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Abstract

Studies have shown that a solid background in mathematics and physics is important to the success of an engineering student. The concept of infinity is one of the most important, and yet difficult links in the mathematics sequence for undergraduate engineering students. The concept of infinity can be taught to K-12 student with hands-on exercises that use an intuitive approach for teaching the concept. However, engineering students require a more mathematically rigorous presentation. This paper presents a method for teaching the topic of infinity in freshman level mathematics course on discrete mathematics for engineering students, based on the ideas of bijection and equivalency within the topic of set theory. We also present some ideas of how the concept of infinity can be targeted in the K-12 environment.

I. Introduction

As part of long-standing efforts to enhance engineering education, the ASEE surveyed prevailing trends in K-12 education[1]. Aiming to determine teachers' attitudes towards engineering as an intellectual and career challenge for their students, the ASEE study reveals an interesting paradox. It discovers that an overwhelming majority of teachers are positive about exposing their students to the discipline of engineering. Agreeing with the statements, such as "Engineers are fun people", "Engineers love their job", "Engineers make people's lives better", the majority of teachers believe that engineering is a noble and challenging profession, gaining in social recognition and tangible rewards. However, when asked about the accessibility of the profession of engineering, the majority of teachers expressed a strong feeling that many of their students have no chance to succeed in the engineering world. The teachers feel that[1] "majoring in engineering in college is harder than majoring in many other subjects... – a feeling they likely pass on to their students."

The dichotomy, revealed in this ASEE study, pinpoints a peculiar inconsistency in grasping the nature of the profession of engineering. Engineers are perceived as smart, wise, knowledgeable professionals who work with tangible objects to solve practical problems. In their work, engineers are engaged in a prolific intellectual activity that demands a great deal of self-imposed discipline and concentration. As a result, they are stereotyped as isolated abstract thinkers with profound insights, often single-minded, awkward, weird and socially inept. In other words, the abstract thinking engineer is often perceived as a "nerd" or "geek", logically contradicting the image of a practical engineer with "hands-on" ideas and the ultimate goal of designing, creating, and developing products and processes to solve practical problems.

Much of the gap between the sense of concrete and abstract in engineering lies in the poor scientific and mathematical background of engineering freshmen. The indispensable disciplines of mathematics and physics, based on non-intuitive models, are sometimes inadequately treated in the K-12 community.

It is well-known that students with a solid background in mathematics and physics have a better chance of succeeding in an engineering program. A study[2] of predictive factors for success in an Electrical Engineering Fundamentals course, using the final course grade as the success metric found that the pre-requisite courses of calculus, physics and differential equations are good predictive factors, as well as math readiness, measured by a math placement test taken by incoming freshmen. Buechler[3] points out the connection between the decrease in mathematical proficiency of college freshmen and the difficulties they experience in mathematically intensive courses. However, this reference concludes that poorly prepared students can succeed in mathematics, and consequently in engineering, if they are properly placed within the mathematics sequence. These students should start with lower level math courses and move on to the next course in the mathematics sequence only after the required mathematical concepts are understood and internalized.

The concept of infinity is one of the most important, and yet difficult links in the mathematics sequence for undergraduate engineering students. A very subtle perception of finite vs. infinite, discrete vs. continuous, and the infinity of a sequence vs. the infinity of the continuum, is required to study limits in calculus, infinite series in Fourier analysis, and distributions in probability. Many important characteristics of large finite sets (e.g., neurons in human brain, large distances) use the ideas and techniques of asymptotic behavior and/or the central limit theorem, and require a rigorous knowledge of the concept of infinity. Dealing with attenuation poles at infinite frequency, or confronting conductivity problems of an infinite slab, bounded by infinite planes, with information extracted from infinitely many pairs of boundary voltage potentials, requires an understanding of infinity well beyond the intuitive.

Take, for example, two digital signals, or mathematical sequences:

$$\{x(n)\} = \{\ldots, x(-2), x(-1), x(0), x(1), \ldots, x(n), \ldots\}$$
$$\{y(n)\} = \{\ldots, y(-2), y(-1), y(0), y(1), \ldots, y(n), \ldots\}$$

Produce a third signal, or sequence, by discrete convolution:

$$z(n) = \sum_{k=-\infty}^{\infty} x(k)y(n-k) \qquad \textit{for } \forall n \in \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$$

We feel that our students have difficulty grasping the meaning of minus infinity in this formula for {z(n)}. An integral from minus infinity to plus infinity can be visualized as a finite area of an unbounded domain. However, we find no plausible metaphor to explain intuitively the minus infinity within the sum given above. We argue that the solid foundation of a rigorous approach to infinity is needed to understand this and other equations of digital signal processing.

Some educators avoid confronting the rigorous approach to teaching the concept of infinity. Instead, they use the notion of infinity without explanations, as if it is intuitively clear. There are many reasons to use the intuitive approach, including the lack of time as well as the lack of mathematical maturity of college freshmen. We do not totally dismiss intuition as a tool to approach the concept of infinity. On the contrary, intuition serves to evoke an informal image of infinity that can be developed and converted into a formal definition of transfinite numbers. In addition, the intuitive approach to infinity is the only one that can be used in K-12 environment. However, the authors contend that engineering students should be exposed to a mathematically rigorous presentation of the infinity concept, based on bijection and cardinality.

Section II of this paper presents a brief review of the history of the concept of infinity. Section III presents the outline of the lectures on infinity that we have used for engineering freshmen. Section IV presents some ideas of how infinity might be taught in K-12 classroom. Section V summarizes this paper and outlines some areas of our future work.

II. The History of the Concept of Infinity

The idea of infinity had been the subject of thoughts and discussions since ancient time. "The Infinite is one of the most intriguing ideas in which the human mind has ever engaged. Full of paradoxes and controversies, it has raised fundamental issues in domains as diverse and profound as theology, physics, and philosophy. The infinite, an elusive and counterintuitive idea, has even played a central role in defining mathematics, a fundamental field of human intellectual inquiry, characterized by precision, certainty, objectivity, and effectiveness in modeling our real finite world.[4]"

The infinity is intuitively interpreted as a dynamic never-ending process. Intuitive understanding is a form of unexplained searching, akin to "I just know it". Our intuition is based on internalized interpretations of sensory stimuli that occur during our lifetime. Simply said, our intuition is based on our accumulated experiences. In "The Critique of Pure Reason" Kant states: "In whatever manner and by whatever means a mode of knowledge may relate to objects, intuition is that through which it is in immediate relation to them... But intuition takes place only in so far as the object is given to us[5]". The object of infinity is not given to us. To attempt to interpret the infinity, we must rely on the experiences of our finite life. Very often we rely on imagination, which can be misleading in relatively simple situations (an example of a misleading activity is given in Exercise 3 of Section IV).

In ancient time, the notion of infinity was considered as a metaphysical unstructured chaotic abyss. There was a common feeling that infinity could not be defined, since eternity (and therefore immortality) is required to accomplish infinitely many steps, and therefore the notion of infinity should be avoided in proper reasoning. Aristotle (384 – 322 BC) distinguished between potential infinity as an ongoing never-ending activity, and actual infinity as a "totality of numbers", a completed thing, a definite entity. Aristotle reluctantly accepted the potential infinity, but totally rejected the actual infinity, claiming that the list of numbers (i.e. positive integers), generated one after one, cannot be completed, therefore cannot be represented in our thoughts.

The concept of actual infinity was considered by Galileo[6]. To understand the feasibility of dividing a solid body into very small parts, Galileo designed clever intuitive geometrical observations, concluding that "the totality of numbers" differs from numbers and cannot be subjected to arithmetical operations: "… the attributes *equal*, *greater*, and *less*, are not applicable to infinite, but only to finite, quantities. When therefore Simplicio introduces several lines of different lengths and asks me how it is possible that the longer ones do not contain more points than the shorter, I answer him that one line does not contain more or less or just as many points as another, but that each line contains an infinite number." (Salviati's monologue, passage [79] of reference 6).

Galileo considered the concept of dividing the infinity as part of investigating a practical engineering problem. A hundred years later Kant commented on the "undividable infinite whole" in the context of an abstract cosmological idea: "If we divide a whole which is given in intuition, ... the division of the parts (subdivision or decomposition) is a regress in the series of ... conditions. The absolute totality of this series would be given only if the regress could reach simple parts. But if all the parts in a continuously progressing decomposition are themselves again divisible, the division ... proceeds in infinitum. ... We are not, however, entitled to say of a whole ... that it is made up of infinitely many parts. For although all parts are contained in the intuition of the whole, the whole division is not so contained, but consists only in the continuous decomposition...[5]". Galileo and Kant reached the same conclusion: infinity is an "undividable whole".

In the 18th century, accelerated by real-world applications, the emerging discipline of Calculus raised the issue of infinity. Still, the scientific community continued to persist in rejecting the idea of actual infinity. However, in the 19th century the founder of set theory and transfinite mathematics Georg Cantor came up with astonishing, counter-intuitive and paradoxical discoveries. Infinity can be described as a quantity, susceptible to understanding through the act of comparison by bijection (a one-to-one and onto mapping). Some infinities are bigger than others, and the infinities can be expressed by transfinite cardinal numbers. Cantor conceptualized particular kinds of actual infinity, namely, transfinite cardinals, ordinals and arithmetic, and created a precise and sophisticated gradation of infinities that influenced and shaped entirely new fields in mathematics. Infinity became a mathematical object, a definite entity. In Section III of this paper we present a small portion of Cantor's work, tailored to engineering students.

Originally met with disbelief, today set theory plays a key role in modern mathematics, its concepts and notions being essential ingredients of mathematical thinking. However, it should be mentioned that some cognitive theories have difficulties recognizing "the reality" of the concept of infinity and its mathematical apparatus, (e.g. radical constructivism[7]).

III. Teaching the Concept of Infinity to Engineering Students

In this section we present our method of teaching the concept of infinity to engineering students. This method has been used in the course of Discrete Mathematics for Software Engineering freshmen at the Sami Shamoon College of Engineering (SCE). Traditionally the concept of infinity is taught within the topic on set theory in courses on discrete mathematics, or calculus, or probability and statistics. The lectures on infinity described here are presented following an introduction that includes the notions of a set and its elements, subsets, universe, complement, empty set, binary operations on sets, and the set of all subsets of a given set (a power set). The topic on set theory generally takes 4 - 6 classroom hours accompanied by 2 - 3 hours of problem solving. The lectures on infinity are structured into the twelve subtopics listed below.

1. Finite and Infinite Sets. A set is *finite* if the number of its elements can be counted. The set of 5 numbers {0, 1, 2, 3, 4}, the class of 37 students, or 2,783,940,064 (almost 3 billion) stars in a distant nebula, are examples of finite sets. Note that the stars of a distant nebula can hardly be counted in reality, although they can be counted in principle. However, we find it difficult to count the set N={1, 2, 3, 4, ..., n, ...} of all positive integers. Suppose we remove the number 1, then we remove 2, then 3, and so on. We continue to remove numbers, but there are always numbers left after each removal. Let us try another way: remove the number 2, then remove 4, continue to remove any even number. We see that after removing any amount of even numbers, there are odd numbers left. There are sets such that we continue to remove elements, but there are elements left after each removal. Sets of this kind are called *infinite*.

2. Bijection. Next we define the notion of bijection. Given two arbitrary sets, A and B, a *function* f is a rule that associates a unique element $b=f(a)$ of B to each element a of A. Instead of saying "a function", we may use the term: "mapping $f\colon A \to B$" between A and B, or from A to B. A mapping $f\colon A \to B$ is said to be *one-to-one* if different elements of A are mapped into different elements of B. A mapping $f\colon A \to B$ is said to be *onto* B if for each element b of B there is an element a of A such that $b=f(a)$. Informally speaking, a mapping $f\colon A \to B$ is onto B, if B can be covered by the elements of A, using the rule $b=f(a)$. A mapping that is not onto B, is said to be *into* B. Finally, a mapping $f\colon A \to B$ is called *bijective* (or a *bijection*) if it is one-to-one and onto B. Informally speaking, f is a bijection if it is onto B, and, in addition, can be reversed and presented as a mapping of B onto A. The definitions of mapping and its properties are accompanied by many examples in class.

3. Equivalent Sets. Two sets, A and B, are *equivalent* if there exists a bijection $f\colon A \to B$. Six people and six chairs are equivalent sets, and the bijection means that each of the six people is sitting on exactly one chair. There are 6! = 720 bijections possible with these two sets, however the existence of only one bijection is enough to establish the equivalence of two sets. We use the notation $A \leftrightarrow B$ and $a \leftrightarrow b$, for equivalent sets and their elements, respectively.

4. Countably Infinite Sets. The notion of equivalence allows us to think about "counting" the number of elements in an infinite set. In reality, we do not count infinite sets. Instead, we compare a given infinite set with another one that is well known and well understood, using the notion of equivalence as a comparison tool. The best-known infinite set is $N = \{1, 2, 3, ..., n, ...\}$. An infinite set is *countably infinite* if it is equivalent to the set $N = \{1, 2, 3, ... , n, ...\}$ of all positive integers. A set is *countable* if it is finite, or countably infinite. A countable finite set can be counted, but, if the set is countably infinite, there is no end to the counting process. Two examples of countably infinite sets are:

- the set of all positive even integers with the bijection $n \leftrightarrow 2n$, and
- the set Z of all integers with the bijection $f\colon Z \to N$, where $f(n) = 2n + 1$ for a nonnegative n, and $f(n) = 2|n|$ for a negative n.

We illustrate the second example of the bijection $f\colon Z \to N$ by arranging the elements of both sets in table 1 below.

Elements of Z	0	-1	1	-2	2	-3	3
Elements of N	1	2	3	4	5	6	7

Table 1.

5. Definition of an Infinite Set via a Proper Equivalent Subset. Two finite sets are equivalent if, and only if, both have the same number of elements. It means that a finite set cannot be equivalent to its proper subset. Recall that a subset M of a set A is a *proper subset*, if there is at least one element in A that does not belongs to M. In other words, M is inside A, but not equal to A. The examples of countably infinite sets cited above show that the situation with infinite sets is different. An infinite set may be equivalent to its proper subset. For example, the bijection $f(n) = 2n$ says that the set $N = \{1, 2, 3, ..., n, ...\}$ of all positive integers is equivalent to its proper subset of even numbers. In a similar way, the bijection $f: Z \rightarrow N$ (Table 1) shows that the set Z of all integers is equivalent to its proper subset N of positive integers. Similarly, the bijection $y = 2^x$ suggests that the set R of all real numbers (the real line) is equivalent to its proper subset of positive real numbers. We establish and discuss the following statement, setting it as a definition of an infinite set: *a set is infinite if and only if it is equivalent to some of its proper subsets*.

6. Cardinal number $\aleph_0$. Now we are ready to introduce the notion of a cardinal number, or cardinality. For a finite set A, the *cardinality*, Card A, is the number of its elements. For example, if $A = \{0, 1, 2, 3, 4\}$, then Card $A = 5$. For the set $N = \{1, 2, 3, 4, ..., n, ...\}$ of all positive integers, we define its cardinality as a special symbol $\aleph_0$, that means N can be counted, but the process of counting will never come to end. We repeatedly emphasize that by writing Card $N = \aleph_0$ we mean that N is countably infinite.

7. Cardinality of Countably Infinite Sets. There are other countably infinite sets, for example, the set Z of all integers. Table 1 gives an idea of how Z can be counted. It seems natural assigning to Z the symbol $\aleph_0$: Card $Z = \aleph_0$. Any countably infinite set A can be counted by using the bijection $A \leftrightarrow N$. Thus, the symbol $\aleph_0$ can be assigned to any countably infinite set A. We write: Card $A = \aleph_0$ for any countably infinite set A, or in other words, for any set A that is equivalent to the set $N = \{1, 2, 3, 4, ..., n, ...\}$ of all positive integers.

8. Equivalent Sets Have the Same Cardinality. Next we ask the question: are all infinite sets we know countable? Or, are there infinite sets that are not countable, i.e., not equivalent to N? What about the real line R? How could we compare the number of points of R with that of N? Is there a bijection between N and R? We agree on the following definition: Card A = Card B if and only if $A \leftrightarrow B$. It means that any two equivalent sets A and B have the same cardinality. In other words, the cardinality of A is a unifying parameter for all sets equivalent to A.

9. Comparison of Cardinals. At this point, the students have internalized the notion of cardinality. They understand that cardinal numbers are created in order to answer the question "how many elements do we have when dealing with infinite sets?" Many different sets are equivalent, or in other words, have the same cardinality. We are now ready for the next step: comparison of cardinals.

Two cardinal numbers can be compared in the following way. Suppose there is a one-to one mapping from *A* into *B*, but there is no one-to-one mapping from *B* into *A* (for finite sets *A* and *B*, this happens only if *A* has less elements than *B*). In that case, we say that cardinality of *A* is less than cardinality of *B*: Card *A* < Card *B*. We emphasize that *B* cannot be mapped one-to-one into *A*, provoking a question about what happens with the cardinalities of A and B if there is also a one-to-one mapping from B into A. If the question arises, we suggest continuing the discussion after the lectures, referring the curious students to Cantor-Schroeder-Bernstein Theorem.

<u>10. There Are "More" Decimals between 0 and 1, than Positive Integers</u>. We know that *N* is a proper subset of *R*. It also seems that *R* has more points than *N*. How about comparing *N* and all decimals between 0 and 1? In other words, let *A* be the open interval of real numbers between 0 and 1: $A = \{x \in R:\ 0 < x < 1\}$. Which one of the sets, *N* or *A*, has more points? Which one of the following statements is true: Card *N* < Card *A*, Card *N* > Card *A*, or Card *N* = Card *A*? Note that only one of these statements can be true.

The question is challenging and counter-intuitive. Most students claim that there are more natural numbers than points between 0 and 1. The confusion is clear: the set $A = \{x \in R:\ 0 < x < 1\}$ is bounded while the set *N* is unbounded.

We are going to show that Card *N* < Card *A*. First, we eliminate the possibility that Card *N* > Card *A*. To remove this possibility, we use the bijection $n \leftrightarrow 1/2^n$ between *N* and the infinite set $M = \{1/2, 1/4, 1/8, ..., 1/2^n, ...\}$. That means: Card *N* = Card *M* = $\aleph_0$. Next, we note that *M* is a proper subset of *A*. Therefore, *N* can be mapped one-to-one into *A* by $n \leftrightarrow 1/2^n$, eliminating the possibility of Card *N* > Card *A*. We are left with either Card *N* < Card *A*, or Card *N* = Card *A*.

Is it possible that Card *N* = Card *A*? In other words, is there a bijection between *N* and *A*? We shall prove that there is no bijection between *N* and *A*, and therefore Card *N* < Card *A*. Suppose on the contrary that there is a bijection $A \leftrightarrow N$. We don't know its formula, but the important thing is that a bijection exists. Using this bijection, we arrange all elements of the set *A*, namely, all real numbers between 0 and 1, in a sequence $A = \{a_1, a_2, a_3, ..., a_n, ...\}$ of decimal numbers as shown in Table 2 below.

1	$\leftrightarrow$	a_1	=	$0.a_{11}a_{12}......a_{1k}.....$
2	$\leftrightarrow$	a_2	=	$0.a_{21}a_{22}......a_{2k}.....$
3	$\leftrightarrow$	a_3	=	$0.a_{31}a_{32}......a_{3k}.....$
...	...	...	...	...
n	$\leftrightarrow$	a_n	=	$0.a_{n1}a_{n2}......a_{nk}.....$
...	...	...	...	...

Table 2.

Initially it appears that all elements of *A* are represented in this sequence. However, we are going to discover an element of *A*, namely, a real number between 0 and 1, that is different from any of the numbers $a_1, a_2, a_3, ..., a_n, ...$. It will contradict the claim that there is a bijection $A \leftrightarrow N$.

We discover a decimal number $b = 0.b_1b_2b_3.....b_n$ such that:

$$b_1 = 1 \text{ if } a_{11} \neq 1, \text{ and } b_1 = 2 \text{ if } a_{11} = 1$$
$$b_2 = 1 \text{ if } a_{22} \neq 1, \text{ and } b_2 = 2 \text{ if } a_{22} = 1$$
$$b_3 = 1 \text{ if } a_{33} \neq 1, \text{ and } b_3 = 2 \text{ if } a_{33} = 1$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$b_n = 1 \text{ if } a_{nn} \neq 1, \text{ and } b_n = 2 \text{ if } a_{nn} = 1$$

The digits of b are either 1 or 2, which means that $0.1111... \leq b \leq 0.2222...$. Therefore, b is between 0 and 1, and it means that b is a member of A.

Recall our assumption that A and N are equivalent (in other words, that that there is a bijection between N and A). Based on this assumption, we have accounted for all elements of A. The number b differs from a_1 by its first digit, from a_2 by its second digit, from a_3 by its third digit, and generally, differs from a_n by its n-th digit. Therefore b is an element of A, different from any of the elements of $A = \{a_1, a_2, a_3, ..., a_n, ...\}$. Therefore, our assumption that $A \leftrightarrow N$ is incorrect. Therefore, A is not equivalent to N. Therefore Card $A \neq$ Card N. The only statement left to be true is that Card $N <$ Card A. We have finally proven that there are "more" decimals between 0 and 1, than positive integers.

<u>11. Cardinality of the Continuum</u>. At this point we introduce a new symbol: the cardinality of $A = \{x \in R\text{: } 0 < x < 1\}$ is denoted by c and named the *cardinality of continuum*. Our next step is to establish the cardinality of the real line. The function:

$$y = f(x) = \pi^{-1}\text{arc tan } x + 1/2$$

defined on the set R of all real numbers, sets a bijection between A and R, which leads to a conclusion that $A = \{x \in R\text{: } 0 < x < 1\}$ and R are equivalent, and therefore have the same cardinality: Card A = Card $R = c$.

The cardinalities $\aleph_0$ and c are the "bread and butter" of any advanced mathematics course, such as calculus, probability, or discrete mathematics. We have students examine different infinite sets and establish their cardinality as $\aleph_0$ or c, using the tool of bijection. They discover that the cardinality $\aleph_0$ is the cardinality of a sequence, the set of rational numbers, and the set of all polynomials with rational coefficients. The cardinality of continuum, c, is the cardinality of the real line, all points on a plane, and all continuous real-valued functions of one or several variables.

<u>12. Continuum Hypothesis.</u> We conclude our lectures on infinity by telling our students that all known infinite subsets of the real line are either countable or of cardinality c. The cardinality $\aleph_0$ is the smallest infinite cardinality, since any infinite set contains a countably infinite subset. (This is a nontrivial statement, involving the Axiom of Choice, however, we do not discuss that issue).

We ask our students to try finding a subset *A* of *R*, such that $\aleph_0 <$ Card $A < c$. In other words, they should find an infinite subset *A* of *R*, such that there is no bijection between *A* and *N*, and there is also no bijection between *A* and any other subset of *R* of cardinality *c*. During the next lecture, we share the story of the Continuum Hypothesis. We explain that mathematics considers "sets" consisting of "elements" of various kinds. The terms "sets", "elements" are regarded as well-understood in the so-called "naive set theory". However, some logical difficulties of hard-core basic mathematics require a more precise and accurate approach referred to as "axiomatic set theory". In general, there is no simple "yes-or-no" answer to some set-theoretical questions. The question of finding a cardinal number strictly between $\aleph_0$ and *c* is one of them. We know that $\aleph_0 < c$, similarly to 5 < 6, or 0.2 < 0.21. But we do not know whether *c* comes next in order after $\aleph_0$, similar to 6 after 5 in the set of integers, or is there at least one cardinal number (and maybe many) between $\aleph_0$ and *c*, similar to 0.2 < 0.2001 < 0.205 < 0.21. This question can neither be proved nor disproved, and both answers to this question are consistent within "axiomatic set theory" (a situation, similar to the parallels axiom of Euclidean Geometry). Probably, new axioms should be invented to solve the Continuum Hypothesis, but then we might again discover cases in which our newly invented axioms do not work.

The story of the Continuum Hypothesis concludes the topic on set theory. Based on our experience, we feel the subtopics 9 and 10 are the most difficult for our students. One of the difficulties is understanding that the set *A* of decimals between 0 and 1 can be counted and presented in the form $A = \{a_1, a_2, a_3, ..., a_n, ...\}$, with no reference to the usual order relation of real numbers. Students expect to "see" that $a_1 < a_2 < a_3 < ...< a_n <......$. We provide the following explanation. First, we point to Table 1 and remind that by counting the whole numbers we "damaged" the usual order relation on *Z*: we count 0, then -1, then 1, then -2, and so on. There is no way to "repair" this damage: *N* has 1 as its smallest number, but *Z* has no smallest number to start the counting process. Second, mathematics allows "existential" statements. It is legitimate to say that some things exist, even if no procedure is given to construct these things. Therefore, just assuming that there exists a bijection between *A* and *N*, we can count *A* using this bijection. Since we do not know the formula of this bijection, we have no idea whether $a_1 < a_2$, or on the contrary, $a_1 > a_2$. The only relevant thing is that a_1 and a_2 are different members of A, counted by a bijection between *A* and *N*, which means: $a_1 \leftrightarrow 1$, $a_2 \leftrightarrow 2$.

In teaching the concepts of infinite cardinal numbers $\aleph_0$ and *c*, we combine the rigorous approach of the naive set theory with the simple intuitive ideas connected to the notion of infinity, using a simple language and many examples.

IV. Teaching Infinity in K-12

The contemporary educational research agrees that humans' mathematical reasoning and conceptualization starts in an informal way. People generally do not think in terms of formal axioms, definitions, theorems and proofs. Responding to a formal definition, students use imagination, metaphorical parallels, conversations with peers and hands-on activities, to explain and clarify the defined concept. Some concepts are well suited for the K-12 environment, and we believe the concept of infinity is one of them. The finite-to-infinite gap can be bridged in the early educational stages. The K-12 students should have greater exposure to the concept of infinity. We suggest starting to teach the notion of infinity in middle school, using the intuitive approach.

To support our claim, we provide a glimpse of research on how people learn mathematical concepts, and in particular, the concept of infinity. We stress that our review is far from being exhaustive.

The way people learn mathematics can be explained within the frame of APOS theory[8]. The APOS theory extends the Piagetian ideas in order to describe the advanced mathematical cognition and epistemology of mathematical concepts. The term APOS is an abbreviation for Action, Process, Object, Schema. Action is a repeatable mental or physical manipulation leading to the transformation of things. When an action is internalized, it becomes a Process. By realizing that an action can mastermind the process, the encapsulated process becomes an Object. A consistent collection of actions, processes and objects constitutes a Schema. Cognitive analysis of conceiving the concept of infinity by internalization and encapsulation, based on APOS theory, can be found in reference 9.

Some mathematical ideas are metaphorical in the sense that they involve inference-invariant transformations between different conceptual domains. Nunez suggests that the various forms of infinities in mathematics, such as infinite sums, infinite sets, transfinite cardinals and ordinals, mathematical induction and limits, are particular forms of actual infinities generated by a specific conceptual mechanism, named the Basic Metaphor of Infinity[4].

The idea of a pathway from Concept Image to Concept Definition is introduced and studied in references 10-11. Concept Image is a dynamic structure, constructed by the learner over years of experience and changes in time under the influence of new perceived stimuli. Different parts of Concept Image are evoked at different times, and therefore Concept Image is not necessarily coherent. Formal Concept Definition is a coherent rigorous verbal framework specifying a concept. A discussion of the conflicts between the infinities of everyday experience and formal infinities based on axioms can be found in reference 12.

The interesting conversations[13] of a knowledgeable father with his six-year old son, Nic, provide an insight on how a child's intuition builds an image of the concept of infinity. In the beginning of the conversations Nic imagines infinity as a very huge number, much bigger than 10, bigger than a million, probably bigger than a "googol" (1 followed by a hundred 0's). Nic's infinity can be operated on like any other number in arithmetic ("infinity" + "infinity" = "two infinity", and there is "half infinity" as well). Nic also invents a number bigger than infinity, namely, "infinity + 1". After learning about the thermometer and temperatures in centigrade, Nic grasps the notion of "minus infinity". By the method of Socratic dialog, Nic is brought to realize that there is a number, named "aleph", and there are "aleph" positive integers, in the same way as, for example, there are 31 students in his class. When asked how much is "aleph" + "aleph", Nic answers that it is again "aleph", saying that he cannot explain why. Seemingly, Nic got the feeling that "aleph" + "aleph" leaves him with the same amount "aleph" of positive integers, which is indeed not similar to 31 + 31 = 62. A consistent view of infinity as an arithmetic entity, such that "infinity" + "infinity" = "two infinity", is replaced in Nic's mind by a conflicting view of one and the only one infinity of "aleph". He then "glues" "plus infinity" with "minus infinity", thus transforming the real line of the temperatures into a circle, which leads him to a conclusion that infinity cannot be reached, because if you could reach infinity by moving to bigger positive numbers (temperatures), then you could pass it, entering the minus numbers again. Nic's intuition, based on his experience with hot bodies, tells him that it is impossible for an object to freeze immediately after being extremely hot.

We believe there is no unique recipe for teaching infinity to young students. We are in the early stages of gathering ideas, materials and data on youngsters' perception of the concept of infinity. However, we suggest two important assumptions. First, a very experienced teacher, who is also a good mathematician, should do the teaching. Second, a designer, acquainted with a specific audience, should create the materials on the infinity for this audience. Materials, well suited for one classroom, may be inappropriate for another classroom.

One of the nicest ways of exposing young students to the idea of infinity is by exploring fractals. There are plenty of age-appropriate materials on fractals. A good collection of ideas, recourses and references on teaching fractals to children can be found in reference 14. In accordance with guidelines for K-12 mathematics and science education, proposed in reference 1, the notion of infinity via fractals can be presented to young students in an aesthetic, appealing and relevant way, by using computers to generate fractals, and by exploring fractals and self-similarity in nature. In this context, we would like to mention a classroom study of third-grade students, aimed to explore and document the emergence of children's perceptions of mathematical similarity[15].

We conclude this section by presenting exercises, based on the intuitive understanding of the concept of countable infinity, tailored to K-12 environment. The exercises also show how the intuitive approach to the notion of infinity can mislead in a relatively simple situation, stressing a need for an experienced teacher and carefully prepared materials on infinity.

Exercise 1 (replica of Zeno's Paradox). A UFO starts moving. In the first second it flies half a mile, during the next second it flies a quarter of mile, during the third second it covers 1/8 of mile, and as it continues, during each additional second of time it flies half of the way it did in the previous second.

> *Question*. Assuming that there are no obstacles, how long will the UFO be flying, and how many miles will it cover?
> *Answer*. The UFO would fly indefinitely. It will cover a number of miles, equal to a sum of infinitely many terms: 1/2 + 1/4 + 1/8 + 1/16 + ... This infinite sum is finite and equal to one.
> *Activity*. Take a rope, cut a half of it, put it aside, suggesting it represents 1/2. Take a half of the remaining half-rope and join it to the first half, explaining that it representing the partial sum 1/2 + 1/4. Cut half of the remaining quarter, join it with the previous pieces, representing the partial sum 1/2 + 1/4 + 1/8. Noticing that less and less of the rope is left, the children are led to conclude, that the sum of infinite many numbers is finite and equal to 1.

Exercise 2. Which one of the following two numbers is bigger: 1 or a repeating decimal 0.9999......?

Answer. The repeating decimal 0.9999... is a sum of infinitely many terms: 9/10 + 9/100 + 9/1000 + 9/10000 + ... , and it is equal to 1.

Activity. Exercise 2 can be illustrated by an activity, similar to the activity of Exercise 1. Take 9/10 of the rope and notice that 1/10 of the rope is left, then take 9/10 of what is left and show the audience the partial sum: 9/10 + 9/100, etc.

Remark. Some students are viewing the repeating decimal 0.9999.... as a sequence rather than a limit. These students will argue that 0.9999.... is less than 1. Their arguments should be considered and carefully analyzed.

The next exercise shows that an intuitive activity as the only basis for understanding the infinity can be misleading.

Exercise 3. Which one of the following two numbers is larger: 1 or a repeating decimal 0.989898.....? (or 0.8888.... for younger students).

Answer: The repeating decimal is the sum of an infinite geometric progression: 0.989898...... = 98/100 + 98/10000 + 98/1000000 + ... = 98/99

Remark: The activity of cutting the rope leads to a conflicting situation. By repeatedly cutting the rope, the limit of 98/99 is unseen. The intuition steaming from the experience with the rope, tells that the repeated decimal 0.989898...... equals to 1. However:

0.9999.... – 0.989898.... = 0.010101....

Since 0.9999.... equals to 1 as well, does it mean that 0.010101.... equals to 0? A skillful teacher of K-12 community will use this seemingly disturbing example to lead her/his students to the next step on the pathway from Concept Image to Concept Definition[10].

The next exercise is aimed to develop gradually the understanding of the concepts of bijection, equivalency and countable infinity.

Exercise 4. (replica of Hilbert's paradox of the Grand Hotel): Dave owns a guest house with an infinite number of rooms, counted as Room 1, Room 2, Room 3, etc. All the rooms are full, but Dave claims there is no need for "No vacancy" sign.

Step 1. When Rachel asks to stay for a night, Dave and the students "create" a free room by moving the tenant of Room 1 into Room 2, the tenant of Room 2 into Room 3, and so on. Rachel moves into Room 1. There are two difficulties in understanding this step. The first is realizing that there is no "last room". The second is visualizing the transfer of each tenant to the next room as a viable simultaneous transposition instead of a never-ending process that cannot be completed. Discussing the feasibility of replacing the tenants, the students discover "the totality of numbers" as an "undividable infinite whole", conducting discussions similar to those cited in reference 6 of Section II.

Step 2. A team of 10 people arrives. The students are encouraged to find a solution. The tenant of Room 1 moves into Room 11, the tenant of Room 2 moves into Room 12, and so on. This step is repeated by changing the number k of the members of the team until the mapping n ↔ n+k is understood and internalized for any fixed k.

Step 3. A new team with infinitely (countably) many members arrives. Dave and the students suggest moving the tenant of Room 1 into Room 2, the tenant of Room 2 into Room 4, the tenant of Room 3 is into Room 6, and so on. Removing an infinite number of occupied even rooms, Dave and the students are left with an infinite number of empty odd rooms for the newly arrived team. This step explores the intuitive definition of an infinite set in the way described in subtopic 1 of Section III.

Step 4. The mapping n ↔ 2n of Step 3 divides the rooms into the sets of even and odd rooms. Using the partition induced by the modulo k relation, the rooms can be divided into any finite number of infinite sets. For example, the modulo 3 relation divides the rooms into three sets. The first set consists of multiples of 3, the second starts with Room 1 and contains rooms numbered (3n+1), the third starts with Room 2 and includes rooms numbered (3n+2). The partition can be explained to young students as movements by jumps of 3. When two teams arrive, each team consisting of an infinitely countable number of members, the tenants are moved in the following way: the tenant of Room n moves into Room 3n. Then the first team occupies Room 1 and rooms numbered (3n+1), and the second team occupies Room 2 and rooms (3n+2). This step is aimed to understand that "infinity" + "infinity" = "3 times infinity" = "the same infinity", as described in reference 13. The step can be repeated, using the modulo k relation for different positive integers k.

Step 5. Suppose an infinite (countable) number of teams, each consisting of an infinite (countable) number of members, arrives to Dave's guest house. The mappings n ↔ kn of Step 4 does not solve the problem of finding rooms for new guests. Discussing the division of integers, Dave and the students discover the notions of prime and composite integers. The discovery leads to the following transposition: the tenant of Room n moves into Room 2^n. The teams occupy the rooms numbered by the degrees of prime numbers: the first team uses the degrees of 3, the second uses the degrees of 5, the third uses the degrees of 7, the next of 11, then of 13, and so on. The arrangement leaves many of the rooms empty. For example, Room 1 and Room 6 are not occupied. The students are encouraged to find a rule for a room to be empty. This advanced step should be used carefully.

The steps 1 and 2 of the last exercise are tailored for very young students. No previous knowledge is needed to start the discussion.

V. Summary and Conclusions.

In this paper we discuss the importance of increasing the time spent teaching prospective and existing engineering students the concept of infinity. Historical information has been provided to motivate the inclusion of the topic on infinity.

Lectures suitable for freshmen engineering students are presented. In teaching the concept of infinity, we try to be as rigorous as possible, keeping in mind that our clients are engineering freshmen and not hard-core mathematicians. We use language that is as simple as possible, combined with many examples. We believe our approach is well suited to the needs of engineering students.

Developing a formal rigorous understanding of a mathematical concept is a long and painstaking process, and the sooner it begins, the better. We suggest starting to discuss the concept of infinity in the K-12 classroom. Activities for K-12 students, useful in understanding of the notion of countable infinity, are presented. We strongly insist that the materials for teaching the concept of infinity should be carefully designed for the particular group of youngsters by experienced educators acquainted with this particular group.

By presenting this work the authors hope to engage faculty teaching mathematics, as well as K-12 professionals, and receive their feedback on the ideas presented.

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