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Teaching physics (mechanics and electromagnetics) by using finite difference technique instead of standard calculus.

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1 Introduction.

Physics is the basis of innumerable technological applications, which shape the face of contemporary society and represents the paradigm of all exact sciences. One would reasonable expect physics to permeate the culture of every even moderately educated man or women. Unfortunately this is not the case and in contrast there is a tendency to allow irrational or simply empirical attitudes to dominate in many areas of life, including the development of new technologies. The delicate task of preventing a general dismissal of physics as the common basis of any technical education belongs naturally to school systems and, in particular, to universities, where the physics subjects are taught, not only in general physics courses, but also as part of various technical courses. However, in of all this a weak point is that often the teaching of general physics follows schemes which follows the ways and methods of many decades ago, even when the contents are up to date. Sometimes modern teaching technologies are adopted but most often only referring to the ways in which doing laboratory, presentations, etc. One may also consider the fact that in recent years several authors pointed out the importance of using numerical calculations in introductory physics courses, owing to the increased availability of personal computers.

Making physics more accessible to students is one of extreme importance, being the most significant achievement of any physics educator. A rigorous understanding of physics generally presumes a rigorous understanding of standard calculus. This raises the question of whether physics can be understood rigorously without using standard calculus. In answering this question, we note that an alternative to calculus is the finite difference approach. The finite difference is a powerful tool, widely used in solving physics and technical problems. One also need to note another advantage of numerical methods is that they permit the treatment of problems for which analytical solutions do not exist, which dramatically enlarges the base of the examples, used to support the course. These motivate our interest in the reformulation of classical mechanics, electromagnetics, acoustics, etc. by using finite difference methods instead of calculus. This reformulation of classical physics, based on the finite difference calculus will be presented in this paper. The finite difference technique is an intuitive modeling technique easy to understand and applied by the vast majority of students. In addition to insight that can be gained on classical mechanics, electrodynamics, or in other branches of physics, there is also the possibility of studying problems of physics that cannot be resolved analytically and to explore many areas of physics. Last but not least, we have to mention, that in order to take full advantage of these potentialities, the numerical methods employed should be understandable to the student, easily programmed, and efficient so that accurate results can be obtained without excessive computational times.

1.1 Finite Difference Methods.

The finite difference method was first developed by A. Thom in the 1920s under the title "the method of squares" to solve non-linear hydrodynamic equations (see Sadiku, 1989 for details). Since then, the method has found applications in solving different physics and engineering problems. The finite difference techniques are based upon approximations which permit replacing differential equations by finite form. These finite difference approximations are algebraic in form; they relate the value of the dependent variable at a point in the solution region to the values at some neighboring points. The finite difference method has been applied successfully to solve many problems of mechanics, acoustics, electromagnetics, fluid dynamics, etc. Any approximation of a derivative in terms of values of a discrete set of points is called *finite difference approximation.* The approach used in obtaining finite difference approximations is based on the using of Taylor's series to approximate the derivatives of a function[6,12]. Following these patterns the definition of time derivative of a particles position $x(t)$ is given by:

$$\frac{dx}{dt} = \lim_{T \to 0} \frac{x(t+T) - x(t)}{T}$$

Virtually all classical equations of physics are defined in terms of $x(t)$, $x^{'}(t)$, and $x^{"}(t)$, so a rigorous understanding of classical mechanics requires knowledge of calculus. If it is out of question that the calculus and advanced mathematics are necessary to understand and presentation of physics, it will also be beneficial to have alternative approaches of making physics more accessible to the students. As we already discussed in first part of this paper our attempt is based on the use of finite-difference calculus instead of the standard infinitesimal calculus. This involves replacing derivative of physical quantities by the finite-difference counter-part, as for example the time derivative is replaced by:

$$Dx(t) = \frac{x(t+T) - x(t)}{T} \tag{1}$$

where T is the smallest interval of time. Note that the finite-difference operator converge to the time derivative as T goes to zero. This replacement will lead, in the case of classical mechanics, to a minor reformulation of momentum, energy and acceleration. This provides the basis for a rigorous mathematical treatment of classical mechanics which is more accessible to the students. According with Lakshminkanthan and Trigante, 1988 the time difference operator has the following properties:

$$D[\alpha \cdot f(x) + \beta \cdot g(x)] = \alpha \cdot Df(x) + \beta \cdot Dg(t) \qquad \alpha, \beta = constant \tag{2}$$

$$D[f(x)\cdot g(x)] = f(x)\cdot Dg(x) + g(x)\cdot Df(x) + TDf(x)\cdot Dg(x) \tag{3}$$

$$d\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\cdot Df(x) - f(x)\cdot Dg(x)}{g(x)(g(x)+TDg(x))} \tag{4}$$

The second-order time-difference operator is define as:

$$D^2x(t) = \frac{x(t-T) - 2x(t) + x(t-T)}{T^2} \tag{5}$$

Similar finite difference operators can be formulate for any other quantities used in classical physics.

2 Mechanics.

With the approach developed in previous section of this paper, we will reformulate some of the equations of classical mechanics. We follow the approaches developed by[1,3,4,5].

2.1 Newton's Laws.

The standard Lagrangian formulation of classical mechanics leads, in the discrete-time case, to the following version of Newton's laws:

$$F(t) = m\cdot a(t) = Dp(t) \tag{6}$$

with the acceleration, using (5) defined by:

$$a(t) = m\frac{x(t+T) - 2x(t) + x(t-T)}{t^2}$$

and with momentum defined by

$$p(t) = m\cdot Dx(t-T)$$

For classical Lagrangian, $L = \frac{1}{2}m(Dx)^2 - V$ (where V accounts for potential energy), the energy function is defined by

$$E = \frac{1}{2}Dx(t)\cdot Dx(t-T) + V(x)$$

and will be conserved when L is not an explicit function of time. Thus all the standard quantities of classical mechanics have their counterparts in the pre-calculus version.

Several formulations of discrete mechanics have been studied both mathematically and physically[(1−5)]. This formulation were explicit and, for $\Delta t > 0$ and $t_k = k \cdot \Delta t, k = 0, 1, 2, \cdots, N$ utilized, in one dimension, the formulas:

$$m \cdot a_k = F(x_k, v_k, t_k), \quad k = 0, 1, \cdots, N \tag{7}$$

$$a_k = [v_{k+1} - v_k]/\Delta t, \quad k = 0, 1, \cdots, N \tag{8}$$

$$[v_{k+1} + v_k]/2 = [x_{k+1} - x_k]/\Delta t, \quad k = 0, 1, \cdots, N \tag{9}$$

where (7) is the discrete Newton's equation, (8) relates velocity and acceleration, and (9) relates distance and velocity. The feasibility of the formulation from the proof of the classical conservation laws[(5)].

2.2 Newton's Equation and Initial-value Problem.

In a more general case, following[(5)] if we consider a particle P of mass m be in motion on x-axis, and if at time t_k, P is at x_k and has velocity v_k, then and equation of the form

$$F(x_k, v_k, t_k) = m \cdot a_k \tag{10}$$

which at time t_k relates the force acting on P to its acceleration, represent the discrete form of Newton's equation. The actual determination of the motion of P from dynamic equation (15) when x_0 and v_0 are given is called an **initial value problem**.

2.2.1 Conservation of energy and momentum.

For discussion in previous sections of this paper, the general iteration procedure to be used for initial-value problem for (10) is given by

$$
\begin{aligned}
v_1 &= v_0 \left(\frac{\Delta t}{m}\right) F(x_0, v_0, t_0) & \\
v_{k+1} &= \left(\frac{\Delta t}{m}\right) \left[\frac{3}{2}F(x_k, v_k, t_k) - \frac{1}{2}F(x_{k-1}, v_{k-1}, t_{k-1})\right] & k = 1, 2, \cdots \\
x_{k+1} &= x_k + \frac{1}{2}\Delta t[v_{k+1} + v_k] & k = 1, 2, \cdots
\end{aligned}
$$

For this 1-D motion[5], we show how the classical conservation laws, in discrete approach can be established. The work W done by force F in moving particle P from x_0 to x_n is defined by

$$W = (x_1 - x_0)F(x_0, v_0, t_0) + \sum_{k=1}^{n-1}(x_{k+1} - x_k)\left(\frac{3}{2}F_k - \frac{1}{2}F_{k-1}\right) \tag{11}$$

or

$$
\begin{aligned}
W &= (x_1 - x_0)a_0 + \sum_{k=1}^{n-1}(x_{k+1} - x_k)\left(\frac{3}{2}a_k - \frac{1}{2}a_{k-1}\right) \\
&= \frac{m}{2}(v_1^2 - v_0^2) + \frac{m}{2}\sum_{k=1}^{n-1}(v_{k+1}^2 - v_k^2)
\end{aligned}
$$

so that

$$W = \frac{1}{2}m(v_n^2 - v_0^2) \tag{12}$$

If we define the kinetic energy K_i at t_i by

$$K_i = \frac{1}{2}mv_i^2 \tag{13}$$

then, finally from (12) and (13), the discrete form of conservation law is given by

$$W = K_n - K_0 \tag{14}$$

The availability of the usual kinetic energy formula (13) and the well-known result (14) leads, in the usual way[5] to the conservation of energy and momentum. These concepts can be generalized to higher dimensions by similar approaches. Two simple cases of this problem are given in the following paragraphs:

2.2.2 1-D Motion under a constant force.

Consider the case of constant force $F = -k/m$. This implies that the Newton's equation formulated above, can be written as:

$$D^2x(t) = -\frac{k}{m} \tag{15}$$

The solution is

$$x(t) = -\left(\frac{k}{2m}\right)t^2 + v_0 t + x(0) \tag{16}$$

and

$$Dx(t) = -\left(\frac{k}{m}\right)t + v_0 \tag{17}$$

which is a similar solution with that given by the calculus-based mechanics.

2.2.3 The harmonic oscillator.

Considering a linear force $F = k(y + a)$, where a is a constant, law of dynamics in the case of the harmonic oscillator becomes:

$$mD^2y(t) = -k(y(t) + a) \tag{18}$$

We can define $x(t) = y(t) + a$ and simplify the problem to

$$x(t+T) - \left[\frac{T^2k}{m} - 2\right]x(t) + x(t-T) = 0$$

The general solution of this problem is given by:

$$x(t) = K_1 \cos\left(\frac{t}{T}\theta\right) + iK_1 \sin\left(\frac{t}{T}\theta\right) + K_2 \cos\left(\frac{t}{T}\theta\right) - iK_2 \sin\left(\frac{t}{T}\theta\right) \tag{19}$$

where $\theta = \cos^{-1}(1 - 1/2\nu^2)$, and $\nu = T(k/m)^{1/2}$.

By choosing the complex coefficients K_1 and K_2 to make $x(t)$ real, we finally get:

$$x(t) = c_1 \cos\left(\frac{t}{T}\theta\right) + c_2 \sin\left(\frac{t}{T}\theta\right) \tag{20}$$

or

$$x(t) = c \cos\left(\frac{t}{T}\theta + \Phi\right)$$

which is very similar with the continuous version of the solution([2]).

3 Electromagnetics.

The computation of electromagnetic fields, needed for an abundance of everyday applications such as antennas, radars, microwaves, and radio has been a subject of intense research since the 1940s. Since most problems that can be solved analytically have already been solved, numerical solutions of electromagnetic problems are gaining popularity among engineers and instructors. The need of introducing computational methods has been expressed again and again(10,11).

3.0.4 Laplace Equation.

One of the most used application of finite difference approach in electromagnetics(9,12) is to apply it to Laplace's equation in two dimensions:

$$\bigtriangledown^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \tag{21}$$

We can use the central difference approximation (see [**?**]) to obtain

$$\frac{\partial^2 \Phi}{\partial x^2} = \frac{\Phi(i+1,j) - 2\Phi(i,j) + \Phi(i-1,j)}{(\Delta x)^2} + O(\Delta x)^2 \tag{22}$$

$$\frac{\partial^2 \Phi}{\partial y^2} = \frac{\Phi(i+1,j) - 2\Phi(i,j) + \Phi(i-1,j)}{(\Delta y)^2} + O(\Delta y)^2 \tag{23}$$

where $x = i\Delta x,\ \ y = j\Delta y$, and $i, j = 0, 1, 2, \cdots,$. If we assume that $\Delta x = \Delta y = h$, to simplify the calculations, substituting (22) and (23) in (21) gives

$$[\Phi(i+1,j) + \Phi(i-1,j) + \Phi(i,j+1) + \Phi(i,j-1)] - 4\Phi(i,j) = 0 \tag{24}$$

or

$$\Phi(i,j) = \frac{1}{4}\left[\Phi(i+1,j) + \Phi(i-1,j) + \Phi(i,j+1) + \Phi(i,j-1)\right] \tag{25}$$

at every point (i,j) in the mesh. The spatial increment h is called the *mesh size*. It is worth noting that equation (25) states that the value of Φ at each point is the average of those at the four surrounding points. The five-point computation molecule for the difference scheme in equation (25) is illustrated in Figure 1 where values of the coefficients are shown. This is a convenient way of displaying finite difference algorithms for Laplace's equation.

3.0.5 Transmission Lines.

The finite difference techniques are well suited for computing the characteristic impedance, phase velocity, and attenuation of several transmission lines, such as: bifilar transmission lines, coaxial cables, micro-strip lines, striplines, rectangular lines, etc. (Note: a stripline represents a flat conductor sandwiched between two ground plane, while a micro-strip is a flat conductor separated by an insulating dielectric from a large conducting ground plane). The knowledge of the basic parameters of these lines is of paramount importance in the design of electronic circuits(10, 11).

For concreteness, following the development of Sadiku, 1991[12], consider the microstrip line. The geometry is deliberately selected to be able to illustrate how one accounts for discrete inhomogeneities and lines of symmetry using finite difference technique. The techniques presented are equally applicable to other transmission lines. Due to the fact that the mode is TEM (neither E or H have components in the direction of propagation), the fields obey Laplace's equation over the line cross-section. The TEM mode assumption provides good approximations if the line dimensions are much smaller than half a wavelength which means that the operating frequency is far below the cutoff frequency for all higher order modes[13]. The finite

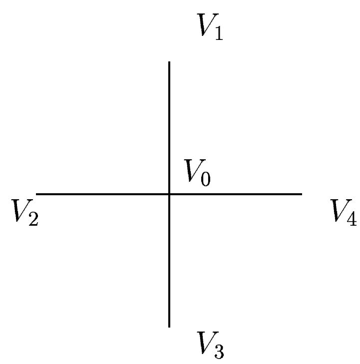

Figure 1: Computational molecule for Laplace's equation.

difference approximation of Laplace's equation has been derived in equation (25), namely:

$$V(i,j) = \frac{1}{4}\left[V(i+1,j) + V(i-1,j) + V(i,j+1) + V(i,j-1)\right] \tag{26}$$

For the sake of conciseness, let us denote

$$V_0 = V(i,j) \quad V_1 = V(i,j+1) \quad V_2 = V(i-1,j)$$
$$V_3 = V(i,j-1) \quad V_4 = V(i+1,j)$$

so that the equation (26) becomes

$$V_0 = \frac{1}{4}\left[V_1 + V_2 + V_3 + V_4\right] \tag{27}$$

with the computation molecule shown in Figure 2. Equation (26) is the general formula to be applied to all nodes in the free space and dielectric region of a micro-strip transmission line. At the dielectric boundary, see the diagram of Figure 2, the boundary condition is given by

$$D_{1n} = D_{2n} \tag{28}$$

must be imposed. This condition is based on Gauss's law for electric field, which finally (see the contour in Figure 2) yields to

$$V_0 = \frac{\epsilon_1}{2(\epsilon_1+\epsilon_2)}V_1 + \frac{\epsilon_2}{2(\epsilon_1+\epsilon_2)}V_3 + \frac{1}{4}V_2 + \frac{1}{4}V_4 \tag{29}$$

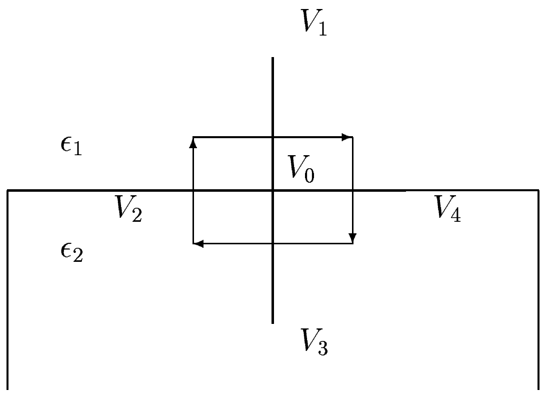

Figure 2: Interface between two media of dielectric permitivities ϵ_1 and ϵ_2

This is the finite difference equivalent of the boundary condition in equation (29). Notice that the discrete inhomogeneity does not affect points 2 and 4 on the boundary but affect points 1 and 3 in proportion to their corresponding electric permitivities.

On the line of symmetry, we impose the condition

$$\frac{\partial V}{\partial n} = 0$$

which implies that $V_2 = V_4$ so the equation (36) becomes

$$V_0 = \frac{1}{4}[V_1 + V_3 + 2V_2]$$

or

$$V(i,0) = \frac{1}{4}[2V(i,1) + V(i-1,0) + V(i+1,0)] \tag{30}$$

By setting the potential at fixed nodes equal to their prescribed values and applying the procedure described above, one can determine the potential at free nodes by iterative methods[(11)]. One this is accomplished, the quantities of interest can be calculated.

4 Discussions and Conclusions.

This paper has presented in simple terms the basic concepts of the finite difference method for solving mechanical and EM problems. Algebras and calculus are the two basic tools of mathematical physics. Classical physics (mechanics, electromagnetics, etc.) can be reformulated using finite difference calculus, instead of the standard calculus. This reformulation is fully rigorous, and in the case of classical mechanics it avoids assuming that space-time is differentiable and thus is conceptually more consistent with intrinsic discrete nature of time and space. Another advantage of the finite difference formulation is the opportunity of involving students in solving non-trivial, real life problems, such as transmission lines, etc., which could be very appealing and challenging to students. Hence, at least in the case of classical mechanics and electromagnetics, replacing the infintesimal calculus with the finite difference calculus seems feasible and desirable.

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