STEM Majors' Cognitive Calculus Ability to Sketch a Function Graph

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Engineering and mathematics undergraduate and graduate students' conceptual function knowledge can have an important impact on their success in conceptually advanced courses. In building blocks of STEM concepts, function concept requires knowledge of sub-concepts such as limit, first derivative, second derivative, and asymptote. In this study, calculus concept knowledge of seventeen undergraduate and graduate engineering and mathematics students enrolled in a Numerical Methods-Analysis course at a large Midwest Research Institution are analyzed based on their responses to a calculus graphing question similar to the research question of Baker, Cooley, and Trigueros (2000). This question is analyzed qualitatively and quantitatively by using the triad classification in Action-Process-Object-Schema (APOS) theory. Mathematics graduate and undergraduate students succeeded the most among all the participants.

Key words: APOS theory, Schema, Triad Classification, Functions, Derivative, Limit, Asymptote, Critical Points.

Introduction

Function concept is an important part of cumulative blocks of concepts in advanced level mathematics and engineering courses. In these advanced courses, topics of single-variable calculus, such as limits, derivatives, integrals, and power series, require function knowledge. The function concept also requires knowledge of limits, derivatives, and asymptotes. Therefore, the following natural questions arise:

- Up to what degrees do the senior undergraduate and graduate Engineering and Mathematics students know the function associated calculus concepts?
- What are the missing calculus sub-concepts in the function concept knowledge?

There are several theories that can be applied to investigate the answers to these questions; however we will focus on APOS theory to find qualitative answers to the question we asked the participants.

Baker, Cooley, and Trigueros (2000) analyzed students' understanding of the calculus concepts on a calculus graphing problem by observing oral and written interview responses of the participating students. In 2007, Cooley, Trigueros, and Baker conducted a more detailed study than in their research in 2000 to observe calculus concept knowledge of students who were considered to be successful by their professors in diverse disciplines, and by using the same theoretical framework of their study in 2000. The motivation behind this study is to analyze conceptual calculus knowledge of the undergraduate and graduate students' at a Midwest research institution who were enrolled in a Numerical Methods or Analysis course by using a theoretical framework similar to that of Baker et al. (2000). Given limit, asymptotes, first and second derivative information, participants in this study are asked to sketch a graph of a function that can be a good fit to the given information.

APOS Theory

In this section, APOS theory and the relevant literature that is employed to evaluate the question and the corresponding interviews to discern the students' conceptual function knowledge are explained.

The philosophy of mathematics affected engineering and mathematics education in the undergraduate curriculum in the 1990s. Piaget's schemes idea in the 1970's, and its development with detailed explanations by Piaget and Garcia in the 1980's, influenced researchers of undergraduate engineering and mathematics education curriculum in the 1990's. Students' conceptual view of the function was defined by Breidenbach, Dubinsky, Hawks, and Nichols in 1992, which relied on Piaget's study of functions in 1977 (Piaget, Grize, Szeminska & Bang, 1977). This formed the action-process-object idea in engineering and mathematics education for the undergraduate curriculum. In 1996, Asiala, Brown, DeVries, Dubinsky, Mathews, and Thomas applied action, process, object, and schema theory (called APOS theory) to mathematical topics (mostly functions), and they explained this theory as the combined knowledge of a student in a specific subject based on Piaget's philosophy. The components of the APOS theory can be briefly explained as follows (Dubinsky and McDonald, 2001):

- An action is a transformation of objects perceived by the as essentially external and as requiring, either individual explicitly or from memory, step-by-step instructions on how to perform the operation...
- When an action is repeated and the individual reflects upon it, he or she can make an internal mental construction called a process which the individual can think of as performing the same kind of action, but no longer with the need of external stimuli...
- An object is constructed from a process when the individual becomes aware of the process as a totality and realizes that transformations can act on it...
- A schema is a ... individuals' collection of actions, processes, objects, and other schemas which are linked by some general principles to form a framework in individual's mind...

In this theory, every concept can be constructed on different concepts and schemas. For example, if a researcher works on the integration of functions, the researcher can base the schemas on functions, limits of functions, derivatives of functions, continuity of functions, and number knowledge of students. All schema combinations can form a schema. We can also say that every concept requires concept knowledge and the construction of a specific concept depends on knowledge of the other concepts. Some of the researchers such as Clark, Cordero, Cottrill, Czarnocha, DeVries, St. John, Tolias, and Vidakovic (1997) did not find the APOS theory applicable in analyzing data in their research. Piaget et al. (1983) introduced the triad stages intra, inter, and trans, used by Baker et al. (2000), to introduce the property and interval schemas to analyze undergraduate students conceptual function knowledge on a calculus graphing problem. In 2007, Cooley, Trigueros, and Baker built on their work from 2000 (Baker et al., 2000) by focusing on the thematization of the schema with the intent to expose those possible structures acquired at the most sophisticated stages of schema development. In their study, the problems were structured in a way that participants were required to respond to the first eight questions and continue with the ninth question only if they succeeded in answering the first eight questions (please see Cooley et al., 2007, pg. 391). The detailed analysis of the collected data indicated participants' success in answering a complex graphing problem, thus schema thematization was possible in their study. Cortés (2004) also observed student difficulties in understanding the function concept by using a questionnaire similar to that of Cooley et al. (2007), with similar results. The questionnaire developed in this study contains different questions than that of Cooley et al. (2007) including analytical calculus concept calculations for a quotient function, answering fill-in-the-blank calculus concept questions, and sketching the graph of a function after calculating calculus concept questions.

In the last decade, APOS theory is widely used in several educational research areas. It is used by Parraguez and Oktac (2010) to lead the students towards constructing the vector space concept, by Mathews and Clark (1997) to observe mean, standard deviation, and the central limit theorem knowledge of successful students who completed an elementary statistics course with a grade of "A", by Kashefi, Ismail, and Yusof (2010) to observe students' obstacles in the learning of two variable functions in calculus, and by Tziritas (2011) to observe students' success in solving related rate problems.

In this work, APOS theory is chosen for data analysis due to its convenience that's also used by other researchers in the literature. It eases to have comparative data analysis results with those questions similarly asked and evaluated in the literature.

Functions & Calculus in STEM

The conceptual knowledge of a student is measured by the researchers based on the student's ability to construct concept-related graphs (conceptual image) and to answer the corresponding algebraic questions. Students' difficulties with the conceptual image are observed by several researchers (Orton, 1983; Selden, Selden, & Mason, 1994). Aspinwall, Shaw, and Presmeg (1997) collected data by observing a student and concluded that incorrectly created derivative images can result in mistakes of analytical reasoning of the student. Given the graph of a function, Ferrini-Mundy and Graham (1994) observed participating students' difficulty in sketching the derivative graph of the given function where many students first tried to find an algebraic representation of the given function. Thompson (1994) observed that senior mathematics undergraduate and graduate students' weak rate of change concept knowledge resulted in weak understanding of the integration concept. Trigueros and Martinez-Planell (2009), and Kashefi et al. (2010) observed students' ability to construct and develop two variable functions by using APOS theory. Kashefi et al. (2010) concluded that in two variable calculus settings students had difficulty in domain, range, and the graphs of two variable functions.

Methodology

Participants and the General Procedure

The participants of this qualitative study are 17 senior undergraduate and graduate students from Engineering and Mathematics disciplines who were enrolled to one of these two courses at a large Midwestern university. Out of these 17 participants, 2 participants are mathematics graduate students, 4 participants are mathematics undergraduate students, 4 of them are engineering graduate students, and 7 of the participants are engineering undergraduate students. All had completed multi-variable calculus courses that cover the content of the given questionnaire. The data was collected during a semester that the main author of this study taught a senior level undergraduate Computer Science Numerical Methods course in 2010. Computer Science undergraduate majors were required to complete this course as a requirement of the Computer Science Bachelor of Science degree. During the same semester, the researcher also graded a senior undergraduate/graduate level Numerical Analysis course offered by the Mathematics Department with students enrolled from various science and engineering disciplines. Each participant was required to complete the same questionnaire that consisted of 15 questions, and interviewed for approximately 40 minutes based on his/her responses to the questionnaire questions. The author video recorded all the interviews and designed the interview questions based on the written responses to the questionnaire questions. Interview data collection is standardized across the participants based on their responses and the detailed data collection procedure will be also explained later in the corresponding section. The goal of the questionnaire and the interview questions is to analyze 17 participants' ability to respond to algebraic, analytic, and geometric function-related calculus concept questions.

Schema Classification

A Scheme is an action which is repeated and can be generalized where the actions are derived from sensory-motor intelligence (Piaget, 1971). The coordination of schemes forms actions which are logical structures. Combination of systems and schemes can form the scheme (Piaget, 1971). The similarity between the schemes in a larger combination of schemes is similar to the set inclusion in mathematics where subsets form the set. The concept knowledge can be formed in a larger combination of schemes.

The schema classification of Baker et al. (2000) is based on the following triad classification:

- **Intra-Interval:** Ability to answer questions regarding the independent intervals where the participant can be confused by the union or intersection of other intervals.
- **Inter-Interval:** Ability to answer questions regarding only sub-domains which consists of two or more intervals but not the entire domain.
- **Trans-Interval:** Ability to answer questions regarding the entire domain.
- **Intra-Property:** Ability to interpret every analytical property independently one at a time.
- **Inter-Property:** Ability to interpret two or more analytical properties simultaneously but not all of them together.

- **Trans-Property:** Ability to interpret all the analytical properties simultaneously.

The schema classification in this work is structured by observing post interview student responses. The data collected in this study suggested following a similar theoretical triad classification to that of Baker et al. (2000). The design of the questions and detailed analysis of the post interview student responses suggested a three-level triad classification of the participants for the question considered in this research is as follows:

Intra-level: Responses reflected only one analytical property on the right interval on independent intervals. The responses in this category indicate mistakes in application of two or more analytical properties in two or more intervals.

Inter-level: Participants were able to apply one or more analytical properties on the right interval, which may consist of the combination of independent intervals; however, the combination of these intervals does not form the entire domain. The responses in this category indicate application mistakes in only one analytical property on a certain interval.

Trans-level: The participants in this category made no mistake in the application of the analytical properties throughout the entire domain.

For example, a participant is considered to be in the intra-level if the second derivative and the asymptote information are not applied correctly on two or more intervals. This is a result of participant confusion by the union or intersection of other intervals and the failure to interpret every analytical property independently one at a time. If there is only one analytical property application mistake, such as the first derivative information on a certain interval which cannot consist of the union of independent intervals, then the response is categorized as inter-level. Students' trans-level triad classification is based on their ability to answer the question correctly in the entire domain.

Research Problems

The main goal of this study is to analyze students' conceptual function knowledge by using action, process, object, and schema theory of Asiala et al. (1996). The question answered by the participants in this study are designed to observe the following:

- Participants' ability to draw the graph of a function by using given limit, first derivative, second derivative, and asymptote information.
- This problem aims to observe advanced level engineering and mathematics students graphing ability by analyzing the given calculus concept information similar to Baker et al. (2000).

Curve Sketching Question

Students' sub-concept knowledge such as limit, vertical and horizontal asymptote, first derivative, and the second derivative to sketch the graph of a function are analyzed by evaluating the responses to the following calculus graphing question. This question is similar to the calculus graphing question of Baker et al. (2000).

Sketching A Curve: Please draw a graph of a function that verifies all of the given information below. Write the necessary values on the coordinate axis.

$$\lim_{x \to -\infty} f(x) = 0, \qquad \lim_{x \to \infty} f(x) = 0,$$
$$\lim_{x \to -3^-} f(x) = -\infty, \qquad \lim_{x \to -2^+} f(x) = \infty,$$
$$\textit{Vertical asymptotes at } x = -3 \textit{ and } x = 2,$$
$$\textit{Horizontal asymptote at } y = 0,$$
$$f'(-2) < 0, \quad f'(1) < 0,$$
$$f''(x) < 0 \quad \textit{when} \quad x < -3,$$
$$f''(x) > 0 \quad \textit{when} \quad x > 2,$$
$$f''(c) = 0 \quad \textit{for an } x = c \textit{ such that } -1 < c < 1.$$

During the interviews, the participants were asked whether they have more information to add to their written responses of this question and to briefly explain their written responses. If they had no additional information and successfully answered the question, they were asked to discuss the solution of the following question. If they had difficulty while answering the question, conceptual questions related to the difficulty encountered were asked. In the case when the participant could not recall the related information, we moved on to discuss the following question response.

Analysis of the Curve Sketching Question

The triad classification for the graphing question based on the post interview data analysis yields the results in the following table:

Graphing Question	Intra-Level	Inter-Level	Trans-Level
Number of Students	3	4	10

It is important to note the triad classification for this question required detailed analysis of each calculus sub-concept problem which resulted in different triad classification; therefore, the following information is useful to understand students' triad classification.

Asymptote Knowledge

Most of the participants showed conceptual knowledge of asymptotes and reflected it in their graphs. A challenge for one of the participants was to see whether the graph can cross the horizontal asymptote or not:

RP 1: …I just kind of drew the picture. I was really confused here (points to the x-axis) because there is a horizontal asymptote at y = 0 and then…I think I remember something from calculus there was like horizontal asymptotes are not as absolute as vertical asymptotes but. I just drew it, I don't know. That is my best guess.

Interviewer: So that is the part that confused you? If you have a horizontal asymptote then decreasing part might cause trouble.

RP 1: Huh, huh [agrees].

Interviewer: …does it make sense when you have a second look now. Would that be okay?

RP 1: Yeah, I think this is okay.

Another participant considered the horizontal asymptote on the interval (-3, 2) and stated that the graph can be a parabola based on the given horizontal asymptote information:

Interviewer: Here you have the given information…can you explain your answer briefly?

RP 17: Well, I can tell limit of f(x) when x approaches one is zero but there is an asymptote at 0 so it is not going to be equal to zero. So it is getting closer and closer without touching the vertical, or the horizontal axis. And the limit as x approaches 3 with negative sign is going to be negative infinity, there is another asymptote…it is not going to touch the line [pointing the vertical line at x=3.] Similarly we have a curve up here [pointing the vertical asymptote at x=2 and the curve on the right of it.] What I couldn't quite figure out was what happened in between. I know something is missing. I was thinking maybe it was something like that [pointing $\frac{1}{(x-2)(x+3)}$ written on the paper.] Maybe it is like a parabola.

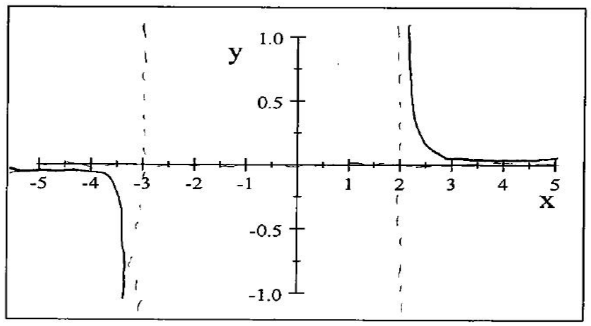

Fig. 1 Answer of RP 17 to the graphing question

Limit Knowledge

The given limiting values in the question were related to the horizontal and vertical asymptote information. 82% (14/17) of the participants were able to apply the limit information to the graph in the right way. 18% (3/17) of the participants were able to apply the vertical asymptote limit information but could not apply the horizontal asymptote limit information. The following response of RP 4 is an example of the intra-level property triad classification. RP 4 made mistakes in applications of the first derivative and the horizontal asymptote information contrary to the correct application of the second derivative information.

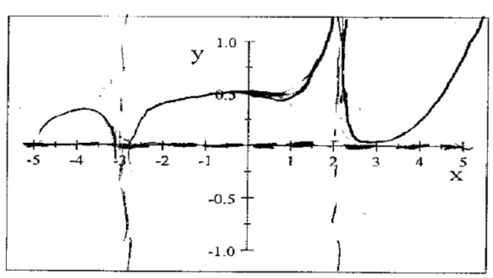

Fig. 2 Answer of RP 4 for the graphing question.

In addition to the response of RP 17, RP 5 tried to find an algebraic representation of a possible function as well:

RP 5: …This is simple [pointing]

$$\lim_{x \to -\infty} f(x) = 0, \qquad \lim_{x \to \infty} f(x) = 0,$$

$$\lim_{x \to -3^-} f(x) = -\infty, \qquad \lim_{x \to -2^+} f(x) = \infty,$$

It is just this function $\frac{1}{(x-2)(x+3)}$ with maybe some constant [pointing the numerator $\frac{c}{(x-2)(x+3)}$.]

The information provided in the question did not include limiting values of the function as x approaches 3 from the right-hand side and as x approaches 2 from the left-hand side; however, all the students except RP 14 drew the graph of a function assuming the limiting values

$$\lim_{x \to -3^+} f(x) = \infty, \qquad \lim_{x \to -2^-} f(x) = -\infty$$

which resulted from the nature of the graph they started constructing. The following graph of RP 14 is an example of the trans-level triad classification.

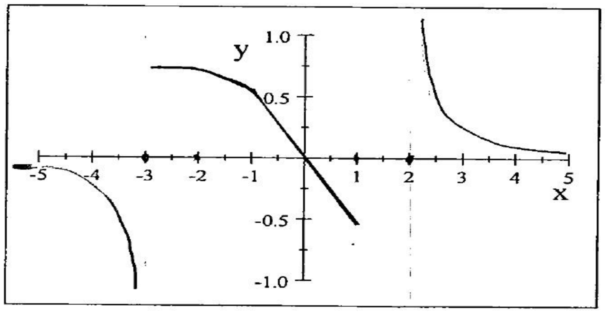

Fig. 3 Graph of RP 14 for the graphing question.

First Derivative Knowledge

Based on the first derivative information

$$f'(-2) < 0, \quad f'(1) < 0$$

given in the question, 53% (9/17) of the participants were able to sketch a true graph where 33% (3/9) of these successful graphs are drawn during the interviews. 24% (4/17) of the students recalled the meaning of the first derivative less than zero but could not draw the graph of the function, and 24% (4/17) of the students could not recall the meaning of the first derivative less than zero and could not draw the graph. The following response of RP 7 is another example of inter-level triad classification for this question. This triad classification is based on the misinterpretation of $f'(1) < 0$.

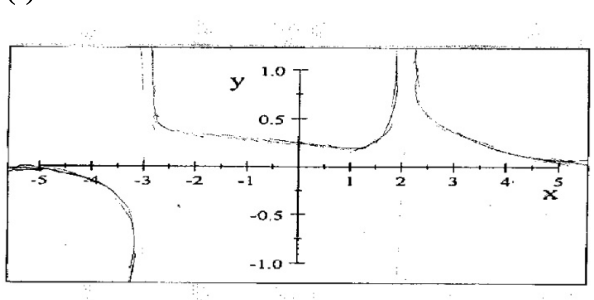

Fig 4. Graph of RP 7 for the graphing question.

Second Derivative Knowledge

Based on the given second derivative information

$$f''(x) < 0 \quad when \quad x < -3,$$
$$f''(x) > 0 \quad when \quad x > 2,$$

all the participants except RP 2 were able to apply the second derivative information on the right interval on their graphs, even if it was not necessarily the correct graph. RP 2 had a concave up graph on the interval (0, ∞) and a concave down graph on the interval (-∞, 0). The following response of RP 2 is another example of intra-level triad classification for this question. The response below indicates successful application of the horizontal asymptote; however, misconception of derivative and asymptote knowledge on an interval.

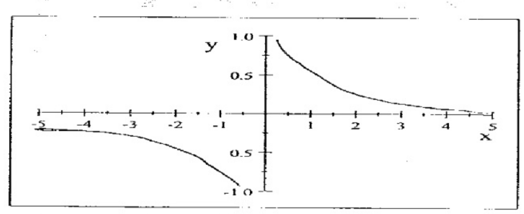

Fig 5. Graph of RP 2 for the graphing question.

The information

$$f''(c) = 0 \text{ for a } x = c \text{ such that } \quad 1 < c < 1$$

given in the question gives the participants freedom of choice of the location of the inflection point anywhere between 1 and 1 on the graph. In particular, a participant claimed that the inflection point has to be at x = 0 by not considering the freedom of choice of the location of the inflection point in the local neighborhood.

Interviewer: You are saying at c = 0 there is a point of inflection. Could it be somewhere else?

RP 16: …no, the point of inflection is the place where it changes concavity…and I don't think it will be a vertical asymptote because it is a point…So it doesn't change it. It changes concavity across the asymptote but it is not a point of inflection.

Interviewer: …it can't be somewhere in between 1 and 1 randomly, it has to be

RP 16: It has to be at zero.

76% (13/17) of the participants were able to apply the inflection point information on the graph whereas 24% of the participants could not construct any graph based on this inflection point information.

Continuity

The information given in the question required the continuity of a possible graph on the intervals (-∞, 3) and (2, ∞). In addition, the continuity of the graph on local neighborhoods of x=-2 and x=1 is also required because of the first derivative information given at these points. All the participants, regardless of a true or false answer, assumed a continuous graph on (-3, 2).

Results and Discussion

In this study, conceptual calculus knowledge of engineering and mathematics undergraduate and graduate students is observed who are either enrolled in or have completed a Numerical Methods or Numerical Analysis course at a large Midwest University. This study is designed to advance the work of Baker et al. (2000) and Cooley et al. (2007). Student success while answering calculus concept questions associated with functions is evaluated by using the concept image and concept definition idea of Vinner (1992) and APOS theory with triad classification similar to Baker et al. (2000). The results of this study give insight about Numerical Methods/Analysis students' success in answering several different function-related calculus concept questions.

The analysis of the graphing question data collected in this article indicated 59% (10/17) of the participants' ability to construct a true graph that reflected all the given information, where 17.6% (3/17) of these participants drew the graphs during the interviews after receiving additional help. On the contrary to the findings of Baker et al. (2000), most of the students did not face difficulty in interpreting the asymptote information or working with the second derivative conditions. However, they faced difficulty while dealing with the first derivative information. Similar to the findings of Asiala et al. (1996), students succeeded in displaying a reasonable understanding of the relationship between the derivative and the slope of the tangent line. Thompson (1994) and Baker et al. (2000) reported misconception of the participants regarding knowledge of the second derivative; however, the findings of this research indicated the contrary for the ability of most participants to correctly interpret the second derivative information. Subject to the post interview data evaluation, among all the participants who could not sketch the correct graph, 29.4% (5/17) failed in reflecting the inflection point information, and 41.2% (7/17) lacked in reflecting the first derivative information to the graph.

The responses to the curve sketching question evaluated in this article indicated lack of first derivative and inflection point knowledge of students' played important roles in their resulting graphs. The first derivative knowledge of the students' appeared to be the major problem in answering this question. Thompson (1994) observed that the rate of change is effective on students' integration. In this study, similar to Thompson's (1994) results, we found the lack of first derivative knowledge affected students function knowledge. Similar to Baker et al. (2000) some of the participants in this study encountered problems with second derivative. Cooley et al. (2007) had a schema thematization in their study; however, because of the complexity of the collected data, a schema thematization is not possible for this study.

Post-interview triad classification for the second, third, and fourth questions indicated trans-level classification for most of the participants, and either intra- or inter-levels of classification for most of the other participants. Trans-level categorization for most of the participants is not surprising for engineering and mathematics majors who are expected to have a well-developed background in mathematics.

In their research, Ferrini-Mundy and Graham (1994) reported many students trying to find an algebraic function expression corresponding to the given graph in order to sketch the corresponding derivative graph. The pre-interview results indicate a similar approach of two of the participants, RP 2 and RP 17, who couldn't construct a graph based on the given information in the graphing question, trying instead to find an algebraic representation for the given information in order to construct a graph.

In conclusion, considering the APOS theory-based data classification, post-interview data collection indicated a uniform triad classification of the participants. The data collected for the graphing problem indicated students' misconception of derivative. Therefore, the main misconception of the participants appeared to be the first and second derivative related information, due to the post-interview responses to the question. Considering the responses to the graphing problem, engineering undergraduate students had the most difficulty, while some of the engineering graduate students also encountered difficulties while answering the proposed questions. Mathematics graduate and undergraduate students had the least misconception of calculus concepts. Particular grouping analysis is not implemented in this work due to the insufficient number of participants participated from each group.

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