
AC 2011-2209: TEACHING MECHANICS WITH MAPLE

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Teaching Mechanics with Maple

Abstract

Classical mechanics is a disciple of theoretical physics and is one of the main constituent parts of physics. The instruction of theoretical physics courses plays an important role in the research of basic science and training physics skill sets. Most students consider that the knowledge of theoretical physics to be very abstract thus causing many difficulties in its study and understanding. Teachers can implement some teaching reform strategies to improve the quality of theoretical physics instruction in the context of classical mechanics. Such strategies may include: stimulate students' interest in learning, perfect students' cognitive structure and knowledge structure of theoretical physics; optimize the system of theoretical physics curriculum, enrich the instruction contents; apply heuristic instruction in theoretical physics teaching; establish virtual theoretical physics experiments and improve assessment and appraisal methods, promote students' all-round development.

Physics is guided by simple principles, but for many topics, physics tends to be obscured in the profusion of mathematics. This paper describes some of the merits of using computer algebra in teaching mechanics. We report on our experience in teaching, during the course of several years, dynamical systems and mechanics courses to second-year engineering students by using symbolic computation. When they enter this course they have already taken one to two semesters of calculus, and during their high-school studies they must have already studied particle kinematics and dynamics. Simulation software and computer algebra systems allow students to experiment with phenomena which are too complex to calculate or too expensive to be reproduced in a laboratory, or are simply not accessible to the senses. A computer algebra system is essentially the ability to manipulate concepts, using computer expressions, which are symbolic, algebraic and not limited to numerical evaluation. A computer algebra systems can perform many of the mathematical techniques which are part and parcel of a traditional physics course. The successful use of computer algebra systems does not imply that the mathematical skills are no longer at a premium: such skills are important as ever. However, computer algebra systems may remove the need for those poorly understood mathematical techniques, which are practiced and taught simply because they serve as useful tools. The conceptual and reasoning difficulties that students have in introductory and advanced physics courses, including calculus-based honors courses, are well-documented by the physics education community. The appropriate use of computer algebra systems can therefore be an important aid in the training of better physicists and engineers. In this paper we will discuss ways in which computer algebra systems like Maple can be used by instructors and students in order to help students make these connections and to use them once they are made. Benefits that accrue to upper-class students able to make effective use of a computer algebra system provide a further rationale for introducing student to use of these systems in our courses, especially for those who plan to major in physics or other technical fields.

Introduction

This work is the third in a series¹⁻³ aimed at extending basic knowledge of mechanics, electromagnetics and other physics areas, and improving understanding, in physics courses.

Moreover, it aids in the ABET goal of integrating computer usage throughout the curricula. In higher education, theoretical physics courses are the main courses of a physics major. The instruction of theoretical physics plays an important role in the research of basic science and training physics talents. Analytical and numerical methods are used complimentary for the understanding of physical phenomena and the solution of technical problems.

The use of computers in numerical simulations of physical problems has a long history of development and success, and is indeed intimately tied to the history and development of large computers. On the other hand, the use of computers in carrying algebraic and other mathematical operations in symbolic form is of much more recent origin, and primarily connected with developments in programming languages and symbolic computation algorithms. Computing is dramatically affecting the way that modern science and engineering are carried out. To date, symbolic computation has played a rather limited, but important, role in the twentieth century advancement of science and engineering. Symbolic computation aims to automate a wide range of the computation involved in mathematical problem solving¹⁻¹². It emphasizes discrete computation on symbols representing mathematical objects. The symbols represent not only numbers but also other mathematical objects like polynomials, rational and trigonometric functions, algebraic numbers, groups, ideals, and tensors. Powerful interactive systems for doing symbolic computation have been designed and built. The software has improved the productivity of scientists and engineers; it has made possible the solving of problems that were previously intractable. However, only the surface has been scratched. We are still confronted with a wide spectrum of challenging problems whose solution will have a crucial influence on our technological problem solving ability.

The applications of symbolic computation ranges over the entire scope of mathematics and its applications, that is, essentially all science and engineering fields. There are three modes of the use: (1) computations that could be carried out by hand but can be done more productively and accurately by a symbolic computation system, (2) computations that are beyond hand calculation but can be done more or less routinely by machine, and (3) calculations that require substantial effort to complete even when using a computer. In addition, there are important applications that are out of reach of present computational methods and hardware. Computer algebra systems (CAS), such as Maple, MathCAD, or Mathematica can assist educators and students to overcome these obstacles and also to improve the lecture presentations via the power of visualization, animation and graphic facilities³⁻¹⁰. Educators and students can use the advantages of symbolic computation in order to be able to concentrate on applying principles of setting equations instead of technical details of solving problems. The conventional approach to a topic places emphasis on theory and formalism, devoting many paragraphs to performing algebraic or calculus operations in deriving equations manually; other than some well known examples, most applications of theory are omitted. One reason that those examples are well known is that they admit analytic solutions. However, they typically represent simplified solutions that generally fail to fully reflect reality. In most real-world situations, the analytic solutions simply do not exist, and one cannot proceed without the assistance of a computer. Although some textbook have sections discussing numerical methods, many of them contain just the theory of numerical methods, and one is required to possess programming skill for practice; this part has generally been neglected. Under a conventional curriculum, a student's ability to calculate and to extract numerical results from formalism is somewhat inadequate. The result is not surprising: a student

may be weak in those areas, and he or she thus achieves only partial comprehension because of technical difficulties. CAS can remedy some deficiency or weakness in the traditional education process and training. Using CAS, one can manipulate equations and diminish tedious work that distracts from the main focus of learning physics. To become proficient problem-solvers, students need to form a coherent and flexible understanding of problem situations with which they are confronted.

Course Details

Classical theoretical physics really began with the work of Sir Isaac Newton. His *Principia Mathematica*, written to answer a question from Robert Hooke, set out to explain how planets move in elliptical orbits and in the process formulated the basic laws of motion and developed the whole mechanism of calculus to solve problems in this new discipline. More than 300 years later Newton's laws and calculus remain an excellent description of how the world works, and have been used to solve a wide range of the simpler problems in mechanics. This course aims to explore some more complex systems. After a few weeks of review we shall go on to develop a reformulation of Newton's mechanics using methods introduced by Euler and Lagrange and refined by Hamilton. The text for the course is *Classical Dynamics of Particles and Systems*, 5th edition, by Thornton and Marion. There are about 12 weekly home-works, mostly taken from Thornton and Marion. These will range from moderate drill problems to some really tough problems that should challenge all of the students. As usual, in all my courses I strongly encourage reasonable collaboration among students in tackling these problems but the solutions that they handed in must be individual and should indicate which parts of the work were collaborative and which parts are their own. I may experiment with small collaboration groups for some problems.

Course Objectives: This is a course on the basic concepts in Classical Mechanics; it serves as the foundation for more advanced courses. The course examines the theoretical framework of mechanics, including the rudiments of Lagrangian and Hamiltonian mechanics. Along the way we will analyze simple but important systems, such as central potentials, two body systems and the motion of a rigid body. The main emphasis of the course is theoretical; therefore, proofs and derivations, not just problem solving, are of fundamental importance. Knowing and understanding proofs and derivations will be expected from the students. The objective for this course is three fold: (1) To provide the students with a strong background in the techniques of classical mechanics at an intermediate level; (2) To introduce the mathematical and computational techniques for setting up and solving differential equations to determine the motion of particles and rigid bodies; and (3) To acquaint the students with the concept of “reading a physics book” and developing self-study techniques. There shall be reading assignments every week along with the standard problem solving assignments some of which may require use of a computer. To note, reading physics is very different from reading a novel, for example.

Motivation: Though classical mechanics has been superseded by relativistic mechanics and quantum mechanics, there is still a large class of interesting phenomena where classical mechanics provides an accurate and complete description. Furthermore, the physical and mathematical tools used in the study of classical mechanics are indispensable for study of

physics in general, and especially for the non-classical mechanics that invalidated or extended the principles of classical mechanics.

Class Schedule: The major topics of this course include:

1. Non-inertial Frames; Non-inertial frame without rotation; the tide force; Newton's laws in a rotating frame; The Coriolis force; Foucault's Pendulum;
2. Rigid body, Rotation around fixed axis; Inertial tensor, principal axes of inertial tensor; Euler equation; Euler angle, spinning top;
3. Hamiltonian Mechanics. Lagrangian equation vs. Hamiltonian equation; Poisson bracket; Canonical Transformation; Conserved Quantities; Liouville's Theorem; Boltzmann transport equation;
4. Continuum Mechanics; Waves; Stress and Strain; Fluid; Bernoulli's equation

Computer Algebra Systems Features and Physics Applications

Computer algebra systems have from their earliest days been concerned with providing tools with which researchers and scientists in other fields can be used in order to determine new results. CAS can have a significant impact on the way mathematics, physics or engineering courses are taught and applied. In teaching mathematics or physics now it is possible to concentrate on mathematical or physics content, rather than on counting numbers or finding solutions of exotic equations or integrals. A computer algebra system in itself is no more than a high level programming language for visualization, and symbolic and numerical computation. Basically, computer algebra systems are programs designed for symbolic manipulation of mathematical objects such as polynomials, vector and matrix manipulations, integrals, equations, etc. Typical actions are simplification or expansion of expressions, solving (systems of) differential or algebraic equations, data analysis and statistical methods, etc. Most CAS allow the user at least to write sequential programs for complex tasks, in a manner similar to writing mathematical equations, and have all features of high-level programming languages available. As well as such features, CAS also have most of the features of numerical systems for visualization (2-D plots, 3-D plots, animations) and numerical computations (numerical equation solving, numerical integration and differentiation). However, numerical systems are typically faster in regard to the numerical handling of floats with fixed precision¹²⁻²⁰. Besides being a tool for the manipulation of formulae, CAS should be able to be used as an expert system knowing all of the mathematics in a good mathematical handbook. The first computer algebra systems, which became available in late 1980s, were mainly of only theoretical interests. Over the last two decades, some of these software packages have evolved into more practical computation and visualization tools that can take over many routine problem solving tasks. At the same time the required hardware has become more affordable.

CAS was from the very beginning a tool for building activities, and was accepted without reservations by physicists and theoretical chemists from earliest days of the symbolic computation. One of the earliest area of CAS applications in physics was celestial mechanics as well classical mechanics, where it become an everyday tool for many researchers. In many applications in this area, such as problems of space dynamics, orbit computation, or the representations of the equations of motion in symbolic form, CAS avoids unreasonably large numerical experiments and simulates effective usage and development of algorithms for qualitative methods of analysis of equations constructed. In these areas, CASs usually suggest substantial aid both on modeling stage (construction of the kinetic energy and the force function for mechanical system, derivation of equations of motion) and during qualitative analysis of obtained equations. This aid is appreciable even for objects of moderate dimension. Another area where CAS was used is special and general relativity, with applications such as classification of Riemann tensor based on studies of the multiplicity roots of a quartic equation or on the equivalence problem. Quantum theory and high energy physics have been other active areas for the applications of symbolic computation. Electromagnetic field theory is one of the areas of physics and engine engineering where symbolic computation is applied on an extended scale due to their capabilities in solving differential equations, visualization and graphic capabilities⁸.

Some of the advantages of using a CAS package are: a) students can write down mathematics in a programming-like way, using symbolic notations; b) less time spent with calculations leaves more time for physical analysis; c) geometric visualization of results; d) learning and becoming proficient in a high-level programming language; and e) there availability of free software applications, using well-documented algorithms. Derive and MathCAD are already implemented on a pocket calculator, and more extensive packages, such as Mathematica and Maple, run on any desktop computer. In several branches of mathematics, physics and engineering, computer algebra systems have been increasingly popular as a tool for constructing proofs, solutions and visualizing the results. Also in introductory mathematics courses at the university level, there is an increasing use of computer algebra software packages in teaching and learning. However, there are fewer examples where computer algebra systems were integrated throughout physics courses. In this study we will argue that a CAS could be used, via several examples, to promote students' understanding of problems and to support the formulation of associations between problem representations and solution information. A didactic approach for using such software to improve learning and the teaching process in physics will be suggested.

There are many commercial and non-commercial products available. The most popular are MathematicaTM and MapleTM which will, in a (hopefully) everlasting contest, continue to evolve. Other systems are REDUCETM, AXIOMTM, MuPADTM or Derive¹³⁻²⁰. All systems can be used for high-school to university mathematics, but they differ in comfort and complexity and each has a different look and feel. All these programs contain a kernel of Maple. The problem with such hybrids is that in general they are fixed to a certain release of the underlying kernel or linked CAS and that normally they could not be used across platforms. Throughout this paper,

Maple V is used as an example CAS, for two reasons: first one of the author preference and the second its availability at our universities.

By using CAS it is possible, instead of training integration rules one exotic cases over and over, to concentrate on meaning of a physics problem and/or on variants of it. We are also no longer limited to trivial examples that work. Students are invited to play with physics they learn that real life examples normally do not lead to closed formulas. They can even visualize the results and different approximations and they also learn to judge the solutions. They are introduced to the multitude of mathematical tools, each with their own advantages and disadvantages and precise applicability. We attempt to devise an instructional approach to promote students' understanding of these problems and to support them in forming associations between problem features and solution methods. The approach is to use symbolic computation packages as tools for problem solving and visualization. A set of modules, such as: harmonic oscillator, pendulum, coupled oscillators, etc. were implemented based on this instructional approach. This approach in teaching physics is unconventional in several aspects: its content reflects the needs of high-tech physicists and engineers, the approach is strongly computer-supported, symbolic computing and other IT tools are systematically applied, problem-solving skills are intensely stressed.

Maple provides many procedures to support such differentiation, and there are many integrated procedures to derive the equations of motion directly. Indeed, the latest version *Maple* 8 has a new package called Variational-Calculus that incorporates all necessary calculations. These procedures are tremendously convenient in practical use, but their shortcomings are that one needs to invoke external procedures for *Maple* to perform such calculation, and that perhaps a novice user may not fully comprehend the underlying physics. We develop a method in *Maple* to solve problems of the calculus of variations. In our approach, we use only basic commands such as `subs`, `diff`, and `dsolve`, without programming or invoking an external library. Although our method is a pedagogic approach that might involve more manual steps, it is a direct attack on this problem, and practically all problems in classical mechanics can be solved once the Lagrangian is found. In most real physical problems there is no analytic solution to differential equations. We particularly emphasize forming plots based on a numeric method by varying physical parameters and initial conditions and observing the graphic outputs, one can develop an intuition about the underlying physics. In this presentation, to illustrate our method we begin with a simple problem, finding the shortest connection between two points in a plane.

Maple Applications in Mechanical Problems

The heart of our approach in teaching classical mechanics with symbolic computation consists of an eclectic set of more than fifty useful and simulating computer algebra worksheets (the recipes) which are systematically organized to correlate with the topics covered in our classical mechanics course syllabus or in any other similar course. Our software applications are designed to complement the mechanics textbooks by showing how Maple or any other computer algebra package can not only solve standard physical models quickly, efficiently, and accurately, but can be used to develop and explore more complex physical systems. The developed worksheets are fully commented and interactive by the authors (see the appendix for a worksheet sample). The developed worksheets (recipes) covered all major topics and problems of classical mechanics. Among the developed worksheets are (with the comments for some of them):

1. **Integrals in Maple;**
2. **Conservation Laws**

a. **Rocket Propulsion:** The problem of mechanical systems in one dimension with varying mass represents an interesting application of Newton's law. It comes up in the rocket problem (variable mass and velocity), the conveyor belt problem, and others. It can find in almost any mechanics textbooks.

b. **Cannon Ball:** We consider the problem of firing cannon balls: this involves two-dimensional motion in a gravitational field including air resistance. We set up Newton's equations for the x and y components of the motion.

c. **One-dimensional Collisions/Compton Scattering:** We used conservation laws for linear momentum and kinetic energy to solve one-dimensional collision problems. In more dimensions the number of degrees of freedom associated with the particle motion exceeds the number of constraints provided conservation laws. This means that we will not be able to obtain complete solutions. Nevertheless, it is interesting to investigate the constraints supplied by the conservation laws. A complete dynamical approach involves solving Newton's equations (or variants thereof, such as Lagrange's, or Hamilton's forms), which takes into account the force law, and which automatically obeys the conservation laws. An example of such a complete solution was the problem of firing a cannonball.

1) The conservation of angular momentum for the motion of objects subject to central forces implies that the motion is confined to a plane. This reduces the three degrees of freedom in the position vectors of the masses to two. The number of degrees of freedom is reduced from 6 to 4.

2) The conservation of energy provides one scalar constraint, i.e., 3 degrees of freedom remain.

3) The conservation of linear momentum in the scattering plane provides 2 constraints.

Thus, the problem cannot be completely determined from the conservation laws. This is illustrated in this worksheet, first in general, and then specifically for the scattering of photons off atomic electrons resulting in an energy transfer to the electron, i.e., Compton scattering.

3. **Moment of Inertia Tensor:** In this worksheet we calculate the moment of inertia matrix in a given coordinate system and show that a coordinate system can be found in which the inertia tensor is represented by a diagonal matrix. This worksheet requires **maple8** to work properly (integral I_2 below will not evaluate in earlier versions, and even in maple8 it integrates with a wrong sign).
4. **Harmonic Oscillator Animations**
5. **Coupled Spring-Mass Systems**
6. **The Mathematical Pendulum**
7. **Work and Force**
8. **The work Integral in Classical Mechanics**
9. **Kepler Problem**

10. The radial Motion Problem.

11. Rolls and Strings A roll (radius R , mass M) has rope wound around it which leads via a pulley to a mass m that falls under gravity. The point of this worksheet is to explain how the motion of the center of mass (CM) is affected by the net force that acts on the roll: the string force, which applies where the string is unwound, and the static friction force which acts at the point of contact (assuming that there is rolling without slipping). Both these forces contribute torques about the CM of the roll. This work shows that the direction of the friction force is difficult to predict. There are situations where it helps to push the CM forward, and acts against the rolling, and there are situations where it helps the rolling, but acts against the motion of the CM. This is a consequence of applying:

(i) the CM theorem: all force vectors acting on the body are applying at the location of the CM, and determine the rate of change of the total linear momentum;

(ii) the angular momentum theorem: the rate of change of angular momentum is controlled by the net torque that applies; one can use always the quantities about the CM, or about the point of contact (in which case friction does not contribute a torque), however, only in the case of pure rolling (no slipping) is the angular velocity about the CM the same as the angular velocity about the point of contact. In the case of pure rolling the part of the roll that touches the ground at the point of contact has no motion, and static friction applies.

(iii) the case of pure rolling is characterized by a fixed relationship between the CM motion and the angular velocity that describes the rotation.

(iv) the mass that falls under gravity undergoes a motion that is composed of motion of the CM and the additional effect of the unwinding of the string.

12. Euler's equations

13. Torque-free gyroscope

14. Wave Equation: The equation (in mechanics) inherits the two time derivatives from Newton's equation. The two spatial derivatives arise as a consequence of the harmonic force experienced by a mass point from its two neighbors: the difference of relative displacements can be expressed as a difference of first derivatives, and thus as a second derivative with respect to location.

15. Classical Differential Scattering Cross Section.

Interested reared can request the full set of the worksheets from one of the author.

Conclusions and Future Work

The paper has focused on the development of a set of teaching and learning modules using symbolic computation for a classical mechanics course. The goal was to support students in gaining intuitive understanding of physics situations, solution methods, or the relations between them and to help them to get a deeper understanding of less intuitive phenomena. Among the distinctive features of Maple or any CAS modules are the use of precise language for specifying problems, visualization support and symbolic and numerical support for solving problems.

Future work will consist in the improvement and extension of already developed modules, the design and implementation of new modules and to extend those modules to other physics branches, such as electromagnetics, fluid mechanics, and thermodynamics. Our long-term goal is the development and design of an e-learning version of the CAS modules^{8,9}.

Finally yet importantly, the students received very well the use of the Maple and symbolic computation techniques in the classical mechanics teaching. At the end of each quarter, the students have been requested to answer (with a five point scale: 1-very poor, 2-poor, 3-satisfactory, 4-good and 5-very good) an anonymous questionnaire containing questions about the usefulness of the use of symbolic computation in teaching and learning mechanics. According to the results, this approach received a 3.9/5.0 rating, comparing with an average rating of 3.4/5.0 for all the courses at our institution.

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Appendix: Illustrative Worksheet (The work integral in classical mechanics)

Let us demonstrate the path independence of the work integral for a conservative curl-free force field, such as the gravitational field of a sphere. Let us define the potential energy for a particle of mass m in the field of the star (or planet) of mass M .

> **V:=-G*M*m/r;**

$$V := -\frac{GMm}{r}$$

> **with(linalg):** Warning, the protected names norm and trace have been redefined and unprotected

> **F:=-grad(V,[r,theta,phi],coords=spherical);**

$$F := -\left[\frac{GMm}{r^2}, 0, 0 \right]$$

> **evalm(F[1]);**

$$\left(-\left[\frac{GMm}{r^2}, 0, 0 \right] \right)_1$$

The - sign on front of the grad throws Maple off, because the grad procedure returns a vector, and not a list. The fix is to put the minus sign with the scalar potential function V.

> **F:=grad(-V,[r,theta,phi],coords=spherical);**

$$F := \left[-\frac{GMm}{r^2}, 0, 0 \right]$$

> **F[1];**

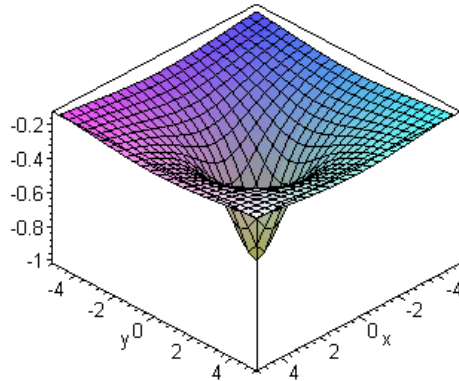
$$-\frac{GMm}{r^2}$$

> **curl(F,[r,theta,phi],coords=spherical);**

$$[0, 0, 0]$$

The force field is curl-free by construction: it has been derived from a scalar potential. For plotting the potential energy we need to switch to Cartesian coordinates. We choose to show the potential at a fixed $z=1$ plane:

```
> PL1:=plot3d(subs(r=sqrt(x^2+y^2+1),G=1,M=1,m=1,V),x=-5..5,y=-5..5,axes=boxed):  
plots[display](PL1);
```



Let us choose a straight-line path:

```
> r_i:=[2,2,1];
```

$$r_i := [2, 2, 1]$$

```
> r_f:=[-2,0,1];
```

$$r_f := [-2, 0, 1]$$

```
> r_t:=r_i+(r_f-r_i)*t;
```

$$r_t := [2, 2, 1] + [-4, -2, 0]t$$

```
> evalm(r_t);
```

$$[2 - 4t, 2 - 2t, 1]$$

Note: here t is a parameter, not necessarily time. Nevertheless, we call the derivative of the parameter representation v_t :

```
> v_t:=diff(r_t,t);
```

$$v_t := [-4, -2, 0]$$

Now the work integral is reduced to a parameter integral using the dotproduct of \mathbf{F} with \mathbf{v}_t , but also realizing that \mathbf{F} needs to be evaluated along the path,

i.e., as $\mathbf{F}(\mathbf{r}_t)$!!!

Also: we have defined the parameter representation in Cartesian coordinates, so we should use these coordinates for the dot product calculation, which means that the force has to be translated into Cartesians. How do we do this?

> **$\mathbf{F}:=\text{grad}(-\text{subs}(r=\text{sqrt}(x^2+y^2+z^2),V),[x,y,z]);$**

$$\mathbf{F} := \left[\begin{array}{c} -\frac{GMm x}{(x^2+y^2+z^2)^{\frac{3}{2}}}, -\frac{GMm y}{(x^2+y^2+z^2)^{\frac{3}{2}}}, -\frac{GMm z}{(x^2+y^2+z^2)^{\frac{3}{2}}} \end{array} \right]$$

We start with the common mistake made by many beginners:

> **$\text{Work}:=\text{int}(\text{dotprod}(\mathbf{F},\mathbf{v}_t),t=0..1);$**

$$\text{Work} := 4 \frac{GMm x}{(x^2+y^2+z^2)^{\frac{3}{2}}} + \frac{2GMm y}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

This is a bad answer for several reasons: 1) the work has to come out as a number (we have specified initial and final points)' 2) we have not evaluated the force along the chosen path.

> **$r_t:=\text{evalm}(r_t);$**

$$r_t := [2 - 4t, 2 - 2t, 1]$$

> **$\text{Work}:=\text{int}(\text{dotprod}(\text{subs}(x=r_t[1],y=r_t[2],z=r_t[3],\mathbf{F}),\mathbf{v}_t),t=0..1);$**

$$\text{Work} := 4 \frac{GMm x}{(x^2+y^2+z^2)^{\frac{3}{2}}} + \frac{2GMm y}{(x^2+y^2+z^2)^{\frac{3}{2}}}$$

This is frustrating: we do apparently the right thing, yet we get the same wrong answer!!! How do we avoid this pitfall? 1) It looks at sub-expressions, and then realize that something hasn't worked. Substitutions into vectors and matrices in Maple require the use of the `op()`:

> **dotprod(subs(x=r_t[1],y=r_t[2],z=r_t[3],op(F)),v_t);**

$$-8 \frac{G M m (-1 + 2 t)}{\left(\frac{3}{2}\right) (9 - 24 t + 20 t^2)} - \frac{4 G M m (-1 + t)}{\left(\frac{3}{2}\right) (9 - 24 t + 20 t^2)}$$

This is the correct integrand: it no longer depends on x , y , z just on the parameter t . Now do the integral:

> **Work:=int(dotprod(subs(x=r_t[1],y=r_t[2],z=r_t[3],op(F)),v_t),t=0..1);**

$$Work := \frac{1}{5} \sqrt{5} G M m - \frac{1}{3} G M m$$

An alternative to using `op()` is to convert the vector produced by `grad` into a list.

> **FL:=convert(F,list);**

$$FL := \left[-\frac{G M m x}{\left(\frac{3}{2}\right) (x^2 + y^2 + z^2)}, -\frac{G M m y}{\left(\frac{3}{2}\right) (x^2 + y^2 + z^2)}, -\frac{G M m z}{\left(\frac{3}{2}\right) (x^2 + y^2 + z^2)} \right]$$

> **Work:=int(dotprod(subs(x=r_t[1],y=r_t[2],z=r_t[3],FL),v_t),t=0..1);**

$$Work := \frac{1}{5} \sqrt{5} G M m - \frac{1}{3} G M m$$

> **evalf(%);**

$$.1138802623 G M m$$

The work integral tells us by how much the kinetic energy has changed between \mathbf{r}_i and \mathbf{r}_f due to the work done by the force on mass m . Now let us compare this result with the potential energy difference:

> **r_i;**

$$[2, 2, 1]$$

We need a function to calculate the magnitude of the position vector:

> **v_mag:=v->sqrt(add(v[i]^2,i=1..3));**

$$v_mag := v \rightarrow \sqrt{\text{add}(v_i^2, i = 1..3)}$$

> **v_mag(r_i),v_mag(r_f);**

$$3, \sqrt{5}$$

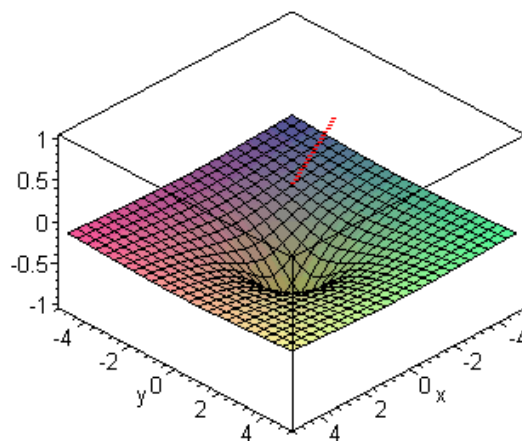
> **subs(r=v_mag(r_i),V)-subs(r=v_mag(r_f),V);**

$$\frac{1}{5}\sqrt{5}GMm - \frac{1}{3}GMm$$

> **Work-%;** 0

> **with(plottools);**

> **l:=line(r_i, r_f, color=red, linestyle=2, thickness=3): plots[display](l,PL1);**



We want to indicate with the height the amount of potential energy, therefore:

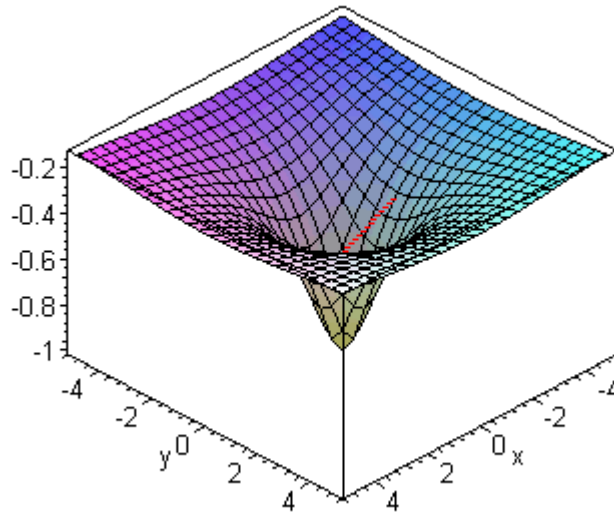
> **r_iV:=r_i: r_iV[3]:=subs(r=v_mag(r_i),G=1,M=1,m=1,V): r_iV;**

$$\left[2, 2, \frac{-1}{3} \right]$$

> **r_fV:=r_f: r_fV[3]:=subs(r=v_mag(r_f),G=1,M=1,m=1,V): r_fV;**

$$\left[-2, 0, -\frac{1}{5}\sqrt{5} \right]$$

> `l := line(r_iV, r_fV, color=red, linestyle=2, thickness=3): plots[display](l, PL1);`



As the particle traverses the potential well it picks up a lot of kinetic energy on the way down, in order to give up a good fraction as it climbs up the hill. We can verify this by calculating the work done at some intermediate point. First recall the total kinetic energy gain:

> `evalf(Work);`

$$.1138802623 \, G M m$$

Now break the path up into two halves, just by looking at what happens between $t=0$ and $t=0.5$, and then between $t=0.5$ and $t=1$:

> `Work1:=int(dotprod(subs(x=r_t[1],y=r_t[2],z=r_t[3],op(F)),v_t),t=0..1/2);`

$$Work1 := \frac{1}{2}\sqrt{2} \, G M m - \frac{1}{3} \, G M m$$

> `Work2:=int(dotprod(subs(x=r_t[1],y=r_t[2],z=r_t[3],op(F)),v_t),t=1/2..1);`

$$Work2 := \frac{1}{5}\sqrt{5} \, G M m - \frac{1}{2}\sqrt{2} \, G M m$$

> `evalf([Work1,Work2,Work1+Work2]);`


```
[.3737734477 G M m, -2598931854 G M m, .1138802623 G M m]
```

Notice the large amount of work done on the particle on the first 'half', then the particle works against the potential well, but still has a net gain of kinetic energy at the final point. It is, however, at a lower value of potential energy at this point, i.e., it gained kinetic energy at the expense of potential energy. Now consider a non-conservative force. It will have no direct physical meaning, i.e., it is a made-up problem to demonstrate what happens when a force field is not conservative.

```
> restart;
```

```
> F:=[x^2+2*y-z,y+z-x,z^3+x^2];
```

$$F := [x^2 + 2y - z, y + z - x, z^3 + 2x]$$

```
> with(linalg):
```

```
Warning, the protected names norm and trace have been redefined and unprotected
```

```
> curl(F,[x,y,z]);
```

```
[-1, -3, -3]
```

This force field is not curl-free, which means that it cannot be derived from a scalar potential. We choose the same path as before.

```
> r_i:=[2,2,1];
```

```
r_i := [2, 2, 1]
```

```
> r_f:=[-2,0,1];
```

```
r_f := [-2, 0, 1]
```

```
> r_t:=evalm(r_i+(r_f-r_i)*t);
```

```
r_t := [2 - 4t, 2 - 2t, 1]
```

```
> v_t:=diff(r_t,t);
```

```
v_t := 0
```

Another one of these Maple-quirks: We defined r_t as a combination of lists, but the `evalm` did something funny.

> `whattype(r_t);`

symbol

The `evalm` turned r_t into a symbol.

> `v_t:=map(diff,r_t,t);`

$v_t := [-4, -2, 0]$

If we had avoided the `evalm`, the simple differentiation would have worked because t appeared as a factor of one of the lists (as carried out in the previous section).

> `Work1:=int(dotprod(subs(x=r_t[1],y=r_t[2],z=r_t[3],F),v_t),t=0..1/2);`

$Work1 := \frac{-49}{6}$

> `Work2:=int(dotprod(subs(x=r_t[1],y=r_t[2],z=r_t[3],F),v_t),t=1/2..1);`

$Work2 := \frac{-31}{6}$

> `Work:=int(dotprod(subs(x=r_t[1],y=r_t[2],z=r_t[3],F),v_t),t=0..1);`

$Work := \frac{-40}{3}$

> `evalf([Work1,Work2,Work1+Work2,Work]);`

$[-8.166666667, -5.166666667, -13.33333333, -13.33333333]$

Of course, the work integral is additive. What we want to demonstrate is that if we connect the two endpoints in some different way, then a different result will be found for the work done on the particle as it moves from A to B. Let us choose as an alternative to the direct connection from A to B a path that goes via the origin $[0,0,1]$ in the plane.

> `r_1:=[0,0,1];`

$r_1 := [0, 0, 1]$

> **r_tp1:=evalm(r_i+(r_1-r_i)*t);**

$$r_{tp1} := [2 - 2t, 2 - 2t, 1]$$

> **v_tp1:=map(diff,r_tp1,t);**

$$v_{tp1} := [-2, -2, 0]$$

> **r_tp2:=evalm(r_1+(r_f-r_1)*t);**

$$r_{tp2} := [-2t, 0, 1]$$

> **v_tp2:=map(diff,r_tp2,t);**

$$v_{tp2} := [-2, 0, 0]$$

Now we are ready to calculate the work integral.

> **Work1:=int(dotprod(subs(x=r_tp1[1],y=r_tp1[2],z=r_tp1[3],F),v_tp1),t=0..1);**

$$Work1 := \frac{-20}{3}$$

> **Work2:=int(dotprod(subs(x=r_tp2[1],y=r_tp2[2],z=r_tp2[3],F),v_tp2),t=0..1);**

$$Work2 := \frac{-2}{3}$$

> **evalf([Work1+Work2,Work]);**

$$[-7.333333333, -13.33333333]$$

The amount work done on the particle as it moves through the force field along a different path is different! What can we do to visualize the force field, and how it affects the particle?

First define the force field in the $z=1$ plane and ignore the z -component of the force, as we are not moving the particle along z (no work is done by this component of the force!).

> **Fzeq1:=subs(z=1,F);**

$$Fzeq1 := [x^2 + 2y - 1, y + 1 - x, 1 + 2x]$$

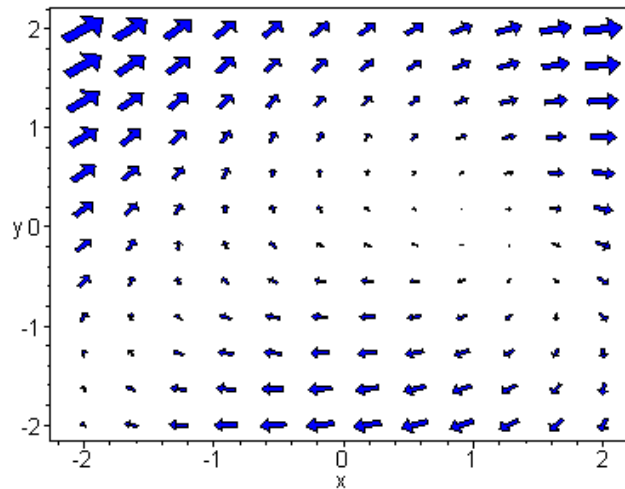
> **F2d:=[Fzeq1[1],Fzeq1[2]];**

$$F2d := [x^2 + 2y - 1, y + 1 - x]$$

> **with(plots):**

Warning, the name changecoords has been redefined

> **PL1:=fieldplot(F2d,x=-2..2,y=-2..2,arrows=thick,color=blue,axes=boxed,grid=[12,12]):**
display(PL1);



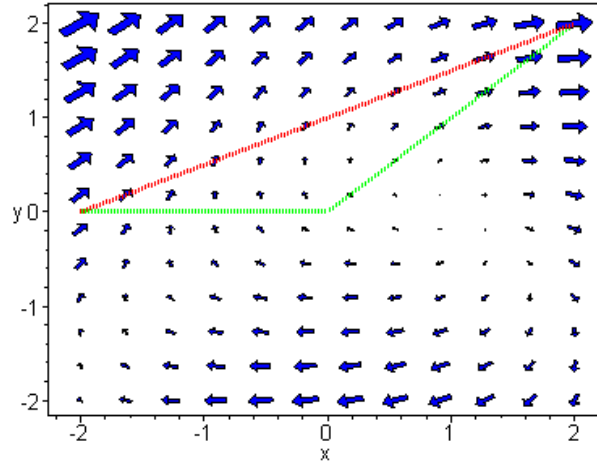
> **with(plottools):**

> **l1 := line([r_i[1],r_i[2]], [r_f[1], r_f[2]], color=red, linestyle=2, thickness=3):**

> **l2 := line([r_i[1],r_i[2]], [r_1[1], r_1[2]], color=green, linestyle=2, thickness=3):**

> **l3 := line([r_1[1],r_1[2]], [r_f[1], r_f[2]], color=green, linestyle=2, thickness=3):**

> **display(l1,l2,l3,PL1);**



The amount of work done on the particle as it moves through this field is different, because in different regions of space the force vectors are allowed to have different lengths and different orientations.

For a force field which is not curl-free, the travel over a closed contour (loop) allows one to gain (or lose) energy. The graph also shows visually why we refer to the field as not being curl-free.

> **evalf([Work1,Work2,Work1+Work2,Work]);**

[-6.666666667, -.666666667, -7.333333333, -13.33333333]

Note that the amount of work done by the force on the particle is negative. We start the particle at [2,2] (top right-hand corner), and it moves against the force field.

The fact that it arrives at location [0,-2] with a negative amount of work (it worked against the force field) means that its kinetic energy will be less than it was at A. It will be less at B than at A by a different amount depending on whether the red or green path is chosen.