS theory found to be inappropriate for evaluating the research question because students' responses didn't reflect a proper APOS setting; therefore participating students' responses are observed by using the schema development idea. The Triad classification in this setting is as follows:

- **Intra Stage:** Students' classified in this category if they were able to recognize the connection between the area concept and the definite integral of the given function but did not necessarily remember other details.
- **Inter Stage:** Students qualified to be in the intra stage are also classified to be in the inter stage if they recognize the need of an absolute value to find the area and explain the concept image properly.
- **Trans Stage:** Inter stage students are qualified to be at the Trans stage if they were able to explain the area and integral connection through approximation of the integral by using rectangles.

Next section is devoted to qualitative and quantitative responses of the participants to the research question.

Research Question & Participant Responses

In this section, we will observe participating students' ability to link integral and area concepts based on their responses to the following question:

Question: What is the connection between definite integral of a function f(x) and the area between the graph of f(x) and the x-axis?

Detailed responses of the participants to this research question are displayed in this section for a comprehensive understanding of the responses with the corresponding statistical results. These researcher-participant conversations are particularly important to understand the moment at which the correct answer triggered in students' mind after a series of questions. A similar questioning methodology can be used as a part of examining students' knowledge when online exams are given in mathematics education. Contrary to the traditional written examination of students with a single question and an expected solution, students' real conceptual knowledge might be examined by a set of follow-up questions after receiving incorrect answer to the actual problem. This method appears to be particularly useful for online education of mathematics.

One of the participants had the correct response to the research question prior to the interview but then had a conflicting idea about the solution during the interview:

I: ...When you find the integral of a function, does it necessarily mean it will give the area or do you need to do something else?

RP 1: …Well, okay. I learned in analysis like that if we have a curve (and draws a curve) every integral you take is pretty much an approximation, I mean you can get a very good approximation but I mean at least it is my understanding of what it is…

I: If it goes below the x-axis, if that function (points to the drawn graph) goes below the x-axis, if you calculate the integral, would that integral be accurate?

RP 1: …yeah. If you calculate the integral of this function you will still get this part (and she highlights the area of the region between the function and the x-axis.)

I: Would that (area) be positive or negative when you calculate the integral?

RP 1: It will be negative.

I: And is there anything you need to do with that?

RP 1: …I don't think so.

I: To be able to find the area?

RP 1: To be able to find the area... I think you take... For the total area you take (shading the area between the x-axis and the curve above the x-axis) this part A and (shading the area between the x-axis and the curve below the x-axis) this part B, and it will be like A-B.

Only one among seventeen participants had the correct written pre-interview response to the research question. 29% (5/17) of the students' had the right response only for positive (or non-negative) functions where negative functions were completely ignored in their responses. While two of these five participants algebraically stated that the integral is the same as the area, three of the students showed that they consider non-negative functions by the curves they sketched:

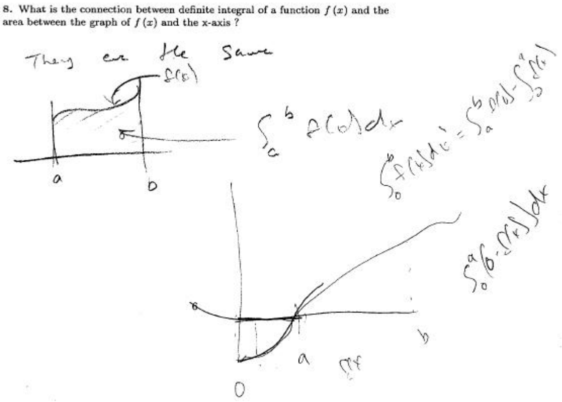


Figure 1: Response of RP 3

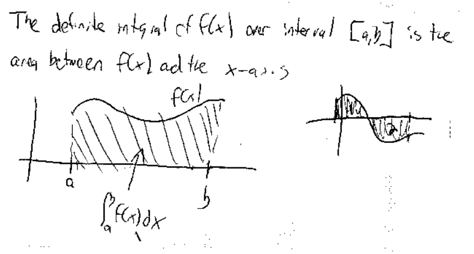


Figure 2: Response of RP 13

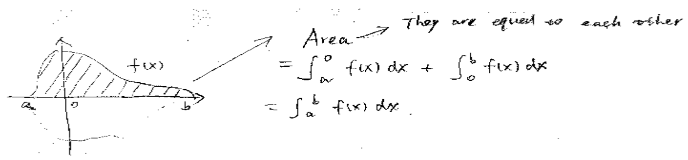


Figure 3: Response of RP 14

The written questionnaire responses indicated 29% (6/17) of the responses not reflecting the general solution:

Participant	Response to the question
RP 1	"The definite integral is the area under (or above) the graph of f(x) measured to the x-axis."
RP 2	"The value of the definite integral of a function f(x) is the area of the region between the graph of f(x) and the x-axis."
RP 5	"They are the same thing"
RP 10	"No difference, this area is calculated by getting this integral of the function."
RP 15	" It's the area"
RP 17	"The definite integral of a function f(x) is the area between f(x) and the x-axis over the interval defined in the definite integral."

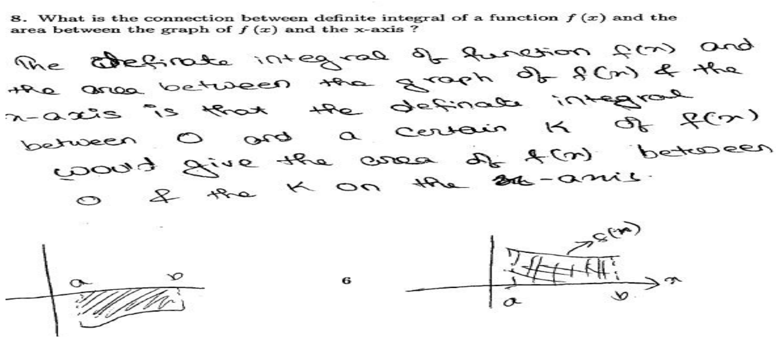
Some of the participants' could not find the correct answer during the interviews:

RP 8: So if the function is (Draws f(x) on the right bottom of Figure 4 drawn below)

I: Let's say that is the case. Would that still be true? Is the area same as the integral?

RP 8: Yes.

I: What would be the integral value? Would it be negative or positive?



8. What is the connection between definite integral of a function $f(x)$ and the area between the graph of $f(x)$ and the x-axis ?

The definate integral of function f(x) and the area between the graph of f(x) & the x-axis is that the definate integral between 0 and a certain K of f(x) would give the area of f(x) between 0 & the K on the x-axis.

Figure 4: Response of RP 8

RP 8: The integral value would be positive.

I: When you calculate the integral…?

RP 8: When you calculate it...

I: So if you have a certain area of the function, does it represent the integral and the other way around? Like, is integral equal to the area of the given function?

RP 9: Not exactly equal. It is approximately equal.

I: Not exactly, just approximately. What makes the difference there, do you recall?

RP 9: Just you can't get as accurate as the actual value.

I: The accurate area?

RP 9: Yeah....

I: What is the connection between the integral of a function between the graph of a function and the x-axis, and you are saying "The definite integral is called the Riemann sum because the area bounded by f(x) and the x-axis is equal to the area under the function f(x) and x-axis." Is it always possible to have the area equal to the integral, the area of the region between the graph of the function and the x-axis?

RP 12: Always equal to 0

I: Is there any possibility of getting something else?

RP 12: No…not from what I remember because basically for any function even if it goes negative or positive (draws a continuous function negative and positive values.) when you take the integral, you are taking that area from the function to the x-axis. That's what the integral is, basically calculation of the area.

I: …If you calculate this region (pointing the region below the x-axis drawn by the participant), integral from here to here of this function, would that integral give you negative or positive value?

RP 12: Positive value...

Overall, 94% (16/17) of the students did not consider the possibility that the function could have negative values as a part of their responses prior to the interviews which appeared to be an interesting result.

The research question is evaluated by using the concept image and concept definition of Hershkowitz et al. (1980). Concept image is the image that corresponds to the area of the function determined by the absolute value of the integral. The concept definition is the absolute value of the definite integral that is desired to be determined to calculate the area.

General Results

In this study, 17 graduate and senior undergraduate engineering and mathematics students' conceptual integral knowledge is investigated based on their responses to a research questions that was asked as a part of a questionnaire consisting of calculus questions. Participants are asked to explain the correspondence between the definite integral of a function and the area of the region between the function and the input axis. The way the research question was structured and the lack of the details in participants' responses appeared to be not appropriate for applying APOS theory to evaluate the research question. Triad classification of the collected data appeared as follows:

- Students' are classified at the intra stage of triad classification if they were able to recognize the connection between the area concept and the definite integral of the given function but did not necessarily remember other conceptual details.
- Students who are qualified to be in the intra stage are also classified to be in the inter stage if they recognize the need of an absolute value to be used with the integral and explain the concept image properly.
- Inter stage students are qualified to be at the Trans stage if they were able to explain the area and integral connection through approximation of the integral by using rectangles.

The use of a mixed data evaluation methodology (by using Triad classification and Concept Image and Concept Definition) implemented in this work yields to a stronger insight about the data, and therefore a better understand of students' thinking process from two different research perspectives. Prior to the interviews, 16/17 (94.11%) of the participants considered only positive functions and ignored the possibility of the area of a region where the function might have negative values. Some of the participants supported their answers with figures drawn for positive valued functions. During the interviews several participants explained the possibility of having a negative valued function where they had the correct response. Some of the participants claimed that the definite integral is just the approximation of the area.

A questioning methodology to the one presented in this work can be used as a part of examining students' knowledge when online exams are given in mathematics and engineering education. Contrary to the traditional written examination of students with a single question and an expected solution, students' real conceptual knowledge might be examined by a set of follow-up questions after receiving incorrect answer to the actual problem. This method could be particularly useful for online education of mathematics.

Integral concept has an important place in engineering and mathematics education, therefore it is important to understand students' level of knowledge. This can help the educators and researchers to find a good way for educating students if there is a lack of knowledge. The results of this study indicated weak conceptual integral and its knowledge of the graduate and senior undergraduate engineering and mathematics students. We invite other researchers to investigate undergraduate students' integral knowledge. Concept image and concept definition idea of Vinner (1992) with the triad classification appears to be a good candidate for evaluating the responses of the participants to the research question evaluated in this work. The design of the questionnaire and the interviews played an important role in the decision of the choice of the methodology to evaluate the collected data.

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