Tensor Concepts in the Engineering Curriculum

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Abstract

Many concepts studied in engineering undergraduate curriculum contain inherent tensorial character, such as moments of inertia in Statics and Dynamics, stresses and strains in Mechanics of Solids, stress and strain rates in Fluid Mechanics, Maxwell's stress tensor in Electromagnetics, and momentum flux in Continuum Mechanics and Physics. This tensor nature is inherent in the study of anisotropic media, thermal conductivity, linear thermal expansion, stiffness, compliance, electrical conductivity, dielectric permittivity, and magnetic permeability. These topics are all taught in undergraduate engineering courses. However, in institutions where the mathematical preparation for these topics is limited to scalar and vector quantities, there exists considerable effort to contort the mathematics to force the physics to fit the restrictions of scalar and vector quantities, whereas the correct and more easy to understand mathematics requires tensor constructs in most useful cases. Such restrictions are applicable for the most mundane cases often leaving students confused for example understanding stress and strain as six component vectors as opposed to introducing the undergraduate student to the correct and more readily understood tensor of rank two. This is not an add-on the already busy schedule of the engineering student and instructor, but is taught in lieu of the approach that is often used.

Keywords as introduced to undergraduate students

Euclidian space is simply the three-dimensional space that we live in. **Scalars** have no directionality, such as temperature and pressure, and can be introduced as tensors of rank zero having one component; **Vectors** have three components in Euclidian space, such as force, velocity and electric field intensity and can readily introduced as tensors of rank one; **Dyadics** are tensors of rank two and in general have nine components in Euclidian space, such as stress, strain and the permittivity between flux densities and field intensities in anisotropic media.

Introduction

Engineering education and engineering practice are dealing with many objects having multiple directionality nature. It is easy for the undergraduate to understand that scalar quantities have no directionality, such as temperature and pressure. In teaching the technical student about quantities such as force and velocity the teacher finds it necessary to introduce a single layer of directionality, by adding a direction to the magnitude of the vector. These concepts are understood at lower-division college level. As the average engineering student enters the Junior level he or she is ready for the next step, namely, that some quantities lend themselves to be better understood as having multiple levels of directionality. For example in mechanics of solids the tensile stress and strain may be different from the sheer stress or strain. Therefore it is convenient to introduce a second level of directionality. Many quantities that have this dual level of directionality are more easily understood by the upper-division and first-year college student when the dyadic is introduced. The dyadic, which is a tensor of rank two, is the mathematically correct and more strait forward way to bring the student into the reality of dual directionality of some quantities. This is not an add-on to the already busy schedule of the engineering student and instructor, but is taught in lieu of the approach that is often used.

In the Statics and in Dynamics courses students meet the concept of moment of inertia, which, if taught as a tensor of the second rank (dyadic) is easier to understand than the contorted six-dimensional vector approach that is often used in order to avoid any discussion of tensors. This adherence In the Kinematics course, students learn that the movement of element of the medium can be subdivided to translation, rotation, and deformation. The last two are symmetric and anti-symmetric parts of the dyadic. In the Fluid Mechanics course students deal with strain rate which is more correctly the dyadic. Thermal and Mass Transfer are dealing with dyadics of conductivity and diffusivity in the case of anisotropic media. Physics and Engineering are using concept of momentum flux density, which is dyadic. Electromagnetics is dealing with Maxwell's stress tensor of the electromagnetic field. Electrical conductivity of crystals described by dyadic, taught to the upper division undergraduate.

Piezoelectric effect is described by tensor of the third rank. In the Mechanics of Solids course students meet concepts of stress (dyadic), strain (dyadic), stiffness and compliance of material (tensors of the fourth rank), taught to the first-year graduate. These latter two examples where the higher-rank tensors play an essential role render the mathematics much more readily understood once the student and instructor has been brought into the realm of the dyadic and beyond.

Development of Tensor Concepts beyond the Sophomore Level

Authors of this report taught elements of tensor algebra and tensor calculus for years in engineering courses at different levels: Undergraduate: Statics, Dynamics, Mechanics of Solids, Electromagnetics, Optical Communications. Graduate: Fluid Mechanics, Theory of Elasticity, Plates and Shells, Mechanics of Composites, Advanced Electromagnetics, Finite Element Methods, Stress Analysis, Structural Analysis, and Advanced Engineering Mathematics.

One of the authors published a textbook [Reference 1] devoted to introducing tensor concepts to undergraduates. When the law of recalculation of the components of a dyadic is derived in the framework of low level course, it is applicable in all courses, where dyadic concept is used and absence of necessity to derive it again and again for particular application gives free lecture time. This time savings allows for including the introduction to tensors in the third math course for engineers as a substitute rather than an add-on as stated above.

The unit vector and vector are simply introduced as in Section 1.1 through 1.1.5 of Reference 1. The dyadic is introduced in Section 1.1.6 together with five examples in undergraduate engineering topics. *"Each of the nine components of the dyadic has a magnitude and a dually directed unitary dyadic called a unit dyad just as does each of the components of a vector have a magnitude and a singly directed unitary vector called the unit vector."* This section importantly introduces the concept of tensor rank: *"The quantitative property of a tensor that specifies its directional compoundedness is 'rank', Thus dyadics are tensors of 'rank two' because of their dual directivity. Similarly, vectors and scalars are also tensors but at rank one and zero, respectively, because vectors have single directivity and scalars have no directivity."*

Section 1.1.7 of Reference 1 introduces tensors and the various ways that tensors are portrayed. Chapter 3 gives an "Elementary Tensor Analysis" that is taught as part of and in lieu of the third math course for engineers with no addition to the already fully loaded schedule.

The components of the invariant vector are recalculated from original orthogonal system of coordinates to components in rotated system of coordinates according the formula:

$$a_j = \sum_{i=1}^{3} \beta_{ji} a_i; \qquad (1)$$

Here β_{ij} is the cosinus of the angle between *i*-th axis of the original (black) system of coordinates and *j*-th axis of rotated (red) system of coordinates (Figure 1).

Equation (1) guarantees that the sets of components a_i and a_j represent one and the same vector. It is invariance with respect to rotation of the system of coordinates. Invariance with respect to translation of the coordinates is achieved by using differences of coordinates. Other invariant objects can be built by using formally a generalization of the Equation (1)

$$a_{ijk...} = \sum_{u=1}^{3} \sum_{v=1}^{3} \sum_{w=1}^{3} ...\beta_{iu}\beta_{jv}\beta_{kw}...a_{uvw...} \qquad (2)$$

Each object is called "tensor". Total quantity of symbols in subscript in the left side of the expression is equal to the *rank of tensor (r)*. Here *i,j,k,...;u,v,w,...* are sets of non-repeated symbols.

The number of components of a tensor in 3D space is $N=r^3$, where r is the tensor rank. Thus the scalar is a tensor of rank 0 and has 1 component, the vector is a tensor of rank 1 and has 3 components, and the Dyadic is a tensor of rank 2 and has 9 components.

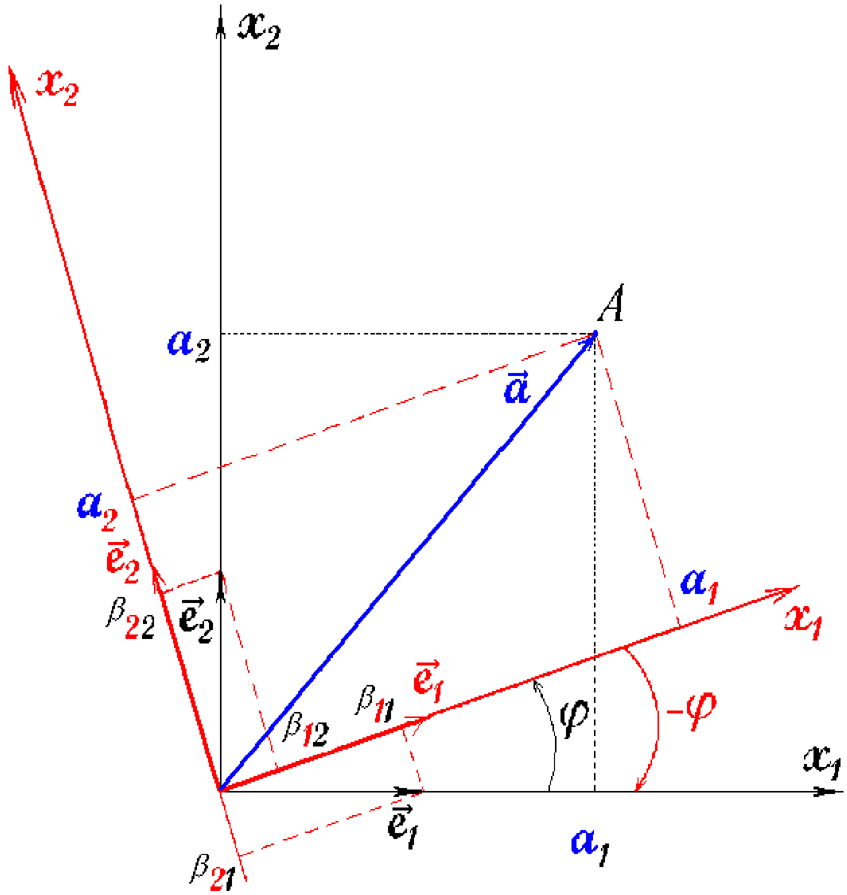


Figure 1. Representations of the same vector in two systems of coordinates and recalculation of components of the vector from one system to another one.

This is the first formal definition of tensor. It will be used for verification of tensor nature of following objects and for evaluation of the rank of them. The physical meaning of dyadic is illustrated by example of electric conductivity of a crystal. Ohm's law for isotropic medium can be written as:

$$\vec{j} = \kappa \vec{E} \quad (3)$$

where $\vec{j}$ is the electrical current density, $\vec{E}$ is the electrostatic field strength, and κ is the specific conductivity of the medium. This law is derived from Ohm's law for a wire:

$$J = \frac{\Delta U}{R} \quad (4)$$

where J is electrical current, ΔU is the difference in voltage, R is the resistance, which is estimated as

$$R = \rho \frac{l}{A} = \frac{1}{\kappa} \frac{l}{A} \quad (5)$$

Here ρ is the material specific resistivity, l is the wire length, and A is the cross sectional area.

$$j = \frac{J}{A} = \kappa \frac{\Delta U}{l}; E = \frac{\Delta U}{l} \qquad (6)$$

The form (3) of the law is independent of the geometry of the electrical conductor. When the electrical field strength vector (gradient of voltage with opposite sign) is applied to isotropic medium, the vector of current density is oriented in the same direction. Another story if the medium is anisotropic (see Figure 2).

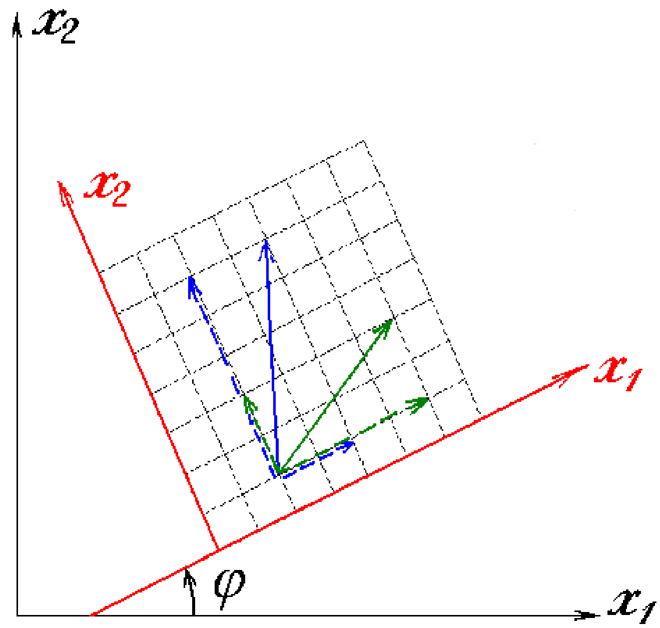


Figure 2. Vector of electric field strength (blue) and vector of density of electric current (green) in a crystal.

Applied vector of electric field strength $\vec{E}$ (blue on Figure 2) can be decompose along the planes of symmetry of the crystal. Let's suppose that the conductivity along x_1 and corresponding component of the electric current density vector (along x_1) green on Figure 2 are relatively large, but the conductivity along x_2 and component of the electric current density vector (along x_2) are small. Then we obtained the obvious components of the direction of the resultant electric current density and the electrical field strength vector, In the red system of coordinates of Figure 2, two different conductivities are characterizing the medium..

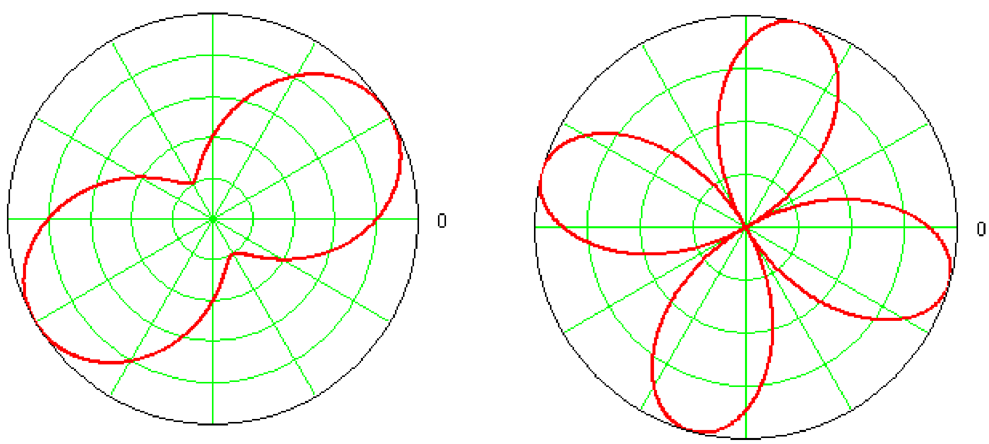
In 3 D case, there will be 3 different conductivities. In this system of coordinates, the directions of corresponding components of two different vectors are matching. In the black system of coordinates, one component of electrical field strength produces three components of electrical current density. As a result, in arbitrary oriented system of coordinates, the linear dependence of electrical current density on the electrical field strength has to be characterized with help of 9 coefficients of proportionality:

$$j_i = \sum_{m=1}^{3} \kappa_{im} E_m \tag{7}$$

It is easy to prove that rotation of the black system of coordinates to the red provides nine coefficients that are transformed as components of dyadic. In general, tensor of rank $r+q$ linearly connects two tensors: one of rank r and the other of rank q. This is the second approach to tensors.

Let's cut the strips from a sheet of anisotropic material. Let's apply gradient of electrical potential to a sample and measure component of the density of electrical current in the same direction. We will obtain a coefficient of proportionality κ or ρ. Then let's change the direction of cutting and repeat experiment. After series of such experiments, we can draw the polar diagram (in 2D case) of κ or ρ. (Figure3a). We can also apply gradient of potential in one direction and can measure the density of electrical current in the perpendicular direction and calculate the ratio of them. After series of such an experiment, we also can draw the polar diagram (Figure 3b). The shapes of the polar diagrams shown reflect the law (2) of transforming of the components of tensor and they are never arbitrary.

The specially oriented coordinate system (as the red one on the Figure 2), when each component of the "vector of the result" has the same direction as the corresponding component of the "vector of cause" is called "principal axes of the dyadic".



a) $\kappa_{11}(\varphi)$ b) $\kappa_{21}(\varphi)$

Figure 3. Polar diagrams of conductivity coefficient of the anisotropic medium.

Let us consider equation of an ellipsoid centered at the origin and oriented under angles with respect to coordinate system (Figure 4). Then we have

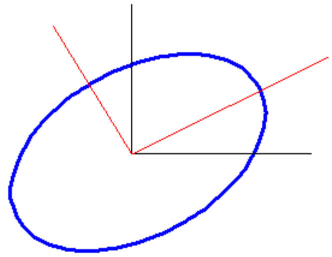


Figure 4. Ellipsoid oriented under angles to coordinates

$$\sum_{i=1}^{3}\sum_{k=1}^{3} A_{ik} x_i x_k = 1 \tag{8}$$

It is easy to prove that in rotation of the system of coordinates, the set of coefficients A_{ik} is transformed exactly according expression (2) for the case of the dyadic. Principal axes of the dyadic are the usual principal axes of ellipsoid. Thus, ellipsoid is the geometrical image of the dyadic, more exactly the symmetric dyadic.

Now let's consider a formal multiplication of two vectors, when each component of one vector is multiplied by each component of another vector. From vectors $\vec{a}$ and $\vec{b}$ we can built a matrix

$$\begin{pmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{pmatrix} \tag{9}$$

Components of this matrix are transformed as components of dyadic, which is easy to prove. Such multiplication is called "tensor multiplication" and it is denoted by the sign $\otimes$

$$\vec{a} \otimes \vec{b} = \left(\sum_{i=1}^{3} a_i \vec{e}_i\right) \otimes \left(\sum_{k=1}^{3} b_k \vec{e}_k\right) = \sum_{i=1}^{3}\sum_{k=1}^{3} a_i b_k \vec{e}_i \otimes \vec{e}_k \tag{10}$$

Formal tensor product of the unit vectors is the vector basis of dyadics and it is called "dyada". Tensor product of two vectors represent symmetric dyadic because $a_i b_k = b_k a_i$. General dyadic can be represented as

$$\overset{\leftrightarrow}{C} = \sum_{i=1}^{3}\sum_{k=1}^{3} C_{ik} \vec{e}_i \otimes \vec{e}_k \tag{11}$$

Tensor of the rank *r* can be represented as

$$\overset{\leftarrow r \rightarrow}{T} = \sum_{i=1}^{3}\sum_{k=1}^{3}\sum_{l=1}^{3} \ldots T_{ikl\ldots} \vec{e}_i \otimes \vec{e}_k \otimes \vec{e}_l \otimes \ldots \tag{12}$$

This is the third approach to the tensors.

Using nabla-operator, which combines vector nature and differentiation nature

$$\vec{\nabla} = \sum_{k=1}^{3} \vec{e}_k \frac{\partial}{\partial x_k} \qquad (13)$$

The following operators are built

$$gradU = \vec{\nabla} \otimes U = vector$$
$$div\vec{A} = \vec{\nabla} \cdot \vec{A} = scalar$$
$$curl\vec{A} = rot\vec{A} = \vec{\nabla} \times \vec{A} = vector$$
$$Grad\vec{A} = \vec{\nabla} \otimes \vec{A} = dyadic \qquad (14)$$

Gradient of the vector field is used in building vector-Laplacian, strain tensor, and the strain rate tensor. Stresses σ_{kl} and strains ε_{ij} relationships as two dyadics are often approximated by linear law, which is generalization of Hook's law:

$$\vec{\vec{\varepsilon}} = \vec{\vec{\varepsilon}}^{\,0} + \vec{\vec{\alpha}}\Delta T + \overset{\leftarrow 4 \rightarrow}{\Xi} : \vec{\vec{\sigma}} \qquad (15)$$

where $\vec{\vec{\alpha}}$ is the thermal expansion dyadic, $\vec{\vec{\varepsilon}}^{\,0}$ is the dyadic of physical and chemical shrinkage, ΔT is the temperature change, $\overset{\leftarrow 4 \rightarrow}{\Xi}$ is the compliance tensor of the fourth rank," : " is the symbol of double scalar multiplication.

By way of introduction to tensors these concepts were presented successfully to juniors, seniors and first-year graduate students during several years at Kansas State University, Lamar University and the University of Texas, Arlington.

Other examples of tensor concepts applications are shown in Reference 2.

Conclusions

We have found that introducing tensors through the dyadic to undergraduate students during their third term of calculus is readily understood and appreciated at that level and can be fitted into the crowded academic schedule when introduced in lieu of some of the contortions that students and instructors must go through in order to avoid the topic. As our teaching experience shows, one lecture on tensors in Statics course (when dyadic of inertia is studied) and two lectures in Mechanics of Solids (when dyadics of strains and stresses are studied) create initial concept of tensors for majority of students. This introduction to tensors becomes natural to the student at this level and serves to open opportunities as the student progresses to higher levels.

References

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