Teaching Fluid Mechanics Using Mathcad

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Abstract

Students are taught that the laws of the conservation of mass and the conservation of momentum are fundamental in fluid mechanics analysis and design. These fundamental principles apply whether the flow is spatially varied or constant, temporally unsteady or steady, and closed conduit or open channel. Thus, the application of these basic principles to such a wide possibility of fluid flow problems presents the student with quite a bit of tasks to accomplish in his/her learning process. In the application of one or both of the basic principles, the student is faced with the need to learn and apply, at times, tedious and time-consuming solution algorithms. As a result, both the teacher and the student are presented with auxiliary tasks, which many times, interrupt and hamper the teaching and learning process. Specifically, these auxiliary tasks include solution algorithms which use: iterative procedures; tedious numerical solutions; diagrams, charts, and nomographs; and other indirect and implicit solution procedures. The final result is that there is not much time left to focus on the modeling of the problem, formulation of the solution, interpretation of the results, changing the assumptions, and going back to modeling of the problem, etc., and thus be able to conduct a sensitivity analysis (or an experimental procedure) in order to find the optimum solution. The use of the mathematical software Mathcad to teach fluid mechanics has proven to greatly reduce the drudgery in solving fluid flow problems. As an illustration of this learning enhancement, Mathcad is used to model the occurrence of critical flow at a change of slope in an open channel flow situation. The Mathcad software performs efficient iterative and numerical solution procedures and direct solutions. The use of Mathcad has been made a requirement for all computational procedures in the Fluid Mechanics courses in the Department of Civil Engineering at Howard University.

Introduction

The laws of the conservation of mass and the conservation of momentum are fundamental in fluid mechanics analysis and design, whether the flow is spatially varied or constant, temporally unsteady or steady, and in a closed conduit or an open channel. The application of these basic principles to such a wide variety of fluid flow problems presents the student with a long list of tasks to accomplish in their learning process. In an effort to keep our students current with the fast-paced technological advances taking place in the scientific field of problem modeling and solution formulation, engineering educators are always in search of improved techniques to teach challenging subjects in Civil Engineering such as fluid mechanics.

Because the solution of many problems in fluid mechanics and hydraulics requires repetitive calculations, using programmed procedures can save considerable time and tedious effort. There are various programming procedures available, which make use of advanced technology: 1) programmable scientific calculators and equation solvers, 2) spreadsheets, 3) mathematics software, 4) applications software, and 5) programming languages [1]. While each procedure may provide certain advantages in varying circumstances, it appears that the mathematics software offers the most useful applications for solving engineering problems in general, as well as for fluid flow problems in particular.

Solution of many fluid flow problems requires solving a set of simultaneous nonlinear equations and /or solving a set of linear or nonlinear ordinary or partial differential equations that may be boundary-value or initial value problems. Because Mathcad is indeed capable of handling such equations *and* is user-friendly, it was the chosen mathematics software used to teach fluid mechanics at both the introductory and intermediate levels to our undergraduate students. Prior to enrolling in this series of fluid mechanics courses, our students are taught Mathcad in the undergraduate courses Computer Essentials and Analysis Methods.

In addition to the fluid mechanics course, currently there are three other courses in the Department of Civil Engineering at Howard University that integrate Mathcad in the teaching and learning process; these courses are statics, dynamics, and mechanics of materials [2,3,4].

Typical Solution Procedure for Fluid Flow Problems

Although the various types of fluid flow problems are vast in number, they each require the student to conduct a number of routine steps in order to reach a solution. The first step is to study the physical problem and determine the flow type: closed conduit or open channel flow, temporally unsteady or steady state flow, and spatially varied or constant flow. The second step is to apply the appropriate fundamental principles to the physical flow situation and thus accurately model the problem with the appropriate equations. The third step is to formulate the appropriate and efficient solution procedure in order to obtain accurate results. The fourth step is to interpret and possibly verify the achieved results. The fifth step is to potentially make changes in the assumed values for one or more of the known variables and repeat steps one through four above. The sixth step is to study the results of the sensitivity analysis (or experimental procedure) accomplished through a series of step five, draw intelligent conclusions.

Traditional Techniques versus Mathcad Capabilities

Whether the student is taught to use traditional or Mathcad techniques, he/she must still follow the typical procedure for fluid flow problems described above. Traditional techniques typically make use of other programmed procedures such as programmable scientific calculators and equation solvers or spreadsheets. These approaches used to solve fluid flow problems require that the student learn and apply tedious and time-consuming solution algorithms. As a result, both the teacher and the student are presented with auxiliary tasks that, many times, interrupt and hamper the teaching and learning process. Specifically, these auxiliary tasks include solution algorithms that use: iterative procedures; tedious numerical solutions; diagrams, charts, and nomographs; and other indirect and implicit solution procedures. The final result is that there is not much time left to focus on the modeling of the problem, formulation of the solution, interpretation of the results, changing the assumptions, and going back to modeling of the newly formulated problem, etc., and thus be able to conduct a sensitivity analysis (or an experimental procedure) in order to find either the optimum solution or various solutions corresponding to various assumptions.

The use of the mathematical software Mathcad to teach fluid mechanics has proven to eliminate the drudgery and significantly enhance the solution of fluid flow problems for both closed conduit and open channel problems. Mathcad is used not only for the actual modeling of the problem and formulating of the solution, but also to derive the fundamental equations that apply to a given fluid flow problem. Applying the fundamental principles that describe the fluid flow, the Mathcad environment facilitates mathematical derivations of the appropriate equations through the symbolic integration and differentiation capabilities and arithmetical calculations. Mathcad is then used to apply the derived equations, to model the problem, using either analytical or numerical solve blocks or set up a differential equation solver. Finally, Mathcad allows a straightforward formulation and presentation of the solution for interpretation, and provides a high degree of ease in the possibility of modeling numerous related flow situations.

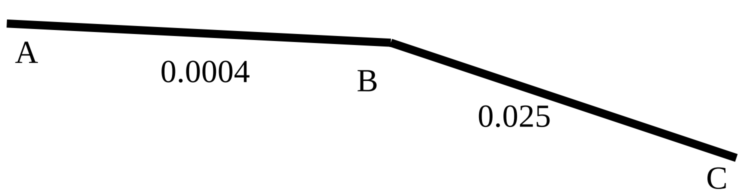
The Role of Mathcad in Teaching the Undergraduate Fluid Mechanics Courses

Mathcad has been extensively used to teach all of the topics covered in both the introductory and intermediate undergraduate fluid mechanics courses, which include fluid properties, fluid statics, fluid kinematics, and fluid dynamics. While the undergraduate topics include both spatially varied and constant flow, in both closed conduit and open channel flow, the undergraduate curriculum assumes steady state flow. Unsteady flow problems are addressed in a graduate course in open channel flow. Students are given Mathcad “worksheets” for lecture notes in addition to receiving detailed chalkboard instruction. Illustrated in the worksheets, Mathcad is used to derive the appropriate equations starting with the laws of the conservation of mass and the conservation of momentum. Numerous examples for each topic are also given in the worksheet. These examples illustrate how to use the appropriate derived equations to model a specific problem using either analytical or numerical “solve blocks” or set up a differential equation solver. Furthermore, the examples clearly show the Mathcad formulation and presentation of the solution for the unknown variables. The examples show how easy it is to make changes in the assumed values for the one or more of the known variables for which Mathcad presents a new solution. Students are assigned projects and homework problems similar to those done in class. Because of the significant amount of time and effort saved in using Mathcad as a teaching and learning tool, we are able to model a larger spectrum and more complex representations of the various fluid flow problems than previously permissible using the traditional techniques.

Illustrative Example of Using Mathcad to Teach Fluid Mechanics

There are a large variety of fluid flow problems from which we can choose an example in order to demonstrate the power of using Mathcad over traditional techniques to solve the problem. Assuming steady state flow, the two general categories are closed conduit flow and open channel flow. For each category we can further assume spatially varied or constant flow. Because there have been several authors [5,6,7] who have already done an excellent job of illustrating the use of Mathcad for spatially varied closed conduit flow problems, we have chosen to illustrate the use of Mathcad for a spatially varied open channel flow problem.

Illustrative Example[8]. Water flows in a rectangular channel that is 5.0 ft wide at a discharge of 16.5 cfs. **a)** Find the surface-water profile through the channel if the channel bottom slope changes from 0.0004 between points A and B (channel section 1) to 0.025 between points B and C (channel section 2) as shown below. Assume Manning’s roughness coefficient n of 0.013. **b)** Find the specific energy diagram.



Before presenting the solution to this example, it is worthy to highlight the significant difference between the two solution approaches, namely the traditional approach typically used for this spatially varied open channel flow problem, versus the Mathcad approach used herein for this problem.

a) Given that the water-surface profile is sought, the unknowns are: the normal depths of flow both upstream of A and downstream of C, the critical depth of flow at B and all the spatially varied flow depths in between points A and B and points B and C.

b) Given that the specific energy diagram is sought, the unknown is: the spatially varied specific energy.

<u>Governing equations [9,10] used to model part a) of the problem:</u>

Continuity Equation:

$$Q = v \cdot A$$

Specific Energy Equation:

$$E(y) = y + \frac{q^2}{2 \cdot g \cdot y^2}$$

from which the critical depth of flow equation is derived for a minimum specific energy:

$$y_c = \sqrt[3]{\frac{q^2}{g}}$$

Momentum Equation:

$$S_f = S_o - \frac{d}{dx}(y) - \frac{v}{g} \cdot \frac{d}{dx}(v)$$

from which the Manning's equation is derived to find normal depths of flow:

$$Q = \frac{1.486}{n} \cdot R(y)^{\frac{2}{3}} \cdot S_f^{\frac{1}{2}} \cdot A(y)$$

and from which the dynamic equation of gradually spatially varied flow(i.e., the resistance equation) is derived (<u>this form is used for the traditional solution approach</u>):

$$\frac{d}{dx}E(y) = S_o - S_f$$

or the form (<u>this form is use for the Mathcad solution approach</u>):

$$\frac{d}{dx}y(x) = \frac{S_o - S_f(y)}{\left(1 - Fr(y)^2\right)}$$

<u>Governing equations used to model part b) of the problem:</u>

Continuity Equation: (see above)

Specific Energy Equation: (see above)

The first difference between the two solution approaches is in the choice of the form of the resistance equation used to model the gradually spatially varied flow problem. The choice made for the typical traditional approach uses the step method, where distance is calculated from depth. The resistance equation is applied in the finite-difference form:

$$\frac{\Delta E}{\Delta x} = \frac{\Delta\left(y + \frac{v^2}{2 \cdot g}\right)}{\Delta x} = S_o - S_f = S_o - \frac{v^2}{C^2 \cdot R}$$

While this modeling choice makes the formulation of the solution very cumbersome, the modeling choice made for the Mathcad approach uses numerical integration in the solution formulation. Use is made of a "differential equation solver" in which rkfixed is called upon; this uses the fourth order Runge-Kutta method to solve the first order differential equation describing the initial-value problem.

A second difference between the two approaches is the solution of the Manning's equation for the normal depths of flow. While the traditional approach uses a systematic trial and error procedure to solve for y, the Mathcad approach uses the "solve block" for its solution formulation. Although the example illustrated herein assumes a rectangular channel cross-section, the power and ease of using Mathcad really soars when a nonrectangular cross section such as a trapezoid is assumed. The traditional solution algorithm assuming a trapezoidal section becomes very tedious and time consuming, whereas using the Mathcad solve block, the solution algorithm is just as easy as for the simpler rectangular section.

A third difference between the two approaches is the solution of the specific energy equation for two alternate depths of flow. Although the example illustrated herein does not ask to find such depths, it is worthy to highlight that this equation is cubic in y in which only the two positive roots are significant. Simply knowing which of the two positive roots you are seeking, (subcritical or supercritical) guides you to provide the appropriate guess value for y for the Mathcad solve block. While the traditional method of solving this equation requires using (programming) a numerical method such as Newton's method of tangents; this solution algorithm has proven to be tedious and time consuming.

A fourth difference is Mathcad's ability to instantly solve for and plot numerous colored graphs, in this case for part **a)** the channel bottom and the water-surface profile, and for part **b)** the specific energy diagram.

<u>Mathcad Solution:</u>

$Q := 16.5 \quad b := 5.0 \qquad q := \frac{Q}{b} \qquad q = 3.3 \qquad S_{o1} := 0.0004 \qquad S_{o2} := 0.025$

$n := 0.013 \qquad g := 32.2$

a) Because the calculation must start at the control (at point B) and proceed in the direction in which the control is being exercised; that is between points B and A, and between points B and C; first we must compute the critical depth of flow at point B. Because the slope between A and B appears to be mild, we expect subcritical flow upstream of point B, and because the slope between B and C appears to be steep, we expect supercritical flow downstream of point B. We will compute the normal depth of flow (i.e., uniform flow) for sections upstream of point B and downstream of point B and confirm this.

$y_c := \sqrt[3]{\frac{q^2}{g}}$ $\qquad y_c = 0.697$

We can use Manning's equation to solve for the uniform depth of flow upstream of point B.

$S_{f1} := S_{o1}$ $\qquad P(y) := 2\cdot y + b$ $\qquad A(y) := b\cdot y$ $\qquad R(y) := \frac{A(y)}{P(y)}$ $\qquad y := 0.697$

Given

$$Q = \frac{1.486}{n}\cdot R(y)^{\frac{2}{3}}\cdot S_{f1}^{\frac{1}{2}}\cdot A(y)$$

$y := \mathrm{Find}(y)$ $\qquad y = 1.505$ $\qquad y_{o1} := y$ $\qquad y_{o1} = 1.505$

Since y is greater than the critical depth, flow *is* subcritical upstream of point B. Once again, we can use the Manning's equation to solve for the uniform depth of

$S_{f2} := S_{o2}$ $\qquad P(y) := 2\cdot y + b$ $\qquad A(y) := b\cdot y$ $\qquad R(y) := \frac{A(y)}{P(y)}$ $\qquad y := 0.697$

Given

$$Q = \frac{1.486}{n}\cdot R(y)^{\frac{2}{3}}\cdot S_{f2}^{\frac{1}{2}}\cdot A(y)$$

$y := \mathrm{Find}(y)$ $\qquad y = 0.382$ $\qquad y_{o2} := y$ $\qquad y_{o2} = 0.382$

flow downstream of point B.

Since y is less than the critical depth, flow *is* supercritical downstream of point B. Starting with the calculation at the control (point B) where the flow is critical, we can compute the surface-water profile for section AB using the resistance equation. The profile begins at point B where flow is critical to point A where the depth of flow is normal.

$y_c = 0.697$ $\quad y_{o1} = 1.505$ $\quad S_{o1} = 4\cdot 10^{-4}$ $\quad P(y) := 2\cdot y + b$ $\quad A(y) := b\cdot y$

$$R(y) := \frac{A(y)}{P(y)} \qquad C(y) := \frac{1.486\, R(y)^{\frac{1}{6}}}{n} \qquad v(y) := \frac{Q}{A(y)} \qquad S_{f1}(y) := \frac{v(y)^2}{C(y)^2\cdot R(y)}$$

$$Fr1(y) := \frac{v(y)}{\sqrt{g\cdot y_c}} \qquad \frac{d}{dx}y(x) = \frac{S_{o1} - S_{f1}(y)}{\left(1 - Fr1(y)^2\right)}$$

Theoretically

$y(x0) = y0 = y_c$

but because when y is critical, the resistance equation indicates that dy/dx equals infinity; there is a discontinuity in the surface- water profile. Therefore, we must begin calculation at a point just upstream of x0=0, say at x0= -20 ft and thus set y0 at y_c + 0.1 for instance. Enter the initial value problem specifics:

$$f(x,y) := \frac{S_{o1} - S_{f1}(y)}{\left(1 - Fr1(y)^2\right)} \qquad x0 := -20 \qquad y0 := y_c + 0.1$$

Enter the desired solution parameters:

$x1 := -4000 \qquad N1 := 100$

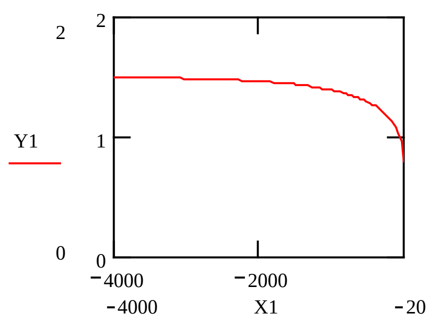
Define solver parameters:

$ic_0 := y0 \qquad D(x, Y) := f\left(x, Y_0\right)$

Solution matrix:

$X1Y1 := \mathrm{rkfixed}(ic, x0, x1, N1, D) \qquad X1 := X1Y1^{\langle 0\rangle} \qquad Y1 := X1Y1^{\langle 1\rangle}$

Plot of the surface-water profile for channel section 1 relative to the elevation datum defined through point B:



Next, starting the calculation at the control (point B) where the flow is critical, we can compute the surface-water profile for section BC using the resistance equation. The profile begins at point B where the flow is critical to point C where the depth of flow is normal.

$y_c = 0.697$ $y_{o2} = 0.382$ $S_{o2} = 0.025$ $P(y) := 2 \cdot y + b$ $A(y) := b \cdot y$

$R(y) := \frac{A(y)}{P(y)}$ $C(y) := \frac{1.486 \cdot R(y)^{\frac{1}{6}}}{n}$ $v(y) := \frac{Q}{A(y)}$ $S_{f2}(y) := \frac{v(y)^2}{C(y)^2 \cdot R(y)}$

$Fr2(y) := \frac{v(y)}{\sqrt{g \cdot y_c}}$ $\frac{d}{dx} y(x) = \frac{S_{o2} - S_{f2}(y)}{\left(1 - Fr2(y)^2\right)}$

Theoretically

$y(x0) = y0 = y_c$

but because when y is critical, the resistance equation indicates that dy/dx equals infinity; there is a discontinuity in the surface- water profile. Therefore we must begin calculation at a point just downstream of x0, say at x0=0.7 ft and thus set y0 at y_c-0.1, for instance. Enter the initial value problem specifics:

$f(x, y) := \frac{S_{o2} - S_{f2}(y)}{\left(1 - Fr2(y)^2\right)}$ $x0 := 0.7$ $y0 := y_c - 0.1$

Enter the desired solution parameters:

$x2 := 70$ $N2 := 100$

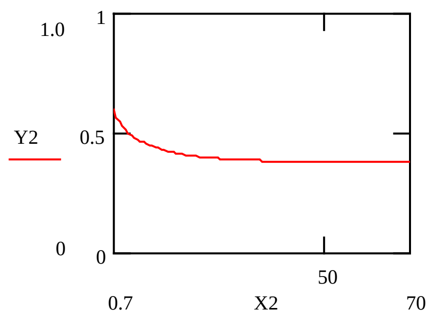
Define solver parameters:

$ic_0 := y0$ $D(x, Y) := f\left(x, Y_0\right)$

Solution matrix:

$X2Y2 := rkfixed(ic, x0, x2, N2, D)$ $X2 := X2Y2^{\langle 0 \rangle}$ $Y2 := X2Y2^{\langle 1 \rangle}$

Plot of the surface-water profile for channel section 2 relative to the elevation datum defined through point B.



Finally we will plot the entire surface-water profile for both channel sections 1 and 2 between points A and C relative to elevations Z1 and Z2 of the respective channel bottom slopes. This is called the hydraulic grade line HGL. First we must reverse the order of the values stored in X1Y1 for channel section 1 so that we can plot the HGL starting from point A to point C.

$X1Y1 := \text{reverse}(X1Y1) \quad X1 := X1Y1^{\langle 0 \rangle} \quad Y1 := X1Y1^{\langle 1 \rangle}$

Next, we can define the elevations Z1 and Z2 for each of the channel bottom slopes, and the elevation ZC defined at point B, the origin (0,0).

$Z1 := -S_{o1} \cdot X1 \quad Z2 := -S_{o2} \cdot X2 \quad ZC := 0$

$X1Z1 := \text{augment}(X1, Z1) \quad X2Z2 := \text{augment}(X2, Z2) \quad XCZC := (0 \ \ 0)$

$XZ := \text{stack}(X1Z1, XCZC) \quad XZ := \text{stack}(XZ, X2Z2) \quad X := XZ^{\langle 0 \rangle} \quad Z := XZ^{\langle 1 \rangle}$

Finally we can define the HGL for the two channel sections, and at point B where flow depth is critical and the elevation of the channel bottom equals zero.

$HGL1 := Y1 + Z1 \quad HGL2 := Y2 + Z2 \quad HGLC := y_c$

$X1HGL1 := \text{augment}(X1, HGL1)$

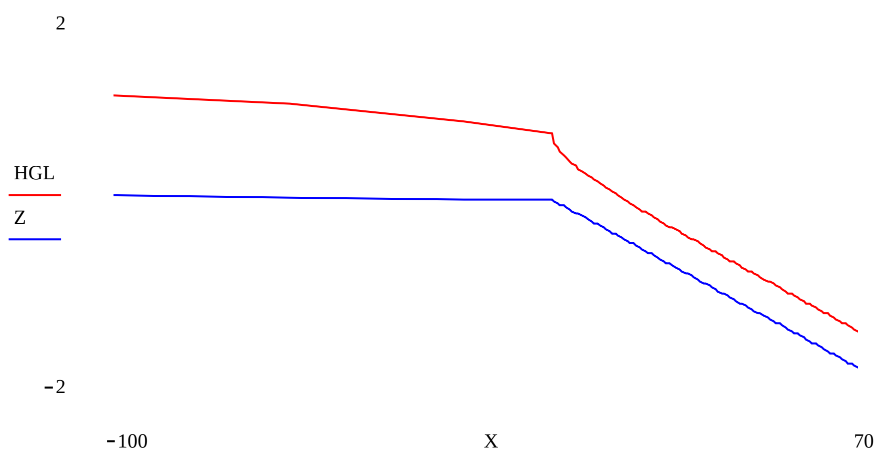
$X2HGL2 := \text{augment}(X2, HGL2)$

$XCHGLC := (0 \ \ HGLC)$

$XHGL := \text{stack}(X1HGL1, XCHGLC)$

$XHGL := \text{stack}(XHGL, X2HGL2) \quad X := XHGL^{\langle 0 \rangle} \quad HGL := XHGL^{\langle 1 \rangle}$

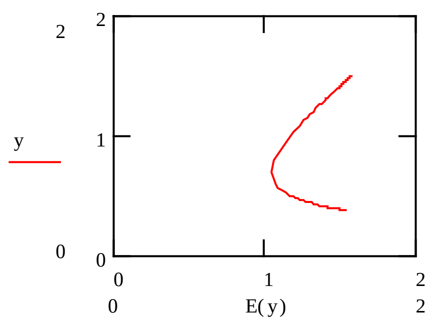
Below is a plot of the entire surface-water profile for the entire channel. Because the upstream slope is mild, the flow upstream of point A and between points A and B is subcritical and the surface-water profile is a type M2 drawdown curve. Because the downstream slope is steep, the flow downstream of point C and between points B and C is supercritical and surface-water profile is a type S2 drawdown curve. At point B where the flow changes from subcritcal to supercritical, the depth must pass through the critical depth. The break in slope at point B is known as a control section.



b) Below is a plot of the specific energy (E-y plot) for a unit discharge q of 3.3 cfs/ft

$XCYC := \begin{bmatrix} 0 & y_c \end{bmatrix}$ $\qquad xy := \text{stack}(X1Y1, XCYC)$

$xy := \text{stack}(xy, X2Y2) \qquad x := xy^{\langle 0 \rangle} \qquad y := xy^{\langle 1 \rangle} \qquad E(y) := y + \dfrac{q^2}{2 \cdot g \cdot y^2}$



Note that the E-y plot for the subcritical flow is displayed on the upper limb, while for the supercritical flow it is displayed on the lower limb. As for the minimum E occurring at critical flow at point B, it is displayed by the crest of the E-y plot.

Base units: Length=ft Mass=lb Time=sec

Conclusions

Using Mathcad to teach fluid mechanics to our undergraduate students allows them to tackle numerous more possibilities for the known variables and to achieve instant results. For instance, the students may, just by reversing the slope values for the two channel sections in the illustrative example above, model the hydraulic jump phenomena. Or by eliminating the second channel section, they can model the free overfall situation. Not to mention using Mathcad's ability to easily manipulate matrices, instantly plot multiple colored graphs, and write reports. Furthermore, the students are exposed to a larger spectrum and more complex problems than was possible with traditional techniques. Finally, the students can concentrate on modeling the problem and formulating the solution rather than laboring over tedious time-consuming solution algorithms.

Acknowledgements

We wish to acknowledge the encouragement and support received from the ECSEL Coalition's Mechanics Learning by Design Group and the Director of the ECSEL Program at Howard University, through funding from the National Science Foundation. The enthusiasm of our students is greatly appreciated.

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