Technical Enrollments and Mathematical Pedagogy

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Problems

In order to make critical decisions about issues such as global warming, health care and pollution, everyone must understand obvious functional relationships. Issues of functional dependence, continuity, rate of increase, concavity, tops and bottoms, all must be understood by more people in a technical democratic society. It is irresponsible to introduce these concepts in terms that are not clear and not student friendly. It is pedagogically unethical and unsound to place delta-epsilon arguments as roadblocks to the study of calculus.

At professional meetings, we are told there is a shortage of engineers and technicians. At the same time, it appears that too many K-12 students are acquiring an aversion to mathematics [5, 16]. The excessively high number of students graduating high school finding mathematics distasteful indicates we are doing something wrong. If we aim to increase enrollments in technological studies, and if it is true that youngsters choose technical studies because they like math, then mathematics must be introduced in a more attractive format. The study of numbers and shapes should be emotionally neutral. There is no reason for either liking or disliking any particular scientific fact. But the discovery of natural facts and patterns of facts is an exhilarating activity and more children should be excited by this exhilaration of discovery. If they are not, we are at fault for failing to provide an effective and more inspiring introduction.

Mathematics in itself has nothing to do with memorization. Calculators and computers can perform general fault-free calculations. Sensitive students must certainly dislike the unnecessary pressure to memorize disconnected and meaningless facts and the pressure to perform calculations with dire consequences for making mistakes. In fact, computational ability does not represent either intelligence or analytical ability. It represents the ability to execute an orderly sequence of rules that obtain the fastest answer even if the student lacks any understanding of the rules. The student really learns that if he just follows the rules, his teacher will reward him with a high grade and the associated praise. In the student's eyes, the entire point is to get it right and continue on to the next question. In the eyes of the teacher and in the eyes of the principal, the system provides well-defined performance goals, which can gain community praise and rewards. However the drill and test environment does not foster creativity or the questioning scientific attitude that should be acquired by students in the STEM disciplines. Any exhilaration in the drill and test system derives merely from the competition and outperforming classmates. This momentary, vicarious thrill is trivial compared with the enduring satisfaction and pride of scientific discovery.

History

In 1807, Fourier published his iconoclastic paper on representations of periodic functions as an infinite series of harmonically related sinusoids. Fourier's paper drastically challenged fundamental concepts of mathematics and resulted in changes including the redefinition of the word ‘function’. Fourier’s ideas enabled the solutions of differential equations but also raised many perplexing and wonderful problems. Over the next century, the finest mathematicians explored these problems. Dirichlet, Cauchy, Cantor, Riemann, Weierstrass and others, in their study of continuity and convergence of series, invented ingenious, counterintuitive counterexamples and produced analytical techniques which culminated in Lebesgue’s magnificent theory of integration about 1906.

During the 1700’s, developments in mathematical theory were dominated by Leonhard Euler. Euler had more mathematical insight, made more mathematical discoveries and had more mathematical fun than anyone else either before or since except maybe Erdos. However, the views of Euler on the nature of mathematics differed sharply from the views of the nineteenth century mathematicians. In fact, he understood a function to be in the constellation of ideas regarding equations in two variables, curves and tables. To quote David M. Bressoud[4]:

> … to the mathematicians before 1807, functions were polynomials, roots, powers, and logarithms; trigonometric functions and their inverses; and whatever could be built up by addition, subtraction, multiplication and division or composition of these functions. Functions had graphs with unbroken curves. Functions had derivatives and Taylor series.

Graphs of these curves clearly showed intercepts, extrema, intersections, rates of change and points of inflection, perhaps without precision. Theorems, which would later require proof in the mathematics community, were considered clearly evident. Any child could clearly see that every continuous curve defined everywhere starting below the x-axis and ending above the x-axis must cross the x-axis somewhere. Similarly, any child could look at a continuous, smooth curve and understand, without symbolic algebra, or derivative notation or deltas and epsilons, that at a maximum point, the tangent line was horizontal. In other words, at the peak of a smooth curve, the tangent line does not tilt. What a wonderful way to approach and view the content of K - 12 algebra and calculus.

To quote Bressoud[4] again, “Fourier’s cosine series flew in the face of everything that was known about the behavior of functions. Something must be dreadfully wrong.”

In order to explore the issues of continuity and convergence of infinite sequences of functions, the followers of Cauchy required that functions be single-valued. Mathematicians settled on the definition of ‘functions’ in abstract terms as single-valued mappings and ordered pairs. Additionally the function concept was extended to include wild domains, discrete, dense, disoriented, multi-dimensional and their combinations. Not only was function redefined, but in addition, continuity, differentiability and integration were consequently defined in terms of deltas and epsilons. For the working mathematician, this approach was right. It was fantastically successful in promoting the development of new mathematics. General techniques were developed that worked for all kinds of wild combinations of functions by applying essentially the same definitions and mechanisms of proof.

Nonetheless these definitions took the focus of calculus away from intuitive smooth curves. These 19th century definitions were and continue to be meaningless for many students and therefore it is time for the STEM community to re-examine the effect on the students of these definitions.

Because points of view and techniques differed so sharply before and after the year 1807, from this point forward in this paper, the mathematics preceding 1807, will be referred to as Euler's calculus and the subsequent developments as Cauchy's analysis. The new analysis of Cauchy and the new definition of functions called for clear descriptions of the domain and range of the functions. But for the algebraic functions of calculus the domains are relatively simple, excluding the isolated zeroes that occur in the denominator and the "excluded intervals", which occur when taking an even root of a negative value.

Euler, being unencumbered by the "monstrosities" developed during the 19th century, could rely on his intuition to develop into "the master of us all." However, the ingenious examples and techniques invented during the 19th century led mathematicians to distrust Euler's intuition and visualizations. By the mid 20th century, visual interpretations were considered misleading and were disparaged in math classes.

The success of Cauchy's analysis led to authoritative abuse. A student who is not considering Fourier series has no reason for these definitions. Why should a student want to study ordered pairs. Why should a student, who may not understand what a transcendental number is, want to follow a delta-epsilon argument? On what basis could a student object to or question the authority of both the teacher and the text? Any student who questioned or objected to the concept of "ordered pairs" was considered as either unintelligent or mathematically immature. If students dropped out, it did not matter; other students, more obedient, would take their place.

By the late 20th century, mathematicians noticed that too many students were dropping out and that changes were needed. The reform movements were initiated. The admission that graphing was needed was a helpful step. Additionally helpful was that tables of functions remained as useful in applications. Digital calculators and computers were now widely available so they could contribute to the reform whether their use constituted conceptual reform or not. These instruments changed mathematics by eliminating much of the drudgery of calculating, allowing us to address problems of conception. However, effective pedagogy requires something more that the mathematicians missed.

Mathematicians believe that they are disseminating the "truth." But nothing is disseminated when the "truth" is not understood. Answers are provided to questions of which are beyond the students' horizon. Issues of convergence of Fourier series do not appear in K-12 math or even in the first calculus course. In addition, issues of functions defined only on the rationals, or say the Cantor ternary set, are not yet in view either. A student unfamiliar with the place of irrationals on the real number line is not in a strong position to approach delta-epsilon arguments. Providing a student with definitions capable of treating all the "monstrosities" of Cauchy's analysis denies him the ability to understand the intuitive problems at hand.

Every mathematical concept has a story that provides its meaning. If the narrative is not related correctly, the student will be left confused. In too many texts, concepts are given abstract definitions, which appear seemingly out of nowhere, that are designed to be the foundation of the proofs that the mathematicians admire and some students may need later. But when the introductory story is needed, it is missing. It is possible to make 'true' statements about a new concept which do not disclose meaningful information to the student. The student, confronted with too many 'true' statements which do not make sense, will walk away and select a non-technical career where the courses may be found to be more understandable and interesting.

On the other hand, do engineering students require the abstract definitions of Cauchy's analysis? No. Most of today's engineering work requires an understanding of continuous smooth curves and surfaces. Euler would feel quite at home in many of today's beginning engineering and physics classes. True, electrical engineers, later on, will need the concepts of Fourier series to study periodic signals. And even the Fourier series, initially, have a visual interpretation and Gibb's curious phenomena can be introduced without a formal proof but with a warning. There are rarely any formal proofs on the P. E. exams. Engineers need to be aware of the interval of convergence of Taylor series. At some later time, engineering students will need to know that the Taylor and Fourier series are forms of functions and will need to learn the properties of these forms and how to manipulate them.

It must be accepted that everything cannot be taught at the same time. Topics must be lined up and presented in an order that is sensible to the student. Show what is simple and clear first. The introduction must be student friendly with the proper context carefully described. The algebra/calculus sequence can be divided on a historically conceptual basis. The first pass could be Eulerian calculus with the visual and intuitive definitions that prevailed during the 18th century. Algebraic curves, finitely multi-valued functions, should be the basic concept. In this study of algebraic curves, a straight line crossing a circle will naturally have two intersections.

A differentiable curve at every point has a unique tangent line. The derivative at a fixed point should be defined as the slope of this tangent line. This definition is not watering down. This definition is clear and more students may be receptive to it. This approach delays the premature introduction of the concepts and notation of limits, deltas and epsilons until after the student is comfortable with the concept of the derivative. After a student has verified the differentiation rules on polynomials and rational functions and used the derivative, he may be better suited to tackle the limit concepts. Until a student understands that every transcendental number can only be written as a limit of sequences of rational numbers, the student is in no position to understand any derivations regarding limits. A student who has just been introduced to the word, limit, is simply in no position to appreciate limits of difference quotients. The limit of the difference quotient is just a method of obtaining the value of the slope, not the derivative itself. There is much to be learned about curves with such an approach without belaboring the difficulties inherent in limit processes. The integral of a positive piecewise monotonic function should be defined as the area under the curve and again the limiting process should be viewed as just a method of obtaining the value.

All references to the concepts and perplexities of the 19th century Cauchy's analysis should be postponed until the series forms are confronted head on, and the issues of convergence naturally associated with limits of functions, sequences, and series forms arise.

Many students have problems mastering mathematical proofs. The meticulous techniques of logical proofs, that mathematicians are trained to perform in graduate schools, are not as valuable to engineers and technicians as understanding the nature and use of functions. An engineer should know that when two continuous functions are added, the resulting function will be continuous. It is not necessary the engineer be able to prove this statement.

Engineers should be taught to recognize when proofs are needed and in such cases to consult a mathematician. In many engineering curricula, it has been decided that most engineers did not have to be expert software programmers and that software should be bought. Treat similarly, the art of mathematical proof.

Bring Back Euler's Analytic Geometry

Suppose we dropped the course pre-calculus and replaced it with a course called "cartesian curves" or analytic geometry. Good courses have a focus and the focus of this course would be an exploration of the relationship between equations in two variables and their curves as graphed in the Cartesian plane. The curves could then be sorted by the forms and operations on their equations. This classification would result in the curves and peculiarities of polynomials, rational functions and the algebraic curves. The curves should be studied in this sequence.

Polynomials have the curves that result from adding and subtracting power functions that have been stretched, compressed and flipped. Rational functions result from dividing polynomials and algebraic curves derive from the implicit form defined by two-variable polynomials set equal to zero. Curves of rational functions and polynomials are single-valued, defined everywhere except in the case of rational functions at the few discrete points where denominators are zero. These two classes of curves are continuous and differentiable (smooth) except where denominators are zero. Algebraic curves are continuous and smooth but may be multi-valued. Simple algebraic curves are not defined at the few discrete points where denominators are zero and also are not defined on those intervals, called excluded, where even roots of negative values must be taken.

There is much to be discovered in such a course. The intersections of algebraic curves with a straight line cannot exceed the degree of the equation. Students should encounter the strategies of finding these intersections. The intuitive properties of these curves can be explored and the methods of determining these properties. Students will learn to apply the rules of differentiation to find slopes of tangent lines, extremes and points of inflection. Students will learn to apply the rules of integration to compute areas and the other usual applications of integration. Numerical methods can be used to confirm the computations when exact integrals prevail and to evaluate the integrals in the many cases where there is no exact form for the solution. Of course, parametric and polar forms for curves could be included. A student could discover and test the power of differential and integral calculus by finding the extremes, arc lengths and areas of the cycloids, cardioids, lemniscates and the other classic curves.

It is poor pedagogy to provide a student with an algebraic proof when this student has just learned the rules of algebra but has not yet obtained confidence in algebraic methods. This plan provides the student with as many opportunities as possible to practice algebra and to discover the power and limits of algebraic methods. Presently, the inexpensive availability of calculators and graphing utilities, such as WINPLOT® eliminates the calculating drudgery that prevailed before the mid-twentieth century. But students still need to become familiar with analytical concepts and strategies. This course in algebraic curves would serve this purpose.

In this course, there is no need for the premature emphasis on domains and ranges. The graphs show the domains and ranges of many of the common examples. Simply, do not divide by zero or take even roots. Curves should be studied in their entirety before the introduction of piecewise defined functions.

The premature introduction of the piecewise-defined functions represents computational clutter. After students understand entire algebraic curves and after students are familiar with interval notation, they will find it easy to cut and reassemble pieces of functions. The techniques of solving, differentiating and integrating piecewise-defined functions will become obvious. The premature computational details involved with the piecewise-defined functions do not clarify the function concept.

Areas of Concern

Mathematics is wonderful. Students must be introduced to that wonder. This study need not be either a chore or a bore. If faculty cannot demonstrate the wonder, at least let the texts be more student friendly. Consider the following areas of concern: tone, language, definitions, organization, strategies, forms and visualizations.

Tone: These questions are sensitive but have to be considered if analytical enrollment trends are to be improved. Are the teachers showing the students the wonders of mathematics in a way that the students can appreciate? Can society expect teachers who do not enjoy or understand math not to transmit their apprehensions to the students. Does the teacher's voice and gestures convey an honest appreciation of the subject? A teacher who thinks math is obscure, abstract and difficult will not be able to hide these concerns from the students. Art teachers are openly enthusiastic about drawing. Why should a student who is sensitive enough to pick up these negative cues consider an analytical career? The nation's slogan should not be, "No child left behind" but instead should be, "Every child should have, in every math class, a teacher who enjoys math and knows wonderful things to do with math." Does our country have enough knowledgeable and capable math teachers?

Are teachers asking the students about the reasoning that led to incorrect answers or simply grading them as incorrect and thereby labeling the students in their minds as incapable, and subsequently, crushing the student's spirits?

Language and Names: Care must be given to names. The principle is that <u>the concept should be clear before it is given a name and the name should clearly describe the concept</u>. A teacher should never say, "Now I am going to tell you about π. Instead, the student should observe that for all circles the ratio of the circumference to the diameter is always the same number, a bit larger than three. Since the number is not a simple fraction or an algebraic number, society has given this number a name, π. Thus the name is introduced in order to discuss a concept which at the moment lacks a name. A student might then observe that the ratio of the area of any circle to the area of its circumscribed square is slightly more than ¾. An ingenious argument can lead to the value $\pi/4$.

Concepts should be named, not after people, but after their place in the study. The Pythagorean Theorem could just as easily be called the right triangle relation, or the slant or diagonal distance formula. Will a student who cannot pronounce Pythagorean care about the triangular relationship? If the name presents an obstacle, is it worth keeping?

Courses should be named after the concepts studied in the course. Drop arithmetic, algebra, pre-calculus, calculus, math 101, and sequential (the New York State name for high school math courses) as names of courses. Name these courses: numbers, continuous functions etc. The study of chemicals is called, appropriately, chemistry. What should a student know after taking sequential II? Is pre-calculus all the junk a student should memorize before he is prepared for calculus? Mathematics course names do not illuminate the course contents.

Consider the two statements which some may consider as saying the same thing:

A. At a maximum of a differentiable function, the derivative is zero.
B. At a peak of a smooth curve on a coordinate system, the tangent line is horizontal.

Statement A can be found in every calculus text. Its understanding relies on the definition of the word function and delta-epsilon arguments required in the definitions of the words differentiable and derivative. Students may not see statement B in a calculus text. A student who interprets the word, function as a curve, and who interprets the word, differentiable, as continuous but without corners, will see that the tangent line at the peak must be horizontal. The following sentence provides a visual justification of statements. Take a horizontal line above the curve and lower it; the first point that the descending line touches the curve is the peak.

Definitions: Mathematical definitions[10, 12] provided in the usual texts are not descriptive. After reading a mathematical definition, many students are left wondering, "What, on earth, does this mean?" These definitions are employed to make proving theorems easy, not conveying concepts.   Concede the conventional definitions are turning students away and seek other student friendly definitions. Examine the following concepts whose definitions are in need of reconsideration: variable, limit, polynomial, inverse function and function.

Students embarking on a study of algebra must confront the word variable, usually defined as a letter representing a member of a set. In a study of rectangles, the length, width, area and perimeter, all belong to the set of positive real numbers. Can an area be added to a length? Can the students be blamed for being confused?

Variables are symbols representing measureable properties of systems. The concept is a notational device for writing the laws of these systems. Consider the set of rectangles as our system to study. The laws are relationships of the system, in this case:

1) The area of any rectangle is the product of its length and width and
2) the perimeter is twice the sum of the length and width.

Call the length, L, the width, W, the area, A and the perimeter, P. Then in algebraic notation these laws become: 1) $A = LW$ and 2) $P = 2(L + W)$. This algebraic notation is a wonderful invention. Not only are laws written more compactly in algebraic notation, but the algebraic notation is easily manipulated. The relationship, $A = LW$, can also be modified by the rules of algebra to $L = A/W$ and $W = A/L$. English sentences cannot be manipulated or combined so easily. There was no need to call out that the variables belong to the set of positive numbers. If a computation resulted in the value of one of these being negative, it would certainly be noticed and a search for the error would follow.

Look at the commutative and distributive laws of arithmetic. How would they appear in ordinary English? Look at the Pythagorean Theorem of geometry. Look at the simple laws of electricity and projectile motion. How would they appear in ordinary English? Consider the laws of factoring polynomials. There is no doubt that writing the laws in variable notation is a significant improvement over common language.

The concept of limits can be introduced naturally without referring to deltas and epsilons. Rational functions exhibit a problem when the numerator and denominator polynomials have common zeroes. Common zeros mean these polynomials have common factors and so these functions will have the form of “zero over zero” and are said to be “undefined” at these isolated points. There are three cases depending on a comparison of the numbers of the common factors in the numerator and denominator.

1) When the numerator and denominator have the same number of common factors the function has a point gap. The common factors can be cancelled and the resulting function evaluated to yield the vertical coordinate of the point gap. The value of the vertical coordinate is called the “limit” of function at the point gap.
2) When the numerator has a greater number of common factors, again the common factors can be cancelled and the resulting function evaluated. The vertical value of the point gap will be zero. The function is said to have a limit of zero at the point gap.
3) When the denominator has a greater number of common factors the function has a vertical asymptote and is described as not having a limit. The function values become unbounded for horizontal coordinates surrounding the problem point. If the number of common denominator factors remaining after cancellation is odd, the values of the function will differ in sign on each side of the problem point like the function $y = 1/(x - 2)$. If the number of common denominator factors remaining after cancellation is even, the values of the function will have the same sign on each side of the problem point like the function, $y = 1/(x - 2)^2$.

Students are told sometime in junior high school that the multi-variable expression, $3a^3b^2 + 5x^4y^2 - 2p^7q$ is a “polynomial.” This expression could be called, more aptly, “sum of products” form. Polynomials are functions and if functions are not under consideration, the word, polynomial is out of place. Polynomials should be classed with, but contrasted to rational functions and algebraic curves. Students do need the idea that the distributive law can be applied to the sum of products form. The use of the word, polynomial, here represents unnecessary verbiage; it is just another big word for students to memorize so that they can appear smart. It does not advance their understanding.

Consider another outrageous definition. Say we are interested in the inverse function of the dependence of area on the radius of a circle, $A = \pi r^2$. At least one very popular pre-calculus text, now in its 8^{th} edition states, to obtain the inverse function, solve for r and interchange the variables A and r. This would yield $A = \text{sqrt}(r/\pi)$. How is a student supposed to make sense of this? How does a teacher explain this to a class and is it fair to require a student to memorize this nonsense? If x represents a variable in the domain of the function, $y = f(x)$, then y will be a variable in the domain of the inverse function f^{-1}. Did anyone in the math community voice a complaint on its appearance in the first edition?

Consider one more example, the important concept of function. Ordered pairs and mappings with domain and ranges do not convey the essence of a ‘single variable continuously controlling a second variable’; the functions which are studied in calculus. Every text clearly states the “functions’ are single valued, with a domain and a range. Students are taught to take pride in acquiring the ability to perform the simplistic vertical line test and therefore be able to distinguish functions from non-functions. We could just as well take the algebraic curve as our basic concept in which case algebraic equations in two variables can be conceived as multi-valued curves. Equations could then naturally represent circles and lemniscates. Vertical and tilted parabolas can be treated similarly.

Students should first gain familiarity with each curve in its entirety. There will be time later, when the need arises, on a second pass, to consider chopping the curves and reassembling pieces. Domains and ranges are not major features of elementary functions or curves. The continuous functions, $y = |x|$, $y = x^{(2/3)}$, $y = \cosh(x) - 1$, $y = |\text{sqrt}(1 + x^2)| - 1$, $y = x^2$, $y = x^2 - 2x + 1$ and $y = e^x \sin^2(x)$ all have the same domains and ranges and yet are very different functions. Functions and their associated curves will not exist when a division by zero or an even root of a negative number is encountered. Bring back excluded intervals. Other than that, there is no need to emphasize domains and ranges until the study of series forms and Cauchy’s analysis. Stress the shape of the curves, the relationship between the variables. For pedagogical reasons, we must resurrect “analytic geometry.”

Organization: Math textbooks are too thick. In providing every professor’s favorite example, these books have developed into a maze where students can easily get lost. How can a student find the most important idea in the course?” What are the limitations of algebra? Must a student read every page in order to acquire an overview and to learn these subjects?

The discovery of the Periodic Table was a great aid to the study of chemistry by providing a chart that predicted where elements were missing and predicted the properties of elements. The real number system as studied in elementary schools has a wonderful algebraic structure [13] of which too many students appear unaware. The structure is based on whole numbers and appends the complete number systems allowing for the operations of negation, division and roots of algebraic equations, yielding the integers, the rational fractions and algebraic numbers. The other numbers needed to complete the real line are called transcendental. These structures are well known by mathematicians but are not employed to help students understand number relationships. The statement that numbers in the real number system and points on the real axis homogeneously correspond to each other (while true) avoids and, therefore, obscures this wonderful, intrinsic algebraic structure.

After studying arithmetic, students should be able to perform the basic number operations on all the kinds and forms of numbers. After studying algebra, students should be able to perform the basic algebraic operations on functions. Do our calculus students know the kinds or forms of numbers or the difference between kinds and forms?

Numbers Systems	Function Systems	variable exponent
Counting Numbers		
Integers	Polynomials	Non-negative
Rational Fractions	Rational Functions	Integers
Algebraic Numbers	Algebraic Curves	Fractions
Transcendentals	Transcendental Functions	

The structure[7] of the functions of algebra parallel the structure of the numbers in arithmetic distinguished by the same algebraic operations; counting number powers of x, (polynomials); negative number powers of x, (rational functions); fractional powers of x (algebraic functions) and other functions having series forms (transcendental). The statement that functions are homogeneous ordered pairs (while true in a sense) also avoids and therefore denies the algebraic structure of the elementary function systems. The techniques of manipulation (except for tabular form), the graphs and uses of the different functions are very different. It is pedagogically misguided to treat all functions uniformly.

But shouldn't the transcendental functions be introduced as soon as possible in preparation for physics? As long as the students understand the structure and understand that the topics are out of order, they should be able to backfill and complete the structures in their mind at their own rate. What is important is that students should not be presented with a hodge-podge of "true" and uniformly incoherent facts but see a progressing development, the exploration of the concept of functions.

Strategies: Solving problems requires a strategy. A student who understands the strategy is not thinking in terms of steps and has no need to memorize steps. But if the student is only shown a sequence of steps, he is forced to rely on these steps in the presented order and may never comprehend the strategy. If there are too many steps, the student will tire eventually and give up in despair. What our society really needs is young students who have the ability to create strategies. Mathematicians understand that strategic thinking is not a buzz word.

Forms: The concept of forms[9] pervades all of mathematics and theoretical engineering, but is not granted its deserved prominence or respect in elementary mathematics. It is natural that mathematical objects can be described in different forms. The concept should be introduced in arithmetic, highlighting the forms of numbers and continued into algebra, highlighting the forms of linear, quadratic and the other functions. The strategy of addition of fractions is easily explained in terms of forms. The manipulations of factoring and completing the square make sense in terms of forms. The concept of forms works in the strategies of solving problems. Perhaps, the form in which the problem is stated can be changed to a form where the answer is obvious. Then the algebraic manipulations must be performed which will change the stated form to the solution form.

Visualizations[15]: Engineering students are taught visual techniques to confront engineering problems. Electrical Engineering students see their signals (voltages and current functions of time) on oscilloscopes. Civil and mechanical engineering students visualize in their minds, loading, shear and moment diagrams in order to predict deflections. Freebody diagrams are essential to the solution of statics problems. It is wrong to deny engineering students the advantages of visualizing math functions in their math courses. But the Dirichlet function and the other "monstrous" functions that scared the 19th century mathematicians cannot be visualized. Limit the first course in calculus to the piece-wise continuous and monotonic functions that the engineering students need and are able to visualize. Let us concede it is unsound to teach everything all at once and let these functions be studied in the order they were discovered by mankind. When function sequences and series are studied, and the wild 19th century functions are revealed, the students will be better prepared to understand the problems posed by these sequences and the methods that were developed to deal with them.

It is true that problems in the fourth and higher dimensions cannot be visualized, but there are many problems where two and three dimensional visualizations can suffice. A conventional technique in multivariable problems is to hold some variables constant so that the problem can be visualized in two or three dimensions.

What should we do?

Although many may view that this paper has been rather harsh in laying bare some of the common defects in our conventional mathematics curriculum, it is no harsher than the article[16] by P. C. Kenschaft which appeared in the newsmagazine MAA FOCUS. She states:

> Too often elementary school teachers teach incorrect "mathematics" and also communicate to their students that that mathematics is too difficult for ordinary mortals. 'If my teacher doesn't understand this than I can't either.' Such intellectual and emotional damage is so devastating that even a teacher who is mathematically competent will find it very difficult to undo. High school and remedial college faculty must overcome much more than lack of knowledge.

Too much that is incoherent is forced on innocent students. How irrationally cruel! Why does our nation accept the combination of seemingly meaningless definitions and unknowing teachers with the rigorous unrelenting testing of 'No Child Left Behind?'

Consider the argument to maintain the status quo. No one appears visibly hurt. Most everyone in our technical workforce has spent some fifteen years in schools competitively memorizing definitions and algorithms in order to obtain the credentials and their first job. The subsequent fifteen years are spent mentally discarding trivia and nonsense and realizing how little was understood. They re-learn and re-organize what is needed to perform their jobs. It is believed, as it appears in other disciplines, that the memorization system works well.

Memorization, in itself, is not harmful. However, students who feel clueless about math will be tempted to employ unethical methods to pass. These methods, if successful, will be continued into their working careers. Is this what our society wants our schools to convey? In addition, our future scientists miss the excitement of discovery and research in their youth. Our scientists lose the opportunity in their youth of playfully exploring strategies and developing the enquiring attitude that a technical society requires. This system of memorization is disastrous.

The thrust to accelerate math education provides another opportunity for disaster. Why should parents take the chance that their children will obtain a better introduction to calculus in K-12 than will be provided in a college calculus course? If the only consideration is that the students will be memorizing a more advanced subject, then it may make no difference. What is gained by hustling students to learn differential equations when they may not yet have mastered the fundamental concepts of college algebra and calculus? Isn't it better that students acquire the attitude that they could invent calculus even if it takes more time?

To imagine ameliorating the harsh ways math is introduced may be just daydreaming. Mathematics at all levels should be seen as play, exploring the universe to understand how things work. Fantasy computer games which are in vogue should appear silly compared to the enduring research of a natural science. Any child with a hundred pennies and an inquisitive mind or some prompting from an adult can undertake mathematical explorations, which constitute genuine research. Under what conditions can rectangles be formed? Squares? Triangles? Hexagons? Do the sequences of numbers form patterns? We will never nurture inquisitive minds if children are never in a position to wonder.

A student, who asks, "Why do I have to learn this?" deserves an answer that makes sense. An answer sometimes provided is "cause you will need it next year," The question is being evaded. No one else may hear this answer but for this student, damage is done. It means every teacher must be sensitive to every question and treat the question with respect. College teachers may not always do this. Teachers who systematically treat student questions with respect should be sought after and rewarded. Teachers should be prepared to honestly answer, "I don't know." or "What can we do to find out?"

Nothing will change overnight. The tone, the names, the definitions, the organization, strategies all must be reviewed and accepted by the communities involved: K-12 teachers, mathematicians, engineers, industry and parents. Do we want our society to be comprised of one large group who is adept at memorizing answers and another much larger group who hate anything analytic? Is it possible to increase the number of curious, inquisitive playful researchers, those who would most likely contribute positively to the nation's pool of mathematicians, engineers and scientists?

Let's be clear. Changing mathematics is not being proposed. The discoveries of the mathematicians stand on solid ground, constantly under review. But mathematicians do not perceive that their emphasis on 'proving' represents an unreasonable barrier to the average student. This emphasis must change. Mathematics plays a bigger role in society than serving as an arena to practice mathematical logic.

There is no reason to expect that the mathematical needs of engineers would coincide with the goals of working mathematicians. Engineering students deserve a different introduction to mathematical ideas than that offered by our conventional textbooks. Engineers and technicians must be able to use mathematical concepts and facts to perform their duties. They do not need to be able to prove everything or to have the material presented in an incomprehensible form, selected to make proving easy.

Calculus is neither abstract nor imaginary. From the student's view calculus is initially unknown, remaining to be discovered and explored. There is nothing mystical about the curves studied in calculus. The curves can intersect, rise, fall, and have peaks and valleys. The curves turn up, turn down and can have tangent lines and inflection points. If two curves intersect at two points, there will be an area between the curves. Pick two points on the curve and there is a distance along the curve between the points. Curves that are rotated can have volumes and surface areas of revolution. Students could be told this at the beginning of the course and that they will learn the theory and techniques associated with computing these features.

In the interests of societal needs and increasing the analytically educated populace, we need to modify the organization, timing and delivery of the presentation of the mathematical concepts and facts. Drop "no child left behind" with its drilling and testing and instead provide our children with an introduction to the spirit of mathematics as it may have developed in Euler's mind. That is, the mathematics to which we introduce the nation's youth should deal with quantitative problems immediately confronted by the students and should be treated as research.

The engineering and technology communities who see declining enrollments may welcome any improvement in the conventional introduction of mathematics. The grade school students, teachers, and even parents might welcome having more understandable texts and activities. The mathematicians who seem to have spearheaded the last reforms are not going to surrender, easily, the verbiage and truths that they so painfully mastered in their graduate schools. The mathematicians have too large an investment in the art of proof and in the status quo. However, ultimately and unfortunately it is the mathematics faculty who select the textbooks. If we care about increasing enrollments then we, the engineers, the technicians and the researchers of the analytic sciences must bring these concerns to the attention of the mathematics societies.

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