AC 2012-3998: TEACHING DEFLECTIONS OF BEAMS: ADVANTAGES OF METHOD OF MODEL FORMULAS VERSUS THOSE OF CONJUGATE BEAM METHOD

Dr. Ing-Chang Jong, University of Arkansas

Ing-Chang Jong is a professor of mechanical engineering at the University of Arkansas. He received a B.S.C.E. in 1961 from the National Taiwan University, a M.S.C.E. in 1963 from South Dakota School of Mines and Technology, and a Ph.D. in theoretical and applied mechanics in 1965 from Northwestern University. He and Bruce G. Rogers coauthored the textbook Engineering Mechanics: Statics and Dynamics, Oxford University Press (1991). Jong was Chair of the Mechanics Division, ASEE, 1996-1997, and received the Archie Higdon Distinguished Educator Award in 2009. His research interests are in mechanics and engineering education.

Teaching Deflections of Beams: Advantages of Method of Model Formulas versus Those of Conjugate Beam Method

Abstract

The *method of model formulas* is a recently published method. It employs a general model loading diagram and derived four key equations as model formulas. These formulas can account for the beam's flexural rigidity, applied concentrated loads, linearly distributed loads, and the boundary or support conditions. No explicit integration is needed in using the model formulas in this method. This method can be applied to solve *most* problems involving beam reactions and deflections encountered in the teaching and learning of mechanics of materials. On the other hand, the *conjugate beam method* is a natural extension of the *moment-area theorems*. It is an elegant, efficient, and powerful method propounded by Westergaard in 1921. Elementary presentation of this method did appear in some early textbooks of mechanics of materials.[2,5] For reasons unknown, this method is currently missing in most such textbooks.

This paper is aimed at providing comparisons of the *method of model formulas* versus the *conjugate beam method* regarding their (*a*) pedagogy and methodology, (*b*) effectiveness in solving problems of deflections of beams and statically indeterminate reactions at supports via several head-to-head contrasting solutions of the same problems, and (*c*) ways to effectively introduce and teach either of these methods to students.

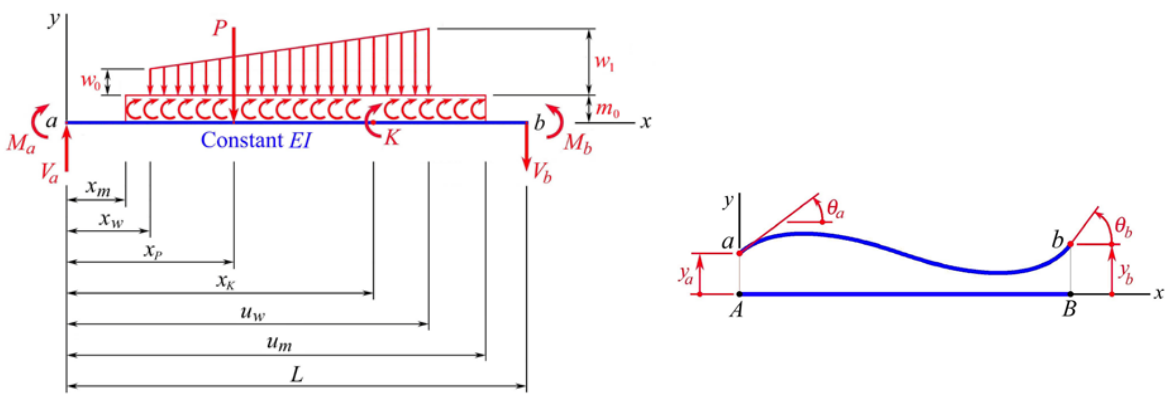
I. Introduction

Beams are longitudinal members subjected to transverse loads. Students usually first learn the design of beams for strength. Then they learn the determination of deflections of beams under loads. Methods used in determining statically indeterminate reactions and deflections of elastic beams include:[1-13] method of integration (*with* or *without* use of singularity functions), moment-area theorems, Castigliano's theorem, method of superposition, method of segments, method of model formulas, and conjugate beam method.

The *method of model formulas* (MoMF)[12] is newly propounded in 2009. A set of *four* model formulas are derived and established for use in this new method. The formulas are expressed in terms of the following: (*a*) flexural rigidity of the beam; (*b*) slopes, deflections, shear forces, and bending moments at both ends of the beam; (*c*) typical applied loads (concentrated force, concentrated moment, linearly distributed force, and uniformly distributed moment) *somewhere* on the beam. To use the MoMF, one must have rudiments of *singularity functions* at play and utilized an excerpt from this method as shown in Fig. 1, courtesy of *IJEE*.[12] This paper includes a one-page of summaries of the rudiments of singularity functions and the sign conventions for beams. Readers, who are familiar with these topics, may *skip* the summaries.

Excerpt from the **Method of Model Formulas**

Courtesy: *Int. J. Engng. Ed.*, Vol. 25, No. 1, pp. 65-74, 2009



(*a*) (*b*)

Positive directions of forces, moments, slopes, and deflections

$$y' = \theta_a + \frac{V_a}{2EI}x^2 + \frac{M_a}{EI}x - \frac{P}{2EI}<x-x_P>^2 + \frac{K}{EI}<x-x_K>^1 - \frac{w_0}{6EI}<x-x_w>^3 - \frac{w_1-w_0}{24EI(u_w-x_w)}<x-x_w>^4 + \frac{w_1}{6EI}<x-u_w>^3 + \frac{w_1-w_0}{24EI(u_w-x_w)}<x-u_w>^4 + \frac{m_0}{2EI}<x-x_m>^2 - \frac{m_0}{2EI}<x-u_m>^2 \quad (1)$$

$$y = y_a + \theta_a x + \frac{V_a}{6EI}x^3 + \frac{M_a}{2EI}x^2 - \frac{P}{6EI}<x-x_P>^3 + \frac{K}{2EI}<x-x_K>^2 - \frac{w_0}{24EI}<x-x_w>^4 - \frac{w_1-w_0}{120EI(u_w-x_w)}<x-x_w>^5 + \frac{w_1}{24EI}<x-u_w>^4 + \frac{w_1-w_0}{120EI(u_w-x_w)}<x-u_w>^5 + \frac{m_0}{6EI}<x-x_m>^3 - \frac{m_0}{6EI}<x-u_m>^3 \quad (2)$$

$$\theta_b = \theta_a + \frac{V_a L^2}{2EI} + \frac{M_a L}{EI} - \frac{P}{2EI}(L-x_P)^2 + \frac{K}{EI}(L-x_K) - \frac{w_0}{6EI}(L-x_w)^3 - \frac{w_1-w_0}{24EI(u_w-x_w)}(L-x_w)^4 + \frac{w_1}{6EI}(L-u_w)^3 + \frac{w_1-w_0}{24EI(u_w-x_w)}(L-u_w)^4 + \frac{m_0}{2EI}(L-x_m)^2 - \frac{m_0}{2EI}(L-u_m)^2 \quad (3)$$

$$y_b = y_a + \theta_a L + \frac{V_a L^3}{6EI} + \frac{M_a L^2}{2EI} - \frac{P}{6EI}(L-x_P)^3 + \frac{K}{2EI}(L-x_K)^2 - \frac{w_0}{24EI}(L-x_w)^4 - \frac{w_1-w_0}{120EI(u_w-x_w)}(L-x_w)^5 + \frac{w_1}{24EI}(L-u_w)^4 + \frac{w_1-w_0}{120EI(u_w-x_w)}(L-u_w)^5 + \frac{m_0}{6EI}(L-x_m)^3 - \frac{m_0}{6EI}(L-u_m)^3 \quad (4)$$

Fig. 1. Model loading and beam deflection formulas for the *method of model formulas*

■ Summary of rudiments of singularity functions

Consistent with the commonly used notations, the argument of a singularity function is enclosed by angle brackets (i.e., < >). The argument of a regular function continues to be enclosed by parentheses [i.e., ( )]. The rudiments of singularity functions include the following:[8,9]

$$< x-a >^n = (x-a)^n \quad \text{if} \quad x-a \geq 0 \quad \text{and} \quad n > 0 \tag{5}$$

$$< x-a >^n = 1 \quad \text{if} \quad x-a \geq 0 \quad \text{and} \quad n = 0 \tag{6}$$

$$< x-a >^n = 0 \quad \text{if} \quad x-a < 0 \quad \text{or} \quad n < 0 \tag{7}$$

$$\int_{-\infty}^{x} < x-a >^n dx = \frac{1}{n+1} < x-a >^{n+1} \quad \text{if} \quad n > 0 \tag{8}$$

$$\int_{-\infty}^{x} < x-a >^n dx = < x-a >^{n+1} \quad \text{if} \quad n \leq 0 \tag{9}$$

$$\frac{d}{dx} < x-a >^n = n < x-a >^{n-1} \quad \text{if} \quad n > 0 \tag{10}$$

$$\frac{d}{dx} < x-a >^n = < x-a >^{n-1} \quad \text{if} \quad n \leq 0 \tag{11}$$

Equations (6) and (7) imply that, in using singularity functions for beams, we take

$$b^0 = 1 \quad \text{for} \quad b \geq 0 \tag{12}$$

$$b^0 = 0 \quad \text{for} \quad b < 0 \tag{13}$$

■ Summary of sign conventions for beams

In the *method of model formulas*, the adopted sign conventions for various model loadings on the beam and for deflections of the beam with a constant flexural rigidity *EI* are illustrated in Fig. 1. Notice the following key points:[8,9]

- A *shear force* is *positive* if it acts upward on the left (or downward on the right) face of the beam element [e.g., $\mathbf{V}_a$ at the left end *a*, and $\mathbf{V}_b$ at the right end *b* in Fig. 1(*a*)].
- At ends of the beam, a *moment* is *positive* if it tends to cause compression in the top fiber of the beam [e.g., $\mathbf{M}_a$ at the left end *a*, and $\mathbf{M}_b$ at the right end *b* in Fig. 1(*a*)].
- If not at ends of the beam, a *moment* is *positive* if it tends to cause compression in the top fiber of the beam just to the right of the position where it acts [e.g., the concentrated moment $\mathbf{K} = K \circlearrowright$ and the uniformly distributed moment with intensity m_0 in Fig. 1(*a*)].
- A *concentrated force* or a *distributed force* applied to the beam is *positive* if it is directed downward [e.g., the concentrated force $\mathbf{P} = P \downarrow$, the linearly distributed force with intensity w_0 on the left side and intensity w_1 on the right side in Fig. 1(*a*), where the distribution becomes uniform if $w_0 = w_1$].

The slopes and deflections of a beam displaced from *AB* to *ab* are shown in Fig. 1(*b*). Note that

- A *positive slope* is a counterclockwise angular displacement [e.g., θ_a and θ_b in Fig. 1(*b*)].
- A *positive deflection* is an upward linear displacement [e.g., y_a and y_b in Fig. 1(*b*)].

■ Methodology and pedagogy of the method of model formulas

The four model formulas in Eqs. (1) through (4) were derived in great detail in the paper that propounded the MoMF.[12] For convenience of readers, let us take a brief overview of how these model formulas are obtained. Basically, it starts out with the loading function q,[9] written in terms of singularity functions for the beam ab in Fig. 1; as follows:

$$\begin{aligned} q = {} & V_a < x >^{-1} + M_a < x >^{-2} - P < x - x_P >^{-1} + K < x - x_K >^{-2} - w_0 < x - x_w >^{0} \\ & - \frac{w_1 - w_0}{u_w - x_w} < x - x_w >^{1} + w_1 < x - u_w >^{0} + \frac{w_1 - w_0}{u_w - x_w} < x - u_w >^{1} \\ & + m_0 < x - x_m >^{-1} - m_0 < x - u_m >^{-1} \end{aligned} \tag{14}$$

By integrating q, one can write the shear force V and the bending moment M for the beam ab in Fig. 1. Letting the flexural rigidity of the beam ab be EI, y be the deflection, y' be the slope, and y'' be the second derivative of y with respect to the abscissa x, which defines the position of the section under consideration along the axis of the beam, one may apply the relation $EIy'' = M$ and readily obtain the expressions for EIy' and EIy via integration. The slope and deflection of the beam are θ_a and y_a at its left end a (i.e., at $x = 0$), and are θ_b and y_b at the right end b (i.e., at $x = L$), as illustrated in Fig. 1. Imposition of these boundary conditions will yield the four model formulas in Eqs. (1) through (4).

The pedagogy of the MoMF lies in teaching and applying the four model formulas in this method. Note that L in the model formulas in Eqs. (1) through (4) is a *parameter* representing the *total length* of the beam segment. In other words, this L is to be replaced by the *total length* of the beam segment to which the model formulas are applied. Furthermore, notice that this method allows one to treat reactions at interior supports (i.e., those *not* at the ends of the beam) as applied concentrated forces or moments, as appropriate. All one has to do is to simply impose the additional boundary conditions at the points of interior supports for the beam segment by using Eqs. (1) and (2). Thus, statically indeterminate reactions as well as slopes and deflections of beams can be determined. A beam needs to be divided into segments for analysis only if (*a*) it is a combined beam (e.g., a *Gerber beam*) having discontinuities in slope at hinge connections between segments, and (*b*) it contains segments with different flexural rigidities (e.g., a stepped beam).

■ Methodology and pedagogy of the conjugate beam method

The *conjugate beam method* (CBM) propounded by Westergaard[1] is a great method and is consistent with the *moment-area theorems*. The support conditions (free end, fixed end, simple support at the end of the beam, simple support not at the end of the beam, and unsupported hinge), rather than the traditional boundary conditions, are heavily used in this method. Earlier textbooks[2,5] included only brief and elementary coverage of this method. Somehow most current prevailing textbooks drop the coverage of this method. The pedagogy of the CBM lies in teaching and applying the ten guiding rules synthesized for this method. For convenience of readers, these rules are listed below, courtesy of *IJEE*.[13]

Guiding rules in the conjugate beam method:[13]

Rule 1: The conjugate beam and the given beam are of the same *length*.

Rule 2: The load on the conjugate beam is the *elastic weight*, which is the bending moment M in the given beam divided by the flexural rigidity EI of the given beam. (This elastic weight is taken to act upward if the bending moment is positive – to cause top fiber in compression – in beam convention.)

For each existing support condition of the given beam, there is a corresponding support condition for the conjugate beam. The correspondence is given by rules 3 through 7 as follows:

	Existing support condition in the **given beam**:	Corresponding support condition in the **conjugate beam**:
Rule 3:	Fixed end	Free end
Rule 4:	Free end	Fixed end
Rule 5:	Simple support at the end	Simple support at the end
Rule 6:	Simple support *not* at the end	Unsupported hinge
Rule 7:	Unsupported hinge	Simple support

Rule 8: The conjugate beam is in static *equilibrium*.

Rule 9: The *slope* of the given beam at any cross section is given by the "shear force" at that cross section of the conjugate beam. (This *slope* is positive, or counterclockwise, if the "shear force" is positive – tending to rotate the beam element clockwise – in beam convention.)

Rule 10: The *deflection* of the given beam at any point is given by the "bending moment" at that point of the conjugate beam. (This *deflection* is upward if the "bending moment" is positive – tending to cause the top fiber in compression – in beam convention.)

II. Teaching and Learning New Methods via Contrast between Solutions

Mechanics is mostly a deductive science, but learning is mostly an inductive process. For the purposes of *teaching* and *learning*, all examples will **first** be solved by the *method of model formulas* (**MoMF**). **Then** the same problems in the examples will be solved by the *conjugate beam method* (**CBM**). For convenient comparison of effectiveness in the solutions by different methods, problems in previous examples[14] will be employed in illustrating solutions by the MoMF and the CBM.

A beam is in *neutral equilibrium* if the force system acting on the beam is statically balanced and the potential energy of the beam in the neighborhood of its equilibrium configuration is constant. This is illustrated in Fig. 2.

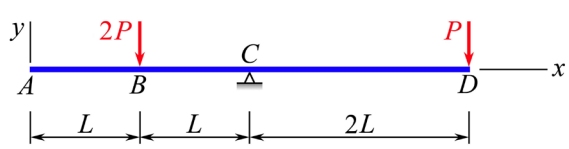


Fig. 2. Beam in neutral equilibrium on a simple support.

Readers are advised to note that **all** methods, *except* the CBM, cannot solve the type of problems involving deflections of *beams in neutral equilibrium*. In this regard, the CBM is more general and powerful! To know more about this feature, readers may see the paper by Jong.[13]

Example 1. A simply supported beam AD with constant flexural rigidity EI and total length L is acted on by a concentrated force $P\downarrow$ at B and a concentrated moment $PL\circlearrowleft$ at C as shown in Fig. 3. Determine (a) the slopes θ_A and θ_D at A and D, respectively; (b) the deflection y_B at B.

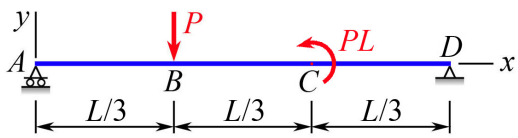


Fig. 3. Given beam AD, simply supported and carrying concentrated loads

Solution. The beam is statically determinate. Its free-body diagram is shown in Fig. 4.

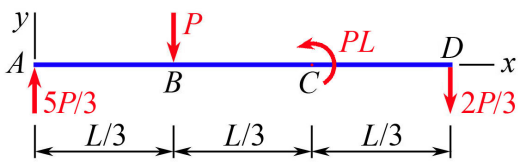


Fig. 4. Free-body diagram of the given beam AD

● **Using MoMF:** In applying the *method of model formulas* to this beam, we must adhere to the sign conventions as illustrated in Fig. 1. At the left end A, the moment M_A is 0, the shear force V_A is $5P/3$, the deflection y_A is 0, but the slope θ_A is unknown. At the right end D, the deflection y_D is 0, but the slope θ_D is unknown. Note in the model formulas that we have $x_P = L/3$ for the concentrated force $P\downarrow$ at B and $x_K = 2L/3$ for the concentrated moment $PL\circlearrowleft$ at C. Applying the model formulas in Eqs. (3) and (4), successively, to this beam AD, we write

$$\theta_D = \theta_A + \frac{(5P/3)L^2}{2EI} + 0 - \frac{P}{2EI}\left(L - \frac{L}{3}\right)^2 + \frac{-PL}{EI}\left(L - \frac{2L}{3}\right) - 0 - 0 + 0 + 0 + 0 - 0$$

$$0 = 0 + \theta_A L + \frac{(5P/3)L^3}{6EI} + 0 - \frac{P}{6EI}\left(L - \frac{L}{3}\right)^3 + \frac{-PL}{2EI}\left(L - \frac{2L}{3}\right)^2 - 0 - 0 + 0 + 0 + 0 - 0$$

These *two* simultaneous equations yield

$$\theta_A = -\frac{14PL^2}{81EI} \qquad \theta_D = \frac{17PL^2}{162EI}$$

Using the value of θ_A and applying the model formula in Eq. (2), we write

$$y_B = y\big|_{x=L/3} = 0 + \theta_A\left(\frac{L}{3}\right) + \frac{5P/3}{6EI}\left(\frac{L}{3}\right)^3 + 0 - 0 + 0 - 0 - 0 + 0 + 0 + 0 - 0 = -\frac{23PL^3}{486EI}$$

We report that

$$\theta_A = \frac{14PL^2}{81EI}\circlearrowright \qquad \theta_D = \frac{17PL^2}{162EI}\circlearrowleft \qquad y_B = \frac{23PL^3}{486EI}\downarrow$$

● **Using CBM:** In using the *conjugate beam method* to solve the problem in this example, we **first** make use of the free-body diagram in Fig. 4 and apply guiding rules **1**, **2**, and **5** in the CBM, as listed at the end of Sect I, to construct the conjugate beam for the given beam as shown in Fig. 5. Note that the bending-moment diagram is drawn **by parts** from both ends toward B.

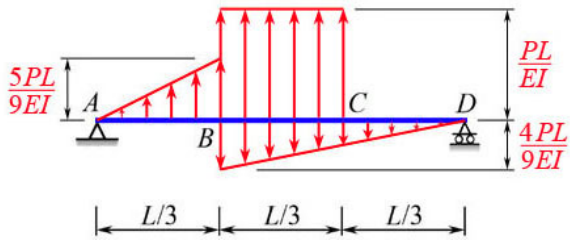


Fig. 5. Conjugate beam for given beam *AD*

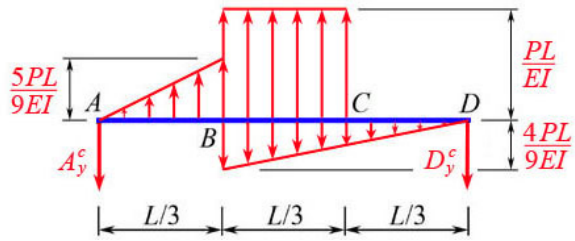


Fig. 6. FBD of conjugate beam *AD*

The free-body diagram of the conjugate beam *AD* is shown in Fig. 6, where the reactions at the simple supports *A* and *D* are assumed to be acting downward; i.e., $\mathbf{A}_y^c = A_y^c \downarrow$ and $\mathbf{D}_y^c = D_y^c \downarrow$. Notice that we have used a superscript ^{c} on the symbols for these reactions to signify that they are associated with the conjugate beam, *not* the given beam.

Next, referring to the conjugate beam in Fig. 6 and applying guiding rule **8** in the CBM, we write

$$+\circlearrowleft \Sigma M_D^c = 0: \qquad L A_y^c + \frac{4L}{9}\cdot\frac{L}{3}\cdot\frac{4PL}{9EI} - \frac{7L}{9}\cdot\frac{L}{6}\cdot\frac{5PL}{9EI} - \frac{L}{2}\cdot\frac{PL^2}{3EI} = 0$$

$$+\uparrow \Sigma F_y^c = 0: \qquad \frac{L}{6}\cdot\frac{5PL}{9EI} + \frac{PL^2}{3EI} - \frac{L}{3}\cdot\frac{4PL}{9EI} - A_y^c - D_y^c = 0$$

These *two* simultaneous equations yield

$$A_y^c = \frac{14PL^2}{81EI} \qquad D_y^c = \frac{17PL^2}{162EI}$$

According to the sign conventions for beams summarized in Sect. I, the "shear forces" at *A* and *D* in the conjugate beam in Fig. 6 are

$$V_A^c = -A_y^c = -\frac{14PL^2}{81EI} \qquad V_D^c = D_y^c = \frac{17PL^2}{162EI}$$

Applying guiding rule **9** in the CBM, we have $\theta_A = V_A^c$ and $\theta_D = V_D^c$. We report that

$$\theta_A = \frac{14PL^2}{81EI} \circlearrowright \qquad \theta_D = \frac{17PL^2}{162EI} \circlearrowleft$$

Applying guiding rule **10** in the CBM, we write

$$y_B = M_B^c = \frac{L}{9}\cdot\frac{L}{6}\cdot\frac{5PL}{9EI} - \frac{L}{3}\cdot A_y^c = \frac{5PL^3}{486EI} - \frac{14PL^3}{243EI} = -\frac{23PL^3}{486EI}$$

We report that

$$y_B = \frac{23PL^3}{486EI} \downarrow$$

<u>Assessment of effectiveness</u>. In this example, we see that the *method of model formulas* enables one to **directly** write the pertinent equations and solve them to obtain the solutions. The *conjugate beam method* does **not** require the use of an excerpt of the model formulas. However, the CBM requires the application of the guiding rules to first construct the conjugate beam for the given beam, then write the equations of equilibrium from the free-body diagram for the

conjugate beam, solve the equations, and apply guiding rules **9** and **10** in the CBM to get the slopes and deflections of the beam. In this example, we observe that the MoMF involves mostly algebraic work from the use of the model formulas in the solution, while the CBM involves guiding rules, more geometry, more statics, and about the same amount of algebraic work. Both MoMF and CBM yield the **same solutions** and are **equally effective** in solving the problem in this example. (Nevertheless, given an opportunity to choose between these two methods to solve a beam deflection problem in the final exam of the author's class MEEG 3013 Mechanics of Materials, in fall 2011, about 75% of the students prefer to use the MoMF.)

Example 2. A cantilever beam AC with constant flexural rigidity EI and total length L is loaded with a distributed load of intensity w in segment AB as shown in Fig. 7. Determine (a) the slope θ_A and deflection y_A at A, (b) the slope θ_B and deflection y_B at B.

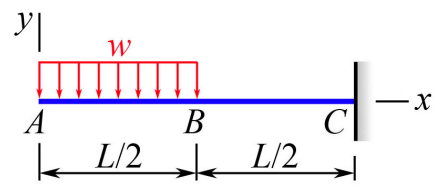


Fig. 7. Cantilever beam AC loaded with a distributed load

Solution. The beam is statically determinate. Its free-body diagram is shown in Fig. 8.

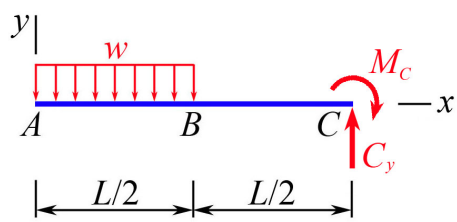


Fig. 8. Free-body diagram of the cantilever beam AC

● **Using MoMF:** In applying the *method of model formulas* to solve the problem, we note that the shear force V_A and the bending moment M_A at the free end A, as well as the slope θ_C and the deflection y_C at the fixed end C, are all zero. Seeing that the uniformly distributed load has $x_w = 0$ and $u_w = L/2$, we apply the model formulas in Eqs. (3) and (4) to the entire beam to write

$$0 = \theta_A + 0 + 0 - 0 + 0 - \frac{w}{6EI}L^3 - 0 + \frac{w}{6EI}\left(L - \frac{L}{2}\right)^3 + 0 + 0 - 0$$

$$0 = y_A + \theta_A L + 0 + 0 - 0 + 0 - \frac{w}{24EI}L^4 - 0 + \frac{w}{24EI}\left(L - \frac{L}{2}\right)^4 + 0 + 0 - 0$$

These *two* simultaneous equations yield

$$\theta_A = \frac{7wL^3}{48EI} \qquad y_A = -\frac{41wL^4}{384EI}$$

Using these values and applying the model formulas in Eqs. (1) and (2), respectively, we write

$$\theta_B = y'\big|_{x=L/2} = \theta_A + 0 + 0 - 0 + 0 - \frac{w}{6EI}\left(\frac{L}{2}\right)^3 - 0 + 0 + 0 + 0 - 0 = \frac{wL^3}{8EI}$$

$$y_B = y\big|_{x=L/2} = y_A + \theta_A\left(\frac{L}{2}\right) + 0 + 0 - 0 + 0 - \frac{w}{24EI}\left(\frac{L}{2}\right)^4 - 0 + 0 + 0 + 0 - 0 = -\frac{7wL^4}{192EI}$$

We report that

$$\theta_A = \frac{7wL^3}{48EI} \circlearrowleft \qquad y_A = \frac{41wL^4}{384EI} \downarrow \qquad \theta_B = \frac{wL^3}{8EI} \circlearrowleft \qquad y_B = \frac{7wL^4}{192EI} \downarrow$$

● **Using CBM:** In using the *conjugate beam method* to solve the problem in this example, we **first** make use of the free-body diagram in Fig. 8 and apply guiding rules **1**, **2**, **3**, and **4** in the CBM, as listed at the end of Sect I, to construct the conjugate beam for the given beam AC as shown in Fig. 9. Notice that the free end at A in the given beam in Fig. 7 becomes a fixed end at A in the conjugate beam in Fig. 9. The free-body diagram of the conjugate beam AC is shown in Fig. 10, where the unknown "shear force" and the "bending moment" at the fixed end A are assumed to act in the positive directions in the beam conventions.

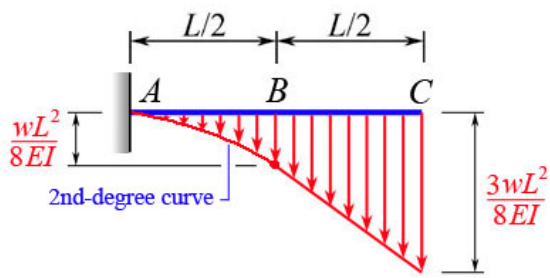


Fig. 9. Conjugate beam for given beam AC

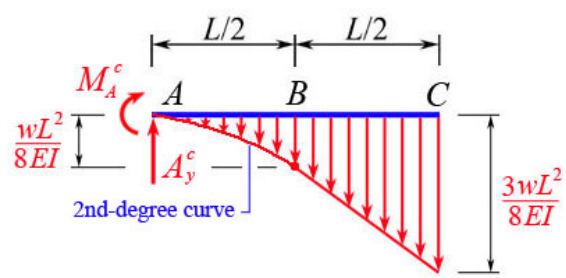


Fig. 10. FBD of conjugate beam AC

Next, referring to Fig. 10 and applying guiding rule **8** in the CBM, we write

$$+\uparrow \Sigma F_y^c = 0: \qquad A_y^c - \frac{L}{6}\cdot\frac{wL^2}{8EI} - \frac{1}{2}\left(\frac{wL^2}{8EI} + \frac{3wL^2}{8EI}\right)\left(\frac{L}{2}\right) = 0$$

$$+\circlearrowleft \Sigma M_A^c = 0: \qquad -M_A^c - \frac{3L}{8}\cdot\frac{L}{6}\cdot\frac{wL^2}{8EI} - \frac{3L}{4}\cdot\frac{L}{2}\cdot\frac{wL^2}{8EI} - \frac{5L}{6}\cdot\frac{L}{4}\cdot\frac{2wL^2}{8EI} = 0$$

These *two* simultaneous equations yield

$$A_y^c = \frac{7wL^3}{48EI} \qquad M_A^c = -\frac{41wL^4}{384EI}$$

Applying guiding rules **9** and **10** in the CBM, we have $\theta_A = V_A^c = A_y^c$, $y_A = M_A^c$. We report that

$$\theta_A = \frac{7wL^3}{48EI} \circlearrowleft \qquad y_A = \frac{41wL^4}{384EI} \downarrow$$

Referring to Fig. 10, we find that the "shear force" and "bending moment" at B of the conjugate beam are

$$V_B^c = A_y^c - \frac{L}{6}\cdot\frac{wL^2}{8EI} = \frac{wL^3}{8EI}$$

$$M_B^c = M_A^c + \frac{L}{2}A_y^c - \frac{L}{8}\cdot\frac{L}{6}\cdot\frac{wL^2}{8EI} = -\frac{7wL^4}{192EI}$$

Applying guiding rules **9** and **10** in the CBM, we have $\theta_B = V_B^c$ and $y_B = M_B^c$. We report that

$$\theta_B = \frac{wL^3}{8EI}\circlearrowleft \qquad y_B = \frac{7wL^4}{192EI}\downarrow$$

Assessment of effectiveness. Again, we see that the *method of model formulas* enables one to **directly** write the pertinent equations and solve them to obtain the solutions. The *conjugate beam method* does **not** require the use of an excerpt of the model formulas. However, the CBM requires the application of the guiding rules to first construct the conjugate beam for the given beam, then write the equations of equilibrium from the free-body diagram for the conjugate beam, solve the equations, and apply guiding rules **9** and **10** in the CBM to get the slopes and deflections of the beam. Both MoMF and CBM yield the **same solutions** and are **equally effective** in solving the problem in this example.

Example 3. A cantilever beam AC with constant flexural rigidity EI and total length $2L$ is propped at A and carries a concentrated moment $M_0\circlearrowleft$ at B as shown in Fig. 11. Determine (*a*) the vertical reaction force $\mathbf{A}_y$ and slope θ_A at A, (*b*) the slope θ_B and deflection y_B at B.

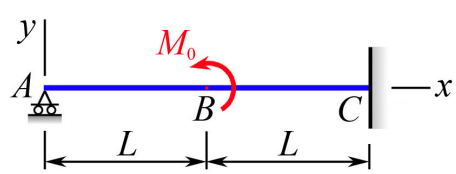


Fig. 11. Cantilever beam AC propped at A and carrying a concentrated moment at B

Solution. The free-body diagram of the beam is shown in Fig. 12, where we note that the beam is statically indeterminate to the *first* degree.

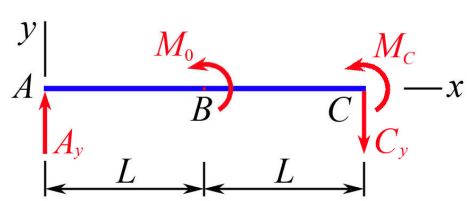


Fig. 12. Free-body diagram of the propped cantilever beam AC

● **Using MoMF:** In applying the *method of model formulas* to this beam, we first note that this beam has a total length of $2L$, which will be the value for the *parameter* L in all of the model formulas in Eqs. (1) through (4). We also note that the deflection y_C and the slope θ_C at C, as well as the deflection y_A at A, are all equal to zero. Applying the model formulas in Eqs. (3) and (4) to this beam, we write

$$0 = \theta_A + \frac{A_y(2L)^2}{2EI} + 0 - 0 + \frac{-M_0}{EI}(2L - L) - 0 - 0 + 0 + 0 + 0 - 0$$

$$0 = 0 + \theta_A(2L) + \frac{A_y(2L)^3}{6EI} + 0 - 0 + \frac{-M_0}{2EI}(2L - L)^2 - 0 - 0 + 0 + 0 + 0 - 0$$

These *two* simultaneous equations yield

$$A_y = \frac{9M_0}{16L} \qquad \theta_A = -\frac{M_0 L}{8EI}$$

Using these values and applying the model formulas in Eqs. (1) and (2), respectively, we write

$$\theta_B = y'\big|_{x=L} = \theta_A + \frac{A_y}{2EI}L^2 + 0 - 0 + 0 - 0 - 0 + 0 + 0 + 0 - 0 = \frac{5M_0 L}{32EI}$$

$$y_B = y\big|_{x=L} = 0 + \theta_A L + \frac{A_y}{6EI}L^3 + 0 - 0 + 0 - 0 - 0 + 0 + 0 + 0 - 0 = -\frac{M_0 L^2}{32EI}$$

We report that

$$\mathbf{A}_y = \frac{9M_0}{16L}\uparrow \qquad \theta_A = \frac{M_0 L}{8EI}\circlearrowright \qquad \theta_B = \frac{5M_0 L}{32EI}\circlearrowleft \qquad y_B = \frac{M_0 L^2}{32EI}\downarrow$$

● **Using CBM:** In using the *conjugate beam method* to solve the problem in this example, we **first** make use of the free-body diagram in Fig. 12 and apply guiding rules **1**, **2**, **3**, and **5** in the CBM, as listed at the end of Sect I, to construct the conjugate beam for the given beam *AC* as shown in Fig. 13. Notice that the simple support at the end *A* in the given beam *AC* in Fig. 11 remains a simple support at the end *A* in the conjugate beam *AC* in Fig. 13; however, the fixed end at *C* in the given beam becomes a free end at *C* in the conjugate beam. The free-body diagram of the conjugate beam *AC* is shown in Fig. 14, where the unknown "shear force" at the end *A* is assumed to act in the positive direction in the beam conventions.

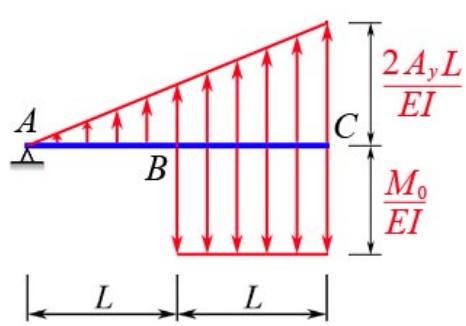


Fig. 13. Conjugate beam for given beam *AC*

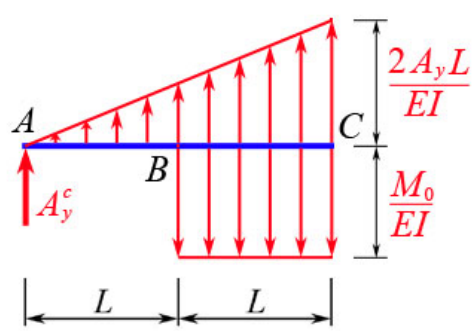


Fig. 14. FBD of conjugate beam *AC*

Next, referring to Fig. 14 and applying guiding rule **8** in the CBM, we write

$$+\circlearrowleft \Sigma M_A^c = 0: \qquad \frac{4L}{3}\cdot L\cdot\frac{2A_yL}{EI} - \frac{3L}{2}\cdot L\cdot\frac{M_0}{EI} = 0$$

$$+\uparrow \Sigma F_y^c = 0: \qquad A_y^c + L \cdot \frac{2A_y L}{EI} - L \cdot \frac{M_0}{EI} = 0$$

The above *two* simultaneous equations yield

$$A_y = \frac{9M_0}{16L} \qquad A_y^c = -\frac{M_0 L}{8EI}$$

Applying guiding rule **9** in the CBM, we have $\theta_A = V_A^c = A_y^c$. We report that

$$\mathbf{A}_y = \frac{9M_0}{16L} \uparrow \qquad \theta_A = \frac{M_0 L}{8EI} \circlearrowright$$

Referring to Fig. 14, we find that the "shear force" and "bending moment" at B of the conjugate beam are

$$V_B^c = A_y^c + \frac{L}{2} \cdot \frac{A_y L}{EI} = \frac{5M_0 L}{32EI}$$

$$M_B^c = L A_y^c + \frac{L}{3} \cdot \frac{L}{2} \cdot \frac{A_y L}{EI} = -\frac{M_0 L^2}{32EI}$$

Applying guiding rules **9** and **10** in the CBM, we have $\theta_B = V_B^c$ and $y_B = M_B^c$. We report that

$$\theta_B = \frac{5M_0 L}{32EI} \circlearrowleft \qquad y_B = \frac{M_0 L^2}{32EI} \downarrow$$

Assessment of effectiveness. As before, we see that the *method of model formulas* enables one to **directly** write the pertinent equations and solve them to obtain the solutions. The *conjugate beam method* does **not** require the use of an excerpt of the model formulas. However, the CBM requires the application of the guiding rules to first construct the conjugate beam for the given beam, then write the equations of equilibrium from the free-body diagram for the conjugate beam, solve the equations, and apply guiding rules **9** and **10** in the CBM to get the slopes and deflections of the beam. Both MoMF and CBM yield the **same solutions** and are **equally effective** in solving the problem in this example.

Example 4. A continuous beam AC with constant flexural rigidity EI and total length $2L$ has a roller support at A, a roller support at B, and a fixed support at C. This beam carries a linearly distributed load and is shown in Fig. 15. Determine (a) the vertical reaction force $\mathbf{A}_y$ and slope θ_A at A, (b) the vertical reaction force $\mathbf{B}_y$ and slope θ_B at B.

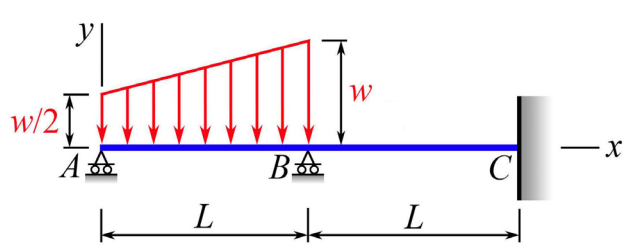


Fig. 15. Continuous beam AC carrying a linearly distributed load

Solution. The free-body diagram of the beam is shown in Fig. 16. We readily note that the beam is statically indeterminate to the *second* degree.

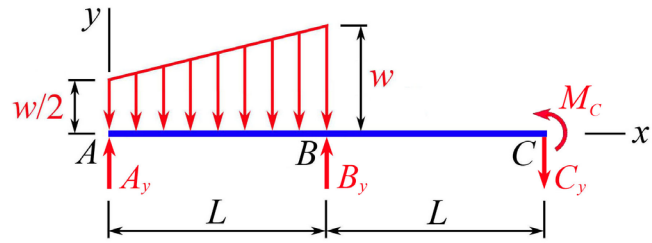


Fig. 16. Free-body diagram of the continuous beam AC

● **Using MoMF:** In applying the *method of model formulas* to this beam, we notice that the beam AC has a total length $2L$, which will be the value for the *parameter* L in all model formulas in Eqs. (1) through (4). We see that the shear force V_A at left end A is equal to A_y, the moment M_A and deflection y_A at A are zero, the deflection y_B at B is zero, and the slope θ_C and deflection y_C at C are zero. Applying the model formulas in Eqs. (3) and (4) to the beam AC and using Eq. (2) to impose the condition that $y_B = y(L) = 0$ at B, in that *order*, we write

$$0 = \theta_A + \frac{A_y(2L)^2}{2EI} + 0 - \frac{-B_y}{2EI}(2L-L)^2 + 0 - \frac{w/2}{6EI}(2L)^3 - \frac{w-(w/2)}{24EIL}(2L)^4$$
$$+ \frac{w}{6EI}(2L-L)^3 + \frac{w-(w/2)}{24EIL}(2L-L)^4 + 0 - 0$$

$$0 = 0 + \theta_A(2L) + \frac{A_y(2L)^3}{6EI} + 0 - \frac{-B_y}{6EI}(2L-L)^3 + 0 - \frac{w/2}{24EI}(2L)^4 - \frac{w-(w/2)}{120EIL}(2L)^5$$
$$+ \frac{w}{24EI}(2L-L)^4 + \frac{w-(w/2)}{120EIL}(2L-L)^5 + 0 - 0$$

$$0 = 0 + \theta_A L + \frac{A_y}{6EI}L^3 + 0 - 0 + 0 - \frac{w/2}{24EI}L^4 - \frac{w-(w/2)}{120EIL}L^5 + 0 + 0 + 0 - 0$$

These *three* simultaneous equations yield

$$A_y = \frac{39wL}{140} \qquad \theta_A = -\frac{3wL^3}{140EI} \qquad B_y = \frac{31wL}{56}$$

Using these values and applying the model formula in Eq. (1), we write

$$\theta_B = y'\big|_{x=L} = \theta_A + \frac{A_y}{2EI}L^2 + 0 - 0 + 0 - \frac{w/2}{6EI}L^3 - \frac{w-(w/2)}{24EIL}L^4 + 0 + 0 + 0 - 0$$
$$= \frac{23wL^3}{1680EI}$$

We report that

$$\mathbf{A}_y = \frac{39wL}{140}\uparrow \qquad \theta_A = \frac{3wL^3}{140EI}\circlearrowright$$

$$\mathbf{B}_y = \frac{31wL}{56}\uparrow \qquad \theta_B = \frac{23wL^3}{1680EI}\circlearrowleft$$

● **Using CBM:** In using the *conjugate beam method* to solve the problem in this example, we **first** make use of the free-body diagram in Fig. 16 and apply guiding rules **1**, **2**, **3**, **5,** and **6** in the CBM, as listed at the end of Sect I, to construct the conjugate beam for the given beam AC as shown in Fig. 17. Because of the incurred complexity of the elastic weights drawn by parts, we here let the conjugate beam carry the sum of two sets of elastic weight as indicated in Fig. 17. Notice that (*a*) the **simple support** at the end A in the given beam AC in Fig. 15 remains a **simple support** at the end A in the conjugate beam AC in Fig. 17; (*b*) the **simple support** at B in Fig. 15, which is *not* at the end of the beam, becomes an **unsupported hinge** at B in Fig. 17; (*c*) the **fixed end** at C in Fig. 15 becomes a **free end** at C in Fig. 17. The free-body diagram of the conjugate beam AC is shown in Fig. 18, which contains three **unknowns**: A_y, B_y, and A_y^c.

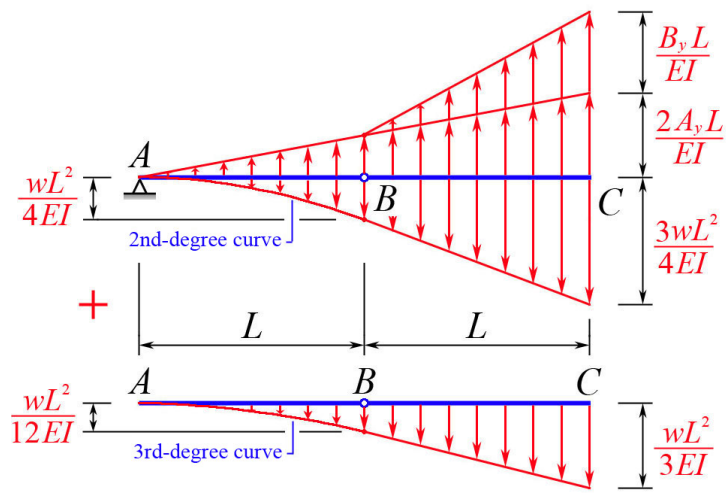


Fig. 17. Conjugate beam for given beam AC

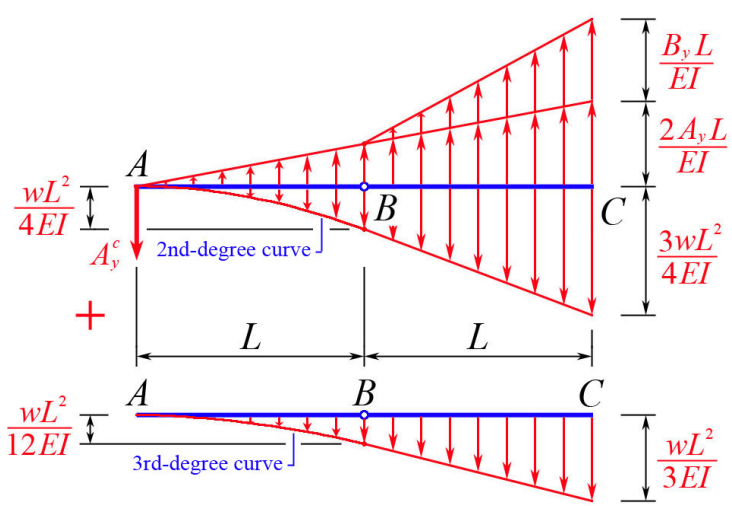


Fig. 18. Free-body diagram of conjugate beam AC

Next, referring to Fig. 18 and applying guiding rule **8** in the CBM, we write

$+\uparrow \Sigma F_y^c = 0$, for the *entire* conjugate beam ABC in Fig. 18:

$$-A_y^c + L\cdot\frac{2A_yL}{EI} + \frac{L}{2}\cdot\frac{B_yL}{EI} - \frac{L}{3}\cdot\frac{wL^2}{4EI} - \frac{L}{2}\cdot\left(\frac{wL^2}{4EI}+\frac{3wL^2}{4EI}\right) - \frac{L}{4}\cdot\frac{wL^2}{12EI} - \frac{L}{2}\cdot\left(\frac{wL^2}{12EI}+\frac{wL^2}{3EI}\right) = 0$$

$+\circlearrowleft \Sigma M_B^c = 0$, for just segment AB — the *left* segment of the conjugate beam in Fig. 18:

$$LA_y^c - \frac{L}{3}\cdot\frac{L}{2}\cdot\frac{A_yL}{EI} + \frac{L}{4}\cdot\frac{L}{3}\cdot\frac{wL^2}{4EI} + \frac{L}{5}\cdot\frac{L}{4}\cdot\frac{wL^2}{12EI} = 0$$

$+\circlearrowleft \Sigma M_B^c = 0$, for just segment BC — the *right* segment of the conjugate beam in Fig. 18:

$$\frac{2L}{3}\cdot\frac{L}{2}\cdot\left(\frac{2A_yL}{EI}+\frac{B_yL}{EI}\right) + \frac{L}{3}\cdot\frac{L}{2}\cdot\frac{A_yL}{EI} - \frac{L}{3}\cdot\frac{L}{2}\cdot\frac{wL^2}{4EI} - \frac{2L}{3}\cdot\frac{L}{2}\cdot\frac{3wL^2}{4EI} - \frac{L}{3}\cdot\frac{L}{2}\cdot\frac{wL^2}{12EI} - \frac{2L}{3}\cdot\frac{L}{2}\cdot\frac{wL^2}{3EI} = 0$$

The above *three* simultaneous equations yield

$$A_y = \frac{39wL}{140} \qquad B_y = \frac{31wL}{56} \qquad A_y^c = \frac{3wL^3}{140EI}$$

Applying guiding rules **9** in the CBM, we have $\theta_A = V_A^c = -A_y^c$. We report that

$$\mathbf{A}_y = \frac{39wL}{140}\uparrow \qquad \theta_A = \frac{3wL^3}{140EI}\circlearrowright \qquad \mathbf{B}_y = \frac{31wL}{56}\uparrow$$

Applying guiding rules **9** in the CBM again, we write

$$\theta_B = V_B^c = -A_y^c + \frac{L}{2}\cdot\frac{A_yL}{EI} - \frac{L}{3}\cdot\frac{wL^2}{4EI} - \frac{L}{4}\cdot\frac{wL^2}{12EI} = \frac{23wL^3}{1680EI}$$

We report that

$$\theta_B = \frac{23wL^3}{1680EI}\circlearrowleft$$

Assessment of effectiveness. Once again, we see that the *method of model formulas* enables one to **directly** write the pertinent equations and solve them to obtain the solutions. In MoMF, the slope and deflection at any position of the beam can always be evaluated by applying the model formulas in Eqs. (1) and (2). The *conjugate beam method* does **not** require the use of an excerpt of the model formulas. However, the CBM requires the application of the guiding rules to first construct the conjugate beam for the given beam, then write the equations of equilibrium from the free-body diagram for the conjugate beam, solve the equations, and apply guiding rules **9** and **10** in the CBM to get the slopes and deflections of the beam. Both MoMF and CBM yield the **same solutions** in this example. Because of the particular loading on this second-degree statically indeterminate beam, the CBM has to "go an extra mile" to construct the rather challenging conjugate beam for the given beam. The writing of moment equilibrium equations for this conjugate beam is also rather challenging. The required algebraic work to get the solutions in these two methods is about the same. All considered, one would likely say that the overall effort required to obtain the solution is **more** in using the CBM than in using the MoMF in this example.

III. Concluding Remarks

In the *method of model formulas*, no explicit integration or differentiation is involved in applying any of the model formulas. The model formulas essentially serve to provide *material equations* besides the equations of static equilibrium of the beam that can readily be written. Selected model applied loads are illustrated in Fig. 1(*a*), which cover most of the loads encountered in undergraduate Mechanics of Materials. Naturally, the MoMF can serve to provide independent checks of solutions obtained using other methods. In the case of a nonlinearly distributed load on the beam, the model formulas may be modified by the user for such a load.

Westergaard's *conjugate beam method* employs support conditions in the solutions of problems involving deflections of beams. This approach works well because boundary conditions have, in fact, been taken into account when the support conditions are specified. The CBM usually requires no explicit integration in the solution, and it requires good skills (*a*) in drawing bending moment diagrams by parts for setting the elastic weights on the conjugate beams, (*b*) in writing equations of statics equilibrium in the process of solution.

The MoMF and the CBM are about equally effective in solving problems involving statically indeterminate reactions and deflections of beams, except that the CBM is a unique method that can be used to solve deflections of beams in *neutral equilibrium*. To know more about this unique feature and capability of the CBM, refer to the paper by Jong.[13] Both of the MoMF and the CBM are suitable for learning by sophomores and juniors; and they have been taught and tested in the course Mechanics of Materials at the author's institution for several years.

References

1. H. M. Westergaard, "Deflections of Beams by the Conjugate Beam Method," *Journal of the Western Society of Engineers*, Vol. XXVI, No. 11, pp. 369-396, 1921.
2. S. Timoshenko and G. H. MacCullough, *Elements of Strength of Materials* (3rd Edition), Van Nostrand Company, Inc., New York, NY, 1949.
3. S. H. Crandall, C. D. Norman, and T. J. Lardner, *An Introduction to the Mechanics of Solids* (2nd Edition), McGraw-Hill, New York, NY, 1972.
4. R. J. Roark and W. C. Young, *Formulas for Stress and Strain* (5th Edition), McGraw-Hill, New York, NY, 1975.
5. F. L. Singer and A. Pytel, *Strength of Materials* (4th Edition), Harper & Row, New York, NY, 1987.
6. A. Pytel and J. Kiusalaas, *Mechanics of Materials*, Brooks/Cole, Pacific Grove, CA, 2003.
7. J. M. Gere, *Mechanics of Materials* (6th Edition), Brooks/Cole, Pacific Grove, CA, 2004.
8. F. P. Beer, E. R. Johnston, Jr., J. T. DeWolf, and D. F. Mazurek, *Mechanics of Materials* (5th Edition), McGraw-Hill, New York, NY, 2009.
9. R. G. Budynas and J. K. Nisbett, *Shigley's Mechanical Engineering Design* (8th Edition), McGraw-Hill, New York, NY, 2008.
10. H. T. Grandin, and J. J. Rencis, "A New Approach to Solve Beam Deflection Problems Using the Method of Segments," *Proceedings of the 2006 ASEE Annual conference & Exposition*, Chicago, IL, 2006.

11. I. C. Jong, "Deflection of a Beam in Neutral Equilibrium à la Conjugate Beam Method: Use of Support, *Not* Boundary, Conditions," *7th ASEE Global Colloquium on Engineering Education*, Cape Town, South Africa, October 19-23, 2008.

12. I. C. Jong, "An Alternative Approach to Finding Beam Reactions and Deflections: Method of Model Formulas," *International Journal of Engineering Education*, Vol. 25, No. 1, pp. 65-74, 2009.

13. I. C. Jong, "Determining Deflections of Elastic Beams: What Can the Conjugate Beam Method Do That All Others Cannot?" *International Journal of Engineering Education*, Vol. 26, No. 6, pp. 1422-1427, 2010.

14. I. C. Jong, and W. T. Springer, "Teaching Deflections of Beams: Comparison of Advantages of Method of Model Formulas versus Method of Superposition," Session W550, *Proceedings of 2011 ASEE Annual Conference & Exposition*, Vancouver, BC, Canada, June 26-29, 2011.