AC 2007-3037: TEACHING FLUX IN THE AGE OF DESKTOP MONTE CARLO

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1. Introduction

Asked to define the scalar flux, too many students reach for an explanation involving surfaces and areas. Something along the lines of "scalar flux is the rate at which particles cross the surface into a sphere ..." This inadequate response is not surprising. Students are often introduced to flux in various special monodirectional cases in which the rate at which particles cross a surface can be evaluated using even scalar flux (along with the implicit knowledge of the directional dependance of the field). That the units of flux involve something per unit area further compounds students' belief that flux has something to do with particles crossing a surface. The misappropriation in radiation transport theory of the word "flux" for what is truly a volumetric concept does not help. My assertion is that no understanding of scalar flux in terms of surface crossing is adequate for our students. They must come to understand scalar flux in terms of path-length rate density.

2. Competing interpretations of scalar flux

Scalar flux, or fluence rate, is a central concept for nuclear engineering analysis. It is defined in several related ways, one of which is under the control of an international standards body. The International Commission on Radiation Units defines[1] fluence as

$$\Phi = \frac{dN}{da} = \frac{\text{number of particles incident on a sphere}}{\text{cross sectional area of sphere}} \tag{1}$$

with the (unstated) understanding that the cross sectional area da is very small compared to macroscopic spatial variations in the particle population and that the period of observation is finite. The fluence rate, called scalar flux in most nuclear engineering literature, is then simply the rate of change of the fluence. This definition centers on the rate at which particles enter a sphere, and it is often asserted that the definition is founded on some idealized measurement with a spherical detector. This is peculiar, because detectors do not count particles that enter them; all detectors measure reaction rates within a detector volume.

The definition of scalar flux as the rate at which particles enter a sphere, divided by the cross sectional area of the sphere, is useless as a concept for computation of the flux, and it makes no obvious connection to the reaction rate density. In nuclear reactor analysis, in shielding analysis, indeed in all applications of radiation, the scalar flux is required as a means to describe the particle population in a manner suited to the computation of reaction rate densities. But reaction rate density is a volumetric concept, while the ICRU definition of flux emphasizes surfaces (of a sphere) and areas (cross section of a sphere). In the development of radiation transport theory, and its approximate cousins like diffusion theory, the flux is introduced in yet another way. Most careful developments move from the particle number density—a concept easily understood—to the flux as particle speed times the number density. This approach is followed because this product naturally arises in computing reaction rate density.

Today it is possible to have students explore neutron and photon particle transport using relatively realistic and relable Monte Carlo codes, such as MCNP5 and TART, that run on their laptop. The most robust method for computing a flux in Monte Carlo is the track-length estimator, in which the average track-length per source particle generated within a fixed volume is estimated. This track-length estimate divided by the volume of the cell is then an estimate of the volume averaged fluence per source particle. Increasingly then, our students are actually computing flux using the definition that is most neglected in textbooks, and the definition that they understand the least.

As educators we are then left with the task of convincing our students that three different definitions of the scalar flux:

1. rate of particles entering a sphere over the cross sectional area of the sphere, in the limit of a the sphere radius going to zero (the ICRU definition);
2. the particle number density times speed;
3. the path-length rate density;

all represent the same quantity. Proof of this equivalence is not so trivial, and brings up subtle questions. Recent IRCU reports assert[1] the equivalence of all three definitions, but offer neither proof nor argument. Many textbooks offer arguments in special cases, typically for vacuum filled spheres or along cylindrical tubes in monodirectional radiation fields.[2] Chilton[3] seems to have been the first to publish a careful argument for the equivalence of the path-length definition and the ICRU definition for general non-convex bodies, although he required further refinement of his argument to address scatter in finite volumes.[4] Rigorous proof that the ICRU's definition of flux is equivalent to the path-length definition is still not completely settled, as testified to by the presence of even recent papers attempting the proof.[5]

From an educational standpoint, there is little compelling reason to establish that either number density times speed or path-length rate density, is equivalent to the ICRU definition, because we neither compute or measure using the ICRU definition. But we cannot ignore the definition provided by the ICRU, because our students need to be aware of the ICRU and its terminology. But more importantly, students often have confusion over the basic concept of scalar flux (fluence rate), and they easily confuse current and flux. The ICRU definition is based on particles entering a sphere—and hence on particles crossing a special surface—and thus requires current to analyze. This only adds to student confusion.

In this paper I will outline approaches to teaching the concept of scalar flux, and provide demonstrations that I use to show the equivalence of the three definitions. The approach includes special cases, and also very general demonstrations that exercise students' understanding of what particles do (not-to-mention their calculus). The approach emphasizes, in the end, the path-length concept. This understanding is most important when combined with an understanding of macroscopic cross-sections as probability per unit path-length, in order to compute reaction rates, and also because it is the definition most used in computing flux in Monte Carlo simulations.

3. Outline of a pedagogical approach

I will describe an approach to teaching flux that relies on building up from the simplest concepts (number density) to the most complex (flux as a path-length density). The approach is pedagogical[6] because the students are entering a new content area and do not, at first, appreciate the relavence of understanding the path-length definition of flux. The students are therefore lead through a set of ideas and exercises that explore the concepts, and the teacher directs the action. There are however several obvious occasions in this program where active learning can be used, and I have generally observed that in-class activities can help students better grasp the mathematics of the path-length rate density concept.[7]

The instruction starts from the concept of number density. This is not a new starting point; many texts in nuclear engineering begin with particle number density and introduce flux from it. For example, Duderstadt & Hamilton[8] proceed this way, but in contrast Shultis & Faw[9] do not. The difference between textbook approaches and that which I present here is in my assertion that significant time and focus should be spent on helping students understand the path-length concept. This idea is seldom emphasized in textbooks, and is generally presented only in special cases. Indeed, many texts (and the ICRU) assert the connection between flux and path-length in a sentence,[9,10] and I have found that this leaves thoughtful students scratching their heads, and less thoughtful students unaware that they cannot defend the idea against hard questioning.

I will use the terms scalar flux and fluence rate essentially interchangeably, although I will generally reserve "fluence rate" for speaking of the ICRU interpretation. Fluence rate is arguably the better term, since "flux" is widely used in many fields to denote quantities related to flow across surfaces. But flux is too well established in nuclear engineering practice to discard. In discussing the subtleties of scalar flux the energy dependance of the scalar flux does not introduce any conceptual difficulties, so it will be ignored throughout this paper. The symbols in Table 1 will be used consistently, and I will consider only steady state radiation fields. Finally, I assume that students have been introduced to the notion of solid angle, including differential solid angle as a patch of area $d^2\Omega = \sin(\theta)d\varphi d\theta$ on a unit sphere using spherical polar coordinates θ, φ.

Table 1: Phase space point and region symbols

$\hat{\Omega}$:	A unit direction vector	Θ:	A range of directions
$\mathbf{r}$:	A point in space	Γ:	A finite region of space of arbitrary shape
v:	Particle speed	T:	A long time period of observation

In outline, our approach to teaching the flux concept proceeds as follows, from simplest concept to most subtle:

1. Particle number density, $n(\mathbf{r},\hat{\Omega})$, as a means to denote the number of particles in a volume and solid angle of interest;
2. angular current, $\mathbf{j}(\mathbf{r},\hat{\Omega}) = \hat{\Omega}vn(\mathbf{r},\hat{\Omega})$ as a means to compute the rate of flow across surfaces;

3. angular flux $\psi(\mathbf{r},\hat{\Omega}) = \|\mathbf{j}(\mathbf{r},\hat{\Omega})\| = vn(\mathbf{r},\hat{\Omega})$ simply as the magnitude of current, and scalar flux as its integral $\phi(\mathbf{r}) = \int_{4\pi} \psi(\mathbf{r},\hat{\Omega})\, d\Omega$;

4. a side trip to ICRU fluence rate, interpreting the scalar flux as the quantity defined by the ICRU;

5. angular flux as a path-length rate density for particles traveling a differential solid angle;

6. and finally, scalar flux as a path-length rate density for particles traveling in any direction.

In the first three steps of this process I emphasize that number density, angular current, and angular flux all contain exactly the same information, and are trivially knowable each from the other. I emphasize to students that this equivalence exists *because* the angular dependance of the radiation field in maintained. It is not until this angular information is integrated away that the trivial connections between these quantities are broken. Most importantly, while net current $\mathbf{J} = \int_{4\pi} \mathbf{j}\, d\Omega$ still provides information about the rate at which particles cross arbitrary surfaces, scalar flux ϕ does not.*

The notion of number density $n(\mathbf{r},\hat{\Omega})$ is easy for students to grasp, as it is very close to the notion of mass density with which they are already very familiar. Both the differential

$$n(\mathbf{r},\hat{\Omega})\, d^3r\, d^2\Omega = \text{number of particles in } d^3r \text{ traveling in directions within } d^2\Omega \text{ about } \hat{\Omega} \tag{2}$$

and the integral

$$\int_\Gamma \int_\Theta n(\mathbf{r},\hat{\Omega})\, d^3r\, d^2\Omega = \text{number of particles in } \Gamma \text{ traveling in directions within } \Theta \tag{3}$$

definitions of number density are easy for students to understand, and should be presented first. They all understand the notion of counting, so the only *concept* here is the notion of a density function. In terms of number density we always *define* angular current rate density as

$$\mathbf{j}(\mathbf{r},\hat{\Omega}) = \hat{\Omega} vn(\mathbf{r},\hat{\Omega}) = \hat{\Omega}\psi(\mathbf{r},\hat{\Omega})\,, \tag{4}$$

the angular flux as

$$\psi(\mathbf{r},\hat{\Omega}) = vn(\mathbf{r},\hat{\Omega}) = \|\mathbf{j}(\mathbf{r},\hat{\Omega})\|\,, \tag{5}$$

and the scalar flux as

$$\phi(\mathbf{r}) = \int_{4\pi} \psi(\mathbf{r},\hat{\Omega}) d\Omega\,. \tag{6}$$

We then "simply" need to develop meaningful physical interpretations of these quantities. This we do by building up from the easily understood number density concept, to the next simplest, angular current. Once angular current is understood it is relatively easy to show that for a smoothly varying particle population ϕ corresponds to the ICRU definition of fluence rate.

*This already provides an interesting MCNP exercise: have students compute the net current and the surface averaged scalar flux on the surface of a water sphere with a gamma source at the center, and examine the difference between them.

4. The program in detail

After introducing number density and defining angular current and angular flux in terms of it, students are presented with the question:

> Suppose we know the angular current $\mathbf{j}$ (or equivalently the number density or the angular flux). We have a macroscopic surface and break it into differential patches of area dA each with normal $\hat{n}$. What is the rate at which particles traveling in solid angle $d^2\Omega$ about $\hat{\Omega}$ cross dA? What is the rate at which the particles cross the macroscopic surface?

The students can be presented this question as a homework, or in class for an active learning activity. Or they can simply be presented with the development of the answer in a standard lecture format. To either develop or understand the result students must *apply* their understanding of number density, since the result is based on multiplying the volume of the differential box $dA \times \hat{n} \cdot \hat{\Omega} vT$ times the number density, and considering the limit as the period of observation $T \to 0$. The result, that $\mathbf{j} \cdot \hat{\Omega}\, dA\, d^2\Omega$ gives the rate at which particles cross dA within solid angle $d^2\Omega$ provides students with the standard interpretation of angular current. The interpretation of the net current $\mathbf{J} = \int \mathbf{j}\, d^2\Omega$ is then an easy exercise, as is the interpretation of the MCNP F1 tally $\int\int |\hat{n} \cdot \mathbf{j}|\, d^2\Omega\, d^2r$.

The angular flux $\psi(\mathbf{r}, \hat{\Omega})$ can be presented as the magnitude of the angular current, and in this regard it can be interpreted as rate of flow across a surface that is *perpendicular to* $\hat{\Omega}$. But there is *no value* in *emphasizing* this. It confuses flux with flow across a surface, a connection that has little utility. The scalar flux $\phi = \int_{4\pi} \psi\, d^2\Omega$ is introduced as a definition, as is the net current $\mathbf{J}$, and the point that neither can be generally computed from the other is made. In other words, once direction information is integrated away, the connection between current-like and flux-like quantities is broken.

The students are now presented with the question:

> In terms of the angular flux $\psi(\mathbf{r}, \hat{\Omega})$ (or equivelently number density or angular current density), what is the rate at which particles enter a sphere of radius R centered at $\mathbf{r}$? What is the limit of this rate of particle entry over the cross sectional area of the sphere, in the limit as $R \to 0$?

This question is answered in detail in Appendix A; having students answer it strengthens their understanding of angular current. The answer to the second part is germane here,

$$\lim_{R \to 0} \frac{\text{rate of particle entry into sphere of radius } R}{\pi R^2} = \int_{4\pi} \psi(\mathbf{r}, \hat{\Omega})\, d^2\Omega = \phi(\mathbf{r})\,.$$

This is the confirmation that ϕ is the same as the ICRU fluence rate.

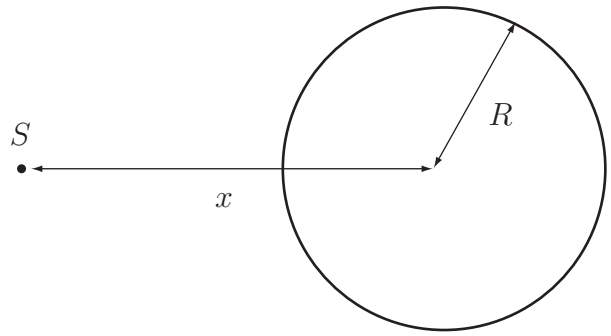

Figure 1: A point source S with a sphere centered a distance $x > R$ away. Compute the volume averaged scalar flux over the sphere.

Students must also be asked not to over interpret this result: for a finite sphere the rate at which particles enter the sphere divided by its cross sectional area is *not* the *average scalar flux* over that sphere. Scalar flux appears only in the limit as the sphere shrinks to zero.

Finally, and most importantly, we need students to explore the flux as a path-length density. An early reference to the path-length concept is in Weinberg & Wigner,[10] in which the argument is that $vT \times n(\mathbf{r},\hat{\Omega})\, d^3r\, d^2\Omega = T\psi(\mathbf{r},\hat{\Omega})\, d^3r\, d^2\Omega$ represents the number of particles in d^3r times the distance they travel in time T. Therefore, it is argued, ψ is the path-length rate density. A question that some students will develop for themselves, and which all should consider, is this: since the particles move during time T, particles generate path-length that *is not within the differential volume* d^3r. This concern is difficult to address when dealing with a differential volume d^3r, because it concerns the order in which two limits are taken ($T \to 0$ vs. the volume going to zero).

The simplest presentations using finite volumes to illustrate the path-length interpretation of flux are based on straight particle tracks,[2,5] which effectively means in vacuum or in a volume so small that interactions might be neglected.[3] Others make appeal to Cauchy's formula (also called Dirac's formula in the reactor physics literature) for the mean chord length of a convex body, but this is not a useful teaching tool unless Cauchy's formula is derived for the students, and even then, it applies only to the very special case of an isotropic flux in the absence of interactions and for a convex body.[10–12]

An example to which Cauchy's formula does not apply, and an example which students can easily undertake as either a homework on an in-class exercise is this:*

> You have an isotropic point source in vacuum, and at a distance x away from it you place a sphere of radius $R < x$. Compute the volume average scalar flux over the sphere. Compute the rate at which particles generate path-length within this sphere, and divide this total distance by the volume of the sphere. (See Fig. 1)

*An easier example is to compute the volume averaged flux over a sphere with a point source at the center, which is again easily related to path-length.

Doing these two calculations involves simply setting up some integrals, and knowing that ϕ goes as $1/(4\pi r^2)$. The integrals that result are relatively elementary, but the integrals in the two approaches (volume average of ϕ vs. path-length computation) do not at first appear related. They are presented in detail in Appendix B. In doing these two integrals students will discover that they yield the same result: the volume average scalar flux over the sphere is equal to the rate at which path-length is generated *only within the sphere* divided by the volume of the sphere. Students should be reminded here that the angular flux across the sphere in this problem is neither isotropic, nor mono-directional, and indeed it's angular variation changes as x is changed (becoming increasingly mono-directional as $x \to \infty$). So the problem is not particularly specialized in the angular flux being analyzed. But it is special in regards the shape of the volume (a sphere) and its content (vacuum).*

This illustration of the relation between volume average scalar flux and rate of path-length generation is more interesting and more challenging to students than similar constructs based on mono-directional flux. But it remains to argue to students that this connection is always true that the volume averaged scalar flux is equal to the volume averaged rate at which path-length is generated by particles,

$$\int_\Gamma \phi(\mathbf{r}) d^3 r = \text{rate at which path-length is generated in } \Gamma. \tag{7}$$

A bright student will observe that the rate at which particles generate path-length is the particle speed, but this is in fact the rate at which *a single* particle generates path-length. The rate at which many particles generate path-length must involve the number of particles, and hence involve number density. But given a region Γ, if we count the particles in it using number density, and multiply by v, we will be counting path-length generated by particles after they leave the region, and not counting path-length generated by particles that enter the region through its surface. Do these two effects *really* compensate each other? Additionally, are we successfully accounting for the path-length generated within the region due to particles born from a volumetric source?

We now challenge the students with this problem:

> Imagine an arbitrary 3-D body Γ, and select a direction $\hat{\Omega}$ and differential solid angle $d^2\Omega$ around it. The flux $\psi(\mathbf{r},\hat{\Omega})$ is known, as is the volumetric source $Q(\mathbf{r},\hat{\Omega})$ throughout Γ (including external sources and inscatter sources). Select a differential tube through this body with cross sectional area dA (See Fig. 2) and compute the path-length generated within this tube during a time interval T by particles traveling in $d^2\Omega$ about $\hat{\Omega}$. By summing up contributions from these differential tubes show that the volume integral $\int_\Gamma \psi(\mathbf{r},\hat{\Omega}) d^3 r\, d^2\Omega$ gives the rate at which path-length is generate within Γ by particles traveling within $d^2\Omega$ about $\hat{\Omega}$.

Note that no special cases are allowed here. The tube is filled with material, particles can in-scatter,

*There is also a good pair of MCNP exercises here: MCNP can be used to compute the volume averaged flux via the pathlength integral, and this result compared to the analytical result. MCNP can be used to compute the number of particles that cross the sphere, and the result related to the solid angle subtended by the sphere.

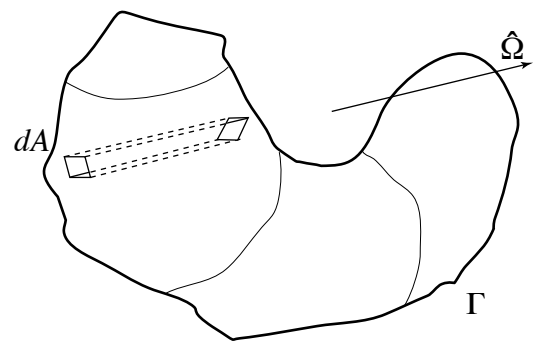

Figure 2: A differential tube of cross-sectional area dA aligned along the direction $\hat{\Omega}$, cutting through an object Γ.

out-scatter and are born within Γ. Appendix C presents a careful development of the answer to the first part of this problem. It parallels elements of both Chilton's[3] and Papiez's developments, but differs from them in explicitly including sources. It relies on building upper and lower bounds for the answer, and considering the limiting case of a Riemann sum. In this sense it reinforces students basic understanding of integral calculus, but no student has ever developed this argument on his own, so this question is usually posed, and then answered, in a lecture format. The answer to the first part of the question is given by the line integral

$$\text{PL from particles traveling in } d\Omega \text{ during } T \text{ along differential tube} = T\int_0^L dl\,\psi(\mathbf{r}_0 + l\hat{\Omega},\hat{\Omega})dAd^2\Omega \tag{8}$$

where the line integral over dl is along the differential tube from entry point $\mathbf{r}_0$ to exit at a distance L down the tube.

This integral can be argued without the level of detail in Appendix C, and I have presented it in class with simple arguments very similar to Chilton.[3] But there are invariably questions about source particles born in dl, or particles that are absorbed in dl and do not make it across the differential length, or in- and out-scatter. The answer to all these questions is that these are second order effects that vanish in the limit of a differential length along the tube. But simply claiming that the effects are second order does not show it to be so, and it is this care that is often lacking in discussions of flux as path-length density. So in class I now show the calculation in a careful way, as presented in the Appendix.

Having dealt with a little differential tube, let's now integrate over all entering area patches, $\hat{n}\cdot\hat{\Omega} < 0$ to get

$$\text{PL generated by particles traveling in } d^2\Omega \text{ during } T = T\int_{\hat{n}\cdot\hat{\Omega}<0} dA \int_0^L dl\,\psi(\mathbf{r}_0 + l\hat{\Omega},\hat{\Omega})\,d^2\Omega$$

but this is a volume integral

$$\text{PL generated by particles traveling in } d\Omega \text{ during } T = T\int_\Gamma d^3r\,\psi(\mathbf{r}_0 + l\hat{\Omega},\hat{\Omega})\,d^2\Omega$$

and so, indeed,

$$\begin{array}{c}\text{rate at which path-length is generated within } \Gamma \\ \text{by particles traveling within } d^2\Omega \text{ about } \hat{\Omega}\end{array} = \int_\Gamma \psi(\mathbf{r},\hat{\Omega}) d^3r\, d^2\Omega \tag{9}$$

Finally, we can ask students to:

Interpret the volume averaged scalar flux in terms of path-length.

Integrating over all directions we discover that the scalar flux integrated over the entire volume is the rate at which path-length is swept out by the radiation while it is inside the region Γ

$$\text{rate of path-length generation within } \Gamma = \int_\Gamma \phi(\mathbf{r}) d^3r$$

or, put another way, the average flux is

$$\bar{\phi} = \frac{1}{V(\Gamma)} \int_\Gamma \phi(\mathbf{r})\, d^3r = \frac{\text{rate of path-length generation within } \Gamma}{V(\Gamma)}$$

This is how an F4 tally in MCNP works. The track-length of particles through a volume is computed, the average track-length per particle is determined, and this is divided by volume to give a volume averaged flux per source particle.

5. Conclusions

Students should understand the scalar flux from three perspectives: as the ICRU definition of the flux concerning an infinitesimally small sphere;[1] as the number density times speed;[10] and as the path-length rate density.[3] This last tends to be neglected, but it is in fact the most important.

It should be emphasized to students that the path-length swept out by particles within a *finite* region gives the volume integral of the flux, and this is true no matter the size or shape or composition of the region. It is a fact about the scalar flux that does not rely on any special angular distribution of particles, or on the absence of interactions. Indeed, it is the way in which Monte Carlo codes compute the flux.

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A. ICRU Flux

The general proof that the quantity defined in Eq. 6 is equivalent to the ICRU definition of flux provides an excellent exercise for students to strengthen their understanding of current. The question to present them is this:

> In terms of the angular flux $\psi(\mathbf{r},\hat{\Omega})$ (or equivelently number density or angular current density), what is the rate at which particles enter a sphere or radius R (located arbitrarily at the origin)? What is the limit of this rate of particle entry over the cross sectional area of the sphere, in the limit as $R \to 0$?

Let's compute first the rate at which particles enter a sphere through a differential patch of area dA with surface outward normal $\hat{n} = \hat{r}$ (refer to Fig. 3)

$$\text{rate of entry via } dA = -\int_{\hat{\Omega}\cdot\hat{n}<0} \mathbf{j}\cdot\hat{n}\, d^2\Omega\, dA \tag{10}$$

$$= -\int_{\hat{\Omega}\cdot\hat{n}<0} \psi(\mathbf{r},\hat{\Omega})\,\hat{\Omega}\cdot\hat{n}\, d^2\Omega\, dA\,. \tag{11}$$

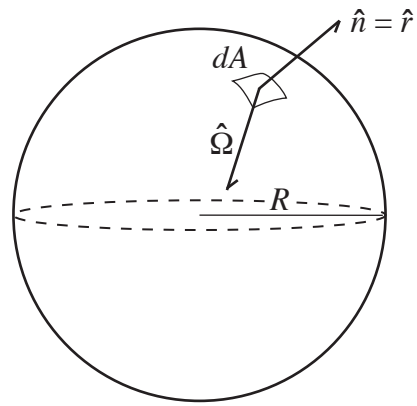

Figure 3: A differential surface area dA with outward normal $\hat{n} = \hat{r}$ on a sphere of radius R. Particles with direction of travel $\hat{\Omega}$ satisfying $\hat{\Omega} \cdot \hat{n} < 0$ can enter the sphere through this patch of surface.

Integrating them over all patches of area dA gives the total rate at which particles enter the sphere as

$$\text{rate of entry into sphere} = -\int_{\text{Sphere}} \int_{\hat{\Omega}\cdot\hat{n}<0} \psi(\mathbf{r},\hat{\Omega})\,\hat{\Omega}\cdot\hat{n}\,d^2\Omega\,dA \tag{12}$$

$$= -\int_{4\pi} d^2\Omega \int_{\hat{\Omega}\cdot\hat{\mathbf{r}}<0} dA\,\psi(\mathbf{r},\hat{\Omega})\,\hat{\Omega}\cdot\hat{r}\,. \tag{13}$$

This is all that can be said in the general case.

To examine now the rate of entry as $R \to 0$ we Taylor expand the angular flux around the center of the sphere, yielding

$$= -\int_{4\pi} d^2\Omega \int_{\hat{\Omega}\cdot\hat{r}<0} dA\left\{\psi(0,\hat{\Omega}) + R\,\hat{r}\cdot\nabla\psi + \ldots\right\}\hat{\Omega}\cdot\hat{r}\,dA\,d\Omega \tag{14}$$

$$= -\int_{4\pi} d^2\Omega\,\psi(0,\hat{\Omega}) \int_{\hat{\Omega}\cdot\hat{r}<0} \left\{\hat{\Omega}\cdot\hat{r}\right\} dA + O(R^3) \tag{15}$$

$$= -\int_{4\pi} d^2\Omega\,\psi(0,\hat{\Omega})\,R^2 2\pi \int_{\pi/2}^{\pi} \left\{\cos(\theta)\right\}\sin(\theta)\,d\theta + O(R^3) \tag{16}$$

$$= -\int_{4\pi} d^2\Omega\,\psi(0,\hat{\Omega})\,R^2 2\pi \int_{-1}^{0} \mu\,d\mu + O(R^3) \tag{17}$$

$$= -\int_{4\pi} d^2\Omega\,\psi(0,\hat{\Omega})\,R^2 2\pi \times \left(-\frac{1}{2}\right) + O(R^3) \tag{18}$$

$$= \pi R^2 \int_{4\pi} d^2\Omega\,\psi(0,\hat{\Omega}) + O(R^3) \tag{19}$$

$$= \pi R^2 \phi(0)\,. \tag{20}$$

So, finally and recognizing that the origin was irrelevant,

$$\phi(\mathbf{r}) = \int_{4\pi} |\mathbf{j}(\mathbf{r},\hat{\Omega})|\,d^2\Omega = \int_{4\pi} \psi(\mathbf{r},\hat{\Omega})\,d^2\Omega = \lim_{R\to 0} \frac{\text{rate of particle entry in to sphere of radius } R}{\pi R^2}\,.$$

This establishes that ϕ as defined in Eq. 6 is equivalent to the to the ICRU 60 definition of fluence rate.[1]

B. Average flux from a point source

A good exercise for students involves having them compute both the volume averaged flux and the rate of path-length generation due to some known flux distribution. For example:

> You have an isotropic point source in vacuum, and at a distance x away from it you place a sphere of radius $R < x$. Compute the volume average scalar flux over the sphere. Compute the rate at which particles generate path-length within this sphere, and divide this total distance by the volume of the sphere. (See Fig. 1)

Note that the scalar flux from the point source is $\phi(r) = S/4\pi r^2$, where r is the distance from the source. Therefore

$$\bar{\phi} = \frac{1}{V}\int_{\text{sphere}} \frac{S}{4\pi r^2}\, d^3r \tag{21}$$

where the integration is over the volume of the sphere with volume $V = 4\pi R^3/3$. Let's set up a spherical coordinate system as shown in Fig. 4. The integral can then be done as

$$\bar{\phi} = \frac{1}{V}2\pi \int_{x-R}^{x+R} r^2 dr \int_0^{\theta(r)} \sin(\theta) d\theta \, \frac{S}{4\pi r^2} \tag{22}$$

$$= \frac{S}{2V}\int_{x-R}^{x+R} \big(1-\cos(\theta(r))\big) dr \tag{23}$$

where $\theta(r)$ is the largest polar angle for any given r value. Writing $\mu(r) = \cos(\theta(r))$ we can use the geometry of the system to write

$$R^2 = (x - r\mu(r))^2 + h^2 \tag{24}$$

$$r^2 = h^2 + r^2\mu^2(r) \tag{25}$$

and hence find

$$\mu(r) = \frac{r^2 + x^2 - R^2}{2rx}\,. \tag{26}$$

Using this in Eq. 23 and performing the integration we have

$$\bar{\phi} = \frac{3S}{8\pi R^3}\left[R + \frac{1}{2x}(x^2 - R^2)\log\left(\frac{x-R}{x+R}\right)\right]. \tag{27}$$

For another approach to this computation, let's compute the total rate at which path-length is generated in the sphere. The rate at which particles from the source enter a differential cone of thickness $d\theta$ is $(S/4\pi) \times 2\pi\sin(\theta)d\theta$. These particles each traverse a path-length of

$$2\sqrt{R^2 - x^2\sin^2(\theta)}\,, \tag{28}$$

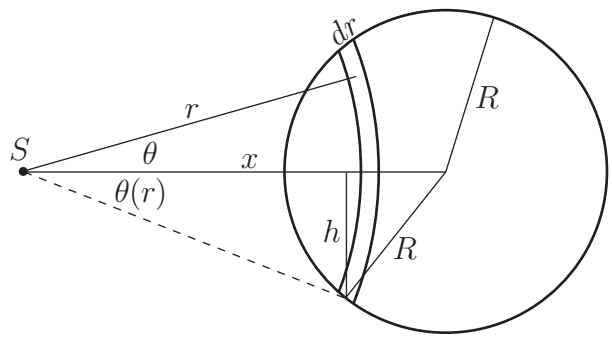

Figure 4: Spherical polar coordinates with origin at the source.

so the rate at which path-length is generated is

$$\dot{\mathrm{PL}} = \int_0^{\theta_{\max}} \frac{S}{4\pi} 2\pi \sin(\theta) 2\sqrt{R^2 - x^2 \sin^2(\theta)} d\theta \tag{29}$$

and the rate of path-length generation divided by the volume $V = 4\pi R^3/3$ of the sphere is

$$\frac{\dot{\mathrm{PL}}}{V} = \frac{3S}{4\pi R^3} \int_0^{\theta_{\max}} \sin(\theta) \sqrt{R^2 - x^2 \sin^2(\theta)}\, d\theta\,. \tag{30}$$

Transforming to the variable $\mu = \cos(\theta)$ we can write this as

$$\frac{\dot{\mathrm{PL}}}{V} = \frac{3S}{4\pi R^3} \int_{\cos(\theta_{\max})}^{1} \sqrt{R^2 - x^2(1-\mu^2)}\, d\mu \tag{31}$$

where $\cos(\theta_{\max}) = \sqrt{1 - R^2/x^2}$. This integral can be evaluated explicitly; the indefinite integral is

$$\int \sqrt{R^2 - x^2(1-\mu^2)}\, d\mu = \frac{x\sqrt{(\mu^2-1)x^2 + R^2}\,\mu + \left(R^2 - x^2\right) \log\left(2\left(x\mu + \sqrt{(\mu^2-1)x^2 + R^2}\right)\right)}{2x} \tag{32}$$

so inserting the limits and using $\cos(\theta_{\max}) = \sqrt{1 - R^2/x^2}$ gives

$$\frac{\dot{\mathrm{PL}}}{V} = \frac{3S}{4\pi R^3} \left[\frac{Rx + \left(R^2 - x^2\right) \log\left(2\left(x + R\right)\right)}{2x} - \frac{\left(R^2 - x^2\right) \log\left(2\sqrt{x^2 - R^2}\right)}{2x} \right] \tag{33}$$

$$= \frac{3S}{8\pi R^3} \left[R + \frac{(R^2 - x^2)}{x} \log\left(\frac{x + R}{\sqrt{x^2 - R^2}} \right) \right] \tag{34}$$

$$= \frac{3S}{8\pi R^3} \left[R + \frac{(x^2 - R^2)}{2x} \log\left(\frac{x - R}{x + R} \right) \right]. \tag{35}$$

This is the same as Eq. 27, computed earlier by averaging the flux over the sphere. Thus, we see that average flux is equivalent to volume averaged path-length rate density.

C. Flux as path-length rate density in a differential tube

It is very important to understand the relationship between path-length density rate and flux. The first question to address is:

> Imagine a differential tube with cross sectional area dA through a body Γ (See Fig. 2) and compute the path-length generated within this tube during a time interval T by particles traveling in $d^2\Omega$ about $\hat{\Omega}$. The flux $\psi(\mathbf{r},\hat{\Omega})$ is known, as is the volumetric source $Q(\mathbf{r},\hat{\Omega})$ throughout the tube (including external sources and inscatter sources).

Let's do this carefully. We want to compute the path length (PL) of particles traveling down the differential tube with cross sectional area dA, oriented parallel to $\hat{\Omega}$. We will focus on particles traveling $d\Omega$ about $\hat{\Omega}$.* We will also focus on a fixed time of observation T. Let the long differential tube be of length L, and divide it into N cells, each of length $\Delta l = L/N$. Then

$$\mathrm{PL} = \sum_i^N \mathrm{PL}_i \tag{36}$$

where PL_i is the path length generated by particles traversing the ith cell. This sum will become the Riemann sum for the line integral in Eq. 8. We will develop an upper bound, and a lower bound, on PL_i, and consider the limit $N \to \infty$ (or equivalently $\Delta l \to 0$).

C.1 Upper bound

The total path length PL generated by particles crossing cell i satisfies

$$\begin{aligned}\mathrm{PL}_i < {} & \text{PL from particles entering cell } i + \text{PL from particles already in cell } i \\ & + \text{PL from particles born in cell } i\end{aligned} \tag{37}$$

where this is an upper bound because some particles that enter are absorbed or outscatter. But every particle that enters travels no more than Δl, so the first term in Eq. 37 is bounded above by $T\psi_i\, dA\, d^2\Omega\, \Delta l$ where ψ_i is the value of ψ at the incoming face of the cell. Similarly, no particle already in the cell travels more than Δl, so the second term in Eq. 37 is bounded above by $(\psi_{im}/v) dA\, \Delta l\, d^2\Omega\, \Delta l$, where ψ_{im} is the largest value of ψ in the cell. Note that this term is not proportional to T. Finally, the path length from particles that are born within the cell is bounded above by $T Q_{im}\, dA\, \Delta l\, d^2\Omega\, \Delta l$ where Q_{im} is the maximum value of $Q(\mathbf{r},\Omega)$ in the cell. Putting these together we have

$$\mathrm{PL}_i < \left(T\psi_i + \frac{\psi_{im}}{v}\Delta l + T Q_{im}\Delta l \right) dA\, d^2\Omega\, \Delta l\,. \tag{38}$$

*This might be objected to, since we are implicitly assuming that certain effects are second order in the directions of travel. These effects can be addressed in formulating the upper and lower bounds developed here, but for the sake of space I will not do so.

C.2 Lower bound

We need a tight lower bound on the path-length, so that we can trap the Riemann sum between the lower and upper bounds. In formulating the lower bound for cell i we can drop the path-length generated by particles born in the cell, and by particles already in the cell. We must explore then the particles that enter the cell during time T. Consider the time interval of observation, $0 \le t \le T$. During the period $0 \le t \le T - \Delta l/v$ all particles that enter the cell have time to contribute a path length of Δl, and the no more than the fraction $(1 - e^{-\Sigma_i \Delta l})$ of these particles will actually do so, where Σ_i is the largest value of the macroscopic cross section in cell i. Therefore,

$$\mathrm{PL}_i > (T - \Delta l/v)(1 - e^{-\Sigma_i \Delta l})\psi_i \, dA \, d^2\Omega \, \Delta l \tag{39}$$

C.3 PL in the differential tube

Using Eqs. 38 and 39 in Eq. 36 we have

$$\sum_{i=1}^{N}\left(T - \frac{\Delta l}{v}\right)(1 - e^{-\Sigma_i \Delta l})\psi_i \, dA \, d^2\Omega \, \Delta l < PL < \sum_{i=1}^{N}\left(T\psi_i + \frac{\psi_{im}}{v}\Delta l + TQ_{im}\Delta l\right) dA \, d^2\Omega \, \Delta l \tag{40}$$

In the limit $N \to \infty$ the first term in the upper bound goes to

$$\sum_{i=1}^{N} T\psi_i \, dA \, d^2\Omega \Delta l \to T\int_0^L \psi(\mathbf{r}_0 + l\Omega, E, \Omega) dl \, dA \, d\Omega \tag{41}$$

where $\mathbf{r}_0$ is the start of the differential tube on the surface of Γ. Further, the second and third terms in the upper bound are

$$\sum_{i=1}^{N}\left(\frac{\psi_{im}}{v}\Delta l + TQ_{im}\Delta l\right)\Delta l \, dA \, d^2\Omega < \left(\frac{\psi_{\max}}{v}N\frac{L}{N} + TQ_{\max}N\frac{L}{N}\right)\frac{L}{N} \, dA \, d^2\Omega \to 0 \tag{42}$$

as long as the flux is globally bounded by $\psi_{\max} < \infty$ and the source is similarly bounded, $Q_{\max} < \infty$.

In the lower bound we have the term

$$\sum_{i=1}^{N} T(1 - e^{-\Sigma_i \Delta l})\psi_i \, dA \, d^2\Omega \, \Delta l \to T\int_0^L \psi(\mathbf{r}_0 + l\Omega, E, \Omega) dl \, dA \, d^2\Omega \tag{43}$$

as $N \to \infty$. The second term in the lower bound contains

$$\sum_{i=1}^{N} \frac{\Delta l}{v}(1 - e^{-\Sigma_i \Delta l})\psi_i \, dA \, d^2\Omega \, \Delta l < \frac{L}{v}\psi_{\max} \, dA \, d^2\Omega \frac{L}{N} \to 0 \tag{44}$$

as $N \to \infty$, and so this term does not contribute in the limit.

So, the upper and lower bounds of Eqs. 38 and 39 when summed over all cells in the long differential tube both limit to the same value, and so the path-length in question is

$$PL = T\int_0^L \psi(\mathbf{r}_0 + l\Omega, E, \Omega) dl \, dA \, d^2\Omega. \tag{45}$$

The key point is that the integral of the angular flux along the length of a tube of differential cross section dA aligned along the direction of particle travel gives the total rate at which path-length is generated in the tube by the particles traveling along, already in, or born within the tube. In seeing this, the volumetric contributions (from particles already in or born into the tube) to path length in any tiny segment of the tube are second order and do not contribute to the path-length generated *within that segment*. However, those volumetric contributions do contribute to the path-length: they contribute to the path-length in cells further along the direction Ω.