AC 2008-1635: THE DIAGRAMMATIC AND MATHEMATICAL APPROACH OF PROJECT TIME-COST TRADEOFFS

Kun-jung Hsu, Leader University

Title of the Paper:
The Diagrammatic and Mathematical Approach of Project Time-cost Tradeoffs

Abstract

A potential project management involving time used of a project can always be tradeoff by additional resources input. Such a tradeoff may come from different options of the activity of the project which can be choice. The situation of "Pay more - Save Time" is common for project management related decision problems. The available technology of shortening the duration of each activity is often the sources of the time-cost tradeoffs problem. And the problem solving processes always rely upon the techniques of critical path method (CPM) calculation and mathematic programming, for example linear programming, or integer programming etc. The paper includes an introduction to the concepts of CPM method, time-cost tradeoff, and the uses of mathematical programming in spreadsheet. The diagrammatic expression of critical path method and mathematical method will be combined in this paper, by which a more clear and efficient exposition of solving the time-cost tradeoffs problem will be exhibited. As a more efficient tool, the paper discusses such new education pedagogy.

Introduction

Prompted by the present emphasis on time-based competition in industry, there are more and more issues focus on the problem of time-cost tradeoffs. A potential project management involving time used of a project can always be tradeoff by additional resources input. Such a tradeoff may come from different options of the activity on the critical path of the project. The situation of "Pay more - Save Time" is common for project management related decision problems. The available technology of shortening the duration of each activity is often the sources of the time-cost tradeoffs problem. And the problem solving processes always rely upon the techniques of critical path method (CPM) calculation and mathematic programming, for example linear programming, or integer programming etc.

Methods of critical path method that are frequently used include Early-Start, Early-Finish, Late-Start, Late-Finish calculation of each activity. Further, one can use forward method and backward method to find the zero float time activity, and define the critical path of the project. The project total duration and its respective total project cost needed thus can be obtained. After advanced evaluation, if an activity on the critical path can be shortened by more resources input, one can obtain the other project time and its respective cost. We can use the same way to calculate each possible combination of project time/cost one by one, and finally obtain a project's time-cost tradeoffs curve. But now including the mathematical programming will be more efficient for the problem.

Using mathematical method to solve the time-cost trade-off problem has been studied extensively in the project management literature. Mathematical approaches convert CPM network and time-cost relationships of the project into constraints and objective functions. Linear programming and integer programming are the two major mathematical approaches used to solve the time-cost trade-off problems in project schedulining. By assuming linear relationship between time and cost for project activities, the linear programming had been developed three decades ago [3, 4], and well-developed later.

The general philosophy of linear programming convert the project time-cost trade-off problems to minimizing the objective cost function, subject to inequality time constraints, and then solve the problem. Computerized "CPM" procedure and the application of project management system had been developed by many researchers, for example, [1], [2], [8], and [7]. Computerized "CPM" procedure using spreadsheets to solve the time-cost tradeoffs problem also already was integrated as parts of the standard OR textbook, for example, [5] and [6].

The advantages of linear programming algorithms used to obtain the optimal solutions include efficiency and accuracy. To simplifying the mathematical formulation and its application in time-cost tradeoff problem, it will be helpful if including the visualization spreadsheet expression. Basically, critical path of a project schedule can be easily calculated in spreadsheet form, the diagrammatic CPM network thus can be extended. But using minimum cost principle to solve time-cost solutions of all possible combinations is time consuming. On the other hand, using mathematical programming to solve the time-cost trade-off problems, or expressed the problem framework in spreadsheet is relative easier. How the diagrammatic expression of critical path method and mathematical method of solving time-cost tradeoff will be a good way for us to combine. Such a new education pedagogy, an efficient tool combines different methods with interaction, thus worth us to address in detail.

The paper begins with a typical introduction of the CPM method. The time-cost tradeoff problem is explored to help facilitating the decision-making process in time-based competition framework. Then the paper integrates the CPM method and mathematical programming in spreadsheet. Finally, how the solutions of time-cost tradeoffs can interact with the respective CPM diagrams were presented.

The Critical Path Method

Let *ES* (Early-Start) represents as the earliest an activity can start; *EF* (Early-Finish) represents as the earliest an activity can finish; *LS* (Late-Start) represents as the latest an activity can start without delaying project completion, *LF*(Late-Finish) represents as the latest an activity can finish without delaying project completion. One can use uses these information and CPM network calculations to determine when each activity must take place in order to finish the project in the least amount of time [9, 10, 11].

Methods of critical path method that are frequently used include Early-Start, Early-Finish, Late-Start, Late-Finish calculation of each activity. These information and technique allow us to identify critical activities which must start and finish on exact dates and non-critical activities whose start and finish times can vary. Because a critical path is the longest paths from project start to finish, and the total float is the maximum time an activity can be delayed without delaying completion of the project. And total float is the maximum amount of time in which an activity can be delayed without interfering with future events. So we need to find the total float (*TF*) equal to zero and find the critical path.

When we do the CPM network calculation, one can use forward pass technique and backward technique to find the zero float time activity, and define the critical path of the project. Forward pass technique is a process of finding earliest start (*ES*) times and earliest finish (*EF*) times for all activities; by which, the forward pass will give us an early-start schedule - the earliest the project can finish with the given logic and activity durations. And backward pass technique is a process of finding latest start (*LS*) times and latest finish (*LF*) times for all activities. Let i represents as beginning node of activity, and j represents as the ending node of activity. One can calculate the total float of an activity (LS_i-ES_i), we can determine the critical path(s). As an illustrative example, Figure 1 showed the network of an example facility project with ten activities. Table 1 showed the normal time vs. crash time scenarios of all activities of the project network, and their time and costs to complete the activities.

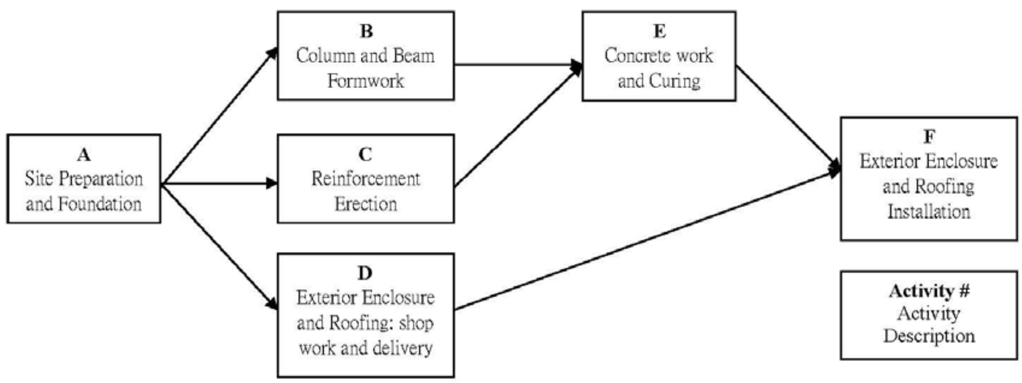

Figure 1: Illustrative example of a building construction project network

Following the critical path method describes above, one can apply Excel to calculate the total float of each activity, thus draw the critical paths of the normal and crash scenarios. Find ES, EF, LS, LF, FF, and TF for the arrow diagram in Figure 2 and Figure 3. Figure 2 showed the critical paths diagram of the normal time; the normal project duration is 130 weeks. And Figure 4 showed the critical paths diagram of the crash time; the minimum project duration is 90 weeks. The double arrows in Figure 2 and Figure 3 indicate the critical paths of the network.

Following the critical path method describes above, the project total duration and its respective total project cost needed can be obtained. After advanced evaluation, if an activity on the critical path can be shortened by more resources input, one can thus obtain the other project time and its respective cost. Basically, we can use the same way to calculate each possible combination of project time/cost one by one, and finally obtain a project's time-cost tradeoffs curve. But now including the mathematical programming will be more efficient for the problem. To simplifying the mathematical formulation and its application in time-cost tradeoff problem, it will be helpful if including the visualization spreadsheet expression.

Table 1: Activity options of the project scenarios: normal time vs. crash time

Activity #	Activity Description	Options	Duration	Cost
A	Site Preparation and Foundation	CREW1+EQUIP1+METHOD1	20	24,600
		CREW2+EQUIP2+METHOD1	28	15,000
B	Column and Beam Formwork	METHOD1	30	39,800
		METHOD2	44	30,000
C	Reinforcement Erection	EQUIPMENT1	32	60,000
		EQUIPMENT2	40	40,000
D	Exterior Enclosure and Roofing: shop work and delivery	METHOD1+RAILROAD	45	80,000
		METHOD2+TRUCK	50	60,000
E	Concrete work and Curing	METHOD1	24	36,000
		METHOD2	34	20,000
F	Exterior Enclosure and Roofing Installation	CRANE1+CREW1	14	45,000
		CRANE2+CREW2	24	40,000

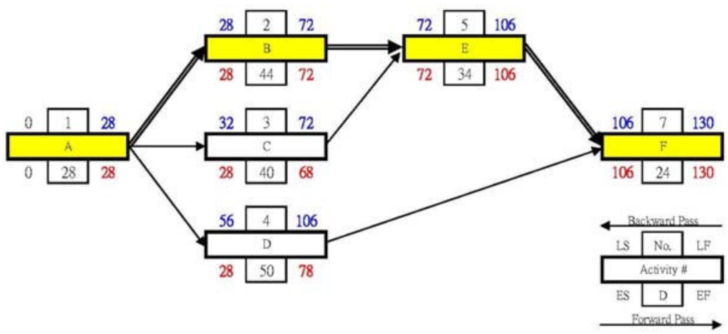

Figure 2: The CPM diagram of the project normal time

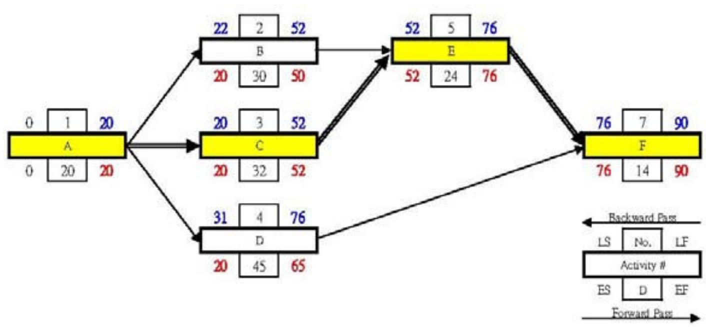

Figure 3: The CPM diagram of the minimum project time

Mathematical Approach of Time-cost Tradeoff problem

Let's denote the normal and crash time-cost points as the coordinates (D, C_D) and (d, C_d) respectively. Supposing the options of the activity can be effective combination, so that all intermediate time-cost trade-offs also are possible and that lie on the line segment between these two points. For the present, it will be assumed that the resources are infinitely divisible, so that all time between *d* and *D* are continuous feasible, and the time-cost relationship of the activity is given by the linear line. The CPM method of time-cost trade-off approach is to determine just which time-cost combination should be used for each activity to meet the scheduled project competition at a minimum cost.

Based on all *normal activity time-cost option,* the minimizing total costs principle of crash time action can be expressed as:

$$\text{Minimizing total project costs} = C = \sum_i C_{Di} + \sum_i \left(S_i \Delta d_i\right) \; ; \tag{1}$$

where Δd_i = the reduction time of activity *i*; S_i represents as the slope of activity *i*. The aggregation of all *normal activity costs* of the total project, $\sum_i C_{Di}$ is constant, so the basic information we need to address in this question is how the minimum total reduction cost. Whenever we crash each possible activity, we choose Δd_i to minimize the total additional crash cost, where the total time of the critical path is T.

To take the project completion time into account, we add an auxiliary variable y_i which expresses the earliest start time of activity *i*. For any activities with predecessor (*i*) /successor (*j*) relationship, we denote $i \rightarrow j$. So all it presents as inequality constraint, $y_j \geq y_i + D_i - \Delta d_i$, for all activity time-cost trade-off relationship. The inequality constraint showed that an activity cannot start until each of its immediate predecessors is finished. The objective function and constraints of all activities for linear programming to approach the time-cost trade-off problem then can be written as follows:

$$\text{Min. } C \tag{2a}$$

$$\text{S.t.} \quad \Delta d_i \leq \Delta d_i^* ; \qquad \text{for all activity } i. \tag{2b}$$

$$y_j \geq y_i + D_i - \Delta d_i ; \qquad \text{for all activity } i, \text{ each precedence } i \rightarrow j. \tag{2c}$$

$$y_{FINISH} \leq T ; \tag{2d}$$

$$\text{and } \Delta d_i^* \geq 0 ; \text{ for all activity } i.$$

where D_i = duration of activity *i*; Δd_i^* = the maximum reduction time of activity *i*.

Illustrative Example of the Diagrammatic and Mathematical Approach

Following the illustrative example showed in Figure 1 and Table 1 in section of Critical Path Method. Using critical path method (CPM), one can attains the maximum normal project duration is 130 weeks, and the minimum project crash duration is 90 weeks. But if we hope to attain a time-cost tradeoff curve, we need to calculate all of the possible combinations for the normal and crash scenarios.

Applying the linear programming method describes in above section, we can use Excel to find the solution of a given project time. Table 2 showed the basic input data of activities for time-cost tradeoffs model. Table 3 showed all the solutions of time-cost and its respect activity time reduction from $T=90\text{-}130$ days, where $\Delta T=1$ day. The simulation results of all time-cost combination were plotted in Figure 4.

Table 2: Basic input data of activities for time-cost tradeoffs model

Activity #	Time		Cost		Maximum Time Reduction	Crash Cost per day Added
	Normal	Crash	Normal	Crash		
A	28	20	15,000	24,600	8	1,200
B	44	30	30,000	39,800	14	700
C	40	32	40,000	60,000	8	2,500
D	50	45	60,000	80,000	5	4,000
E	34	24	20,000	36,000	10	1,600
F	24	14	40,000	45,000	10	500

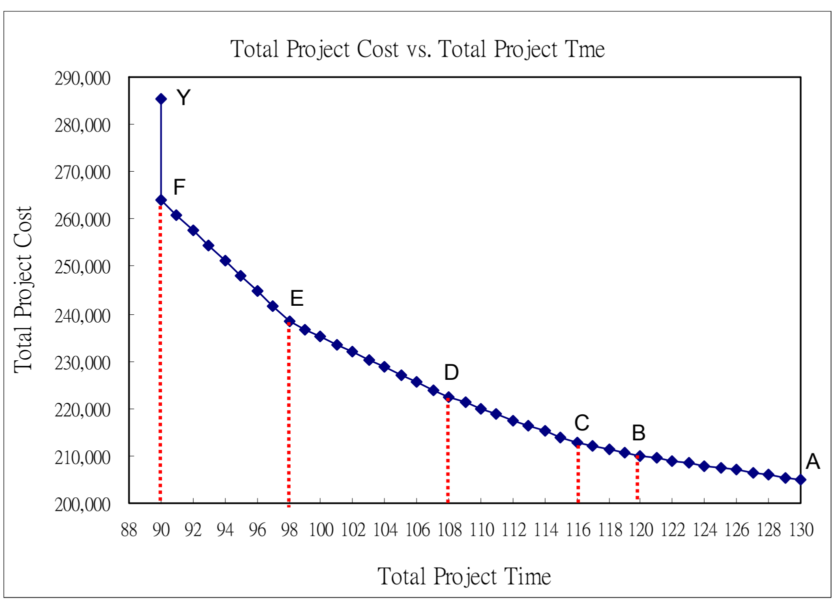

Figure 4: Time-cost tradeoff curve of the illustrative example

Table 3: Simulation results of all time-cost tradeoffs for the illustrative example

Project Finish Time	Time Reduction of Activity #						Total Cost
	A	B	C	D	E	F	
130	0	0	0	0	0	0	205,000
129	0	0	0	0	0	1	205,500
…	…	…	…	…	…	…	…
120	0	0	0	0	0	10	210,000
119	0	1	0	0	0	10	210,700
…	…	…	…	…	…	….	…
116	0	4	0	0	0	10	212,800
115	1	4	0	0	0	10	214,000
…	…	…	…	…	…	…	…
108	8	4	0	0	0	10	222,400
107	8	4	0	0	1	10	224,000
…	…	…	…	…	…	…	…
98	8	4	0	0	10	10	238,400
97	8	5	1	0	10	10	241,600
…	…	…	…	…	…	…	…
90	8	12	8	0	10	10	264,000
90							285,400

If we hope to reveal the story behind each linear segment of the time-cost tradeoff curve in Figure 4, we need to combine the information of Table 2, Table 3, and CPM diagram of the specific project duration now. For example, among activities on critical path of the normal time network showed in Figure 2, activity-F has the least crash cost per day added (S_i=-500, see Table 2). So Table 3 and Figure 5 showed that in order to crash the project time from 130 days to 120 days, the project manager need to spend more resources in activity-F. This is the case of linear segment $\overline{AB}$ in Figure 4. Whenever the maximum time reduction of activity-F is exhausted, the strategy of crashing the project time shifts to the second lower unit crash cost, activity-B (S_i=-700, see Table 2, Table 3, and Figure 6).

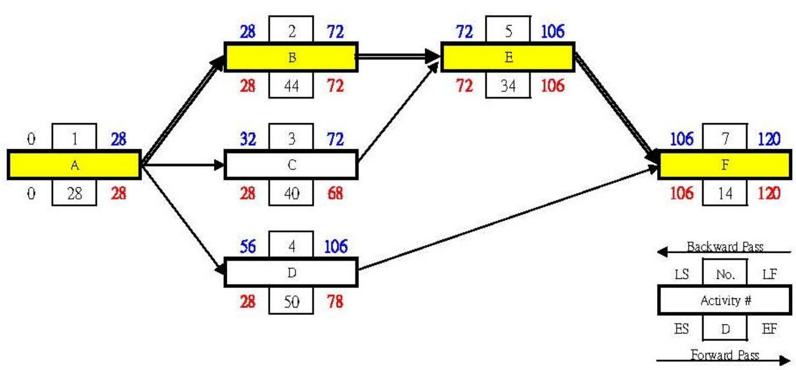

Figure 5: The CPM diagram of the project time 120 days

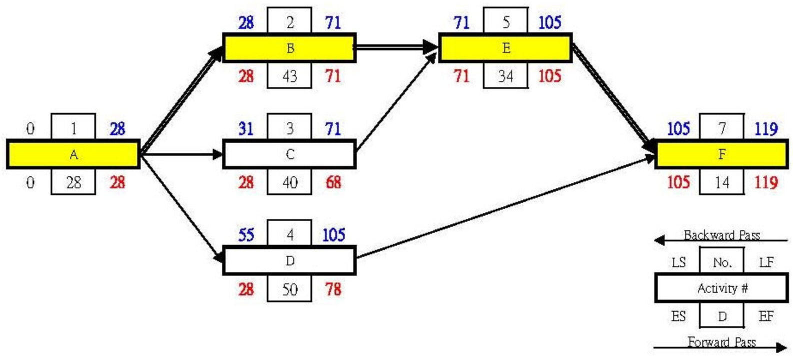

Figure 6: the CPM diagram of the project time 119 days

But whenever one hopes to crash the project time from 116 days to 115 days, activity-B will be not a good choice. The CPM diagram Figure 7 showed that if we crash activity one day more, activity-B should not an activity on the critical path and more. In this case, even activity-B has least unit crash cost, it doesn't work for crashing the project time. The solution of crashing project time as T = 115 now shift to the second lower unit crash cost of the critical path activity-A (Figure 8, Table 4). Table 3 showed that the solutions of crashing project time from 116 days to 108 days are crashing the activity-A, which in the linear segment $\overline{CD}$ in Figure 4. After the maximum time reduction of activity-A in exhausted, the crashing shifts to activity-E (the case of $\overline{DE}$ in Figure 4). One can uses the same concept described above and extends it to explore the rest segments of Figure 4, which need combine the information of LP solutions of time-cost tradeoffs and CPM diagram of specific project duration.

Table 4: Comparison different cases of specific activity time reduction: T=116→115

	T=116		T=115		T=116	
Activity	Time	Cost	Time	Cost	Time	Cost
A	28	15,000	27	16,200	28	15000
B	40	32,800	40	32,800	39	26500
C	40	40,000	40	40,000	40	40000
D	50	60,000	50	60,000	50	60000
E	34	20,000	34	20,000	34	20000
F	14	45,000	14	45,000	14	45000
Total cost		212,800		214,000		206500

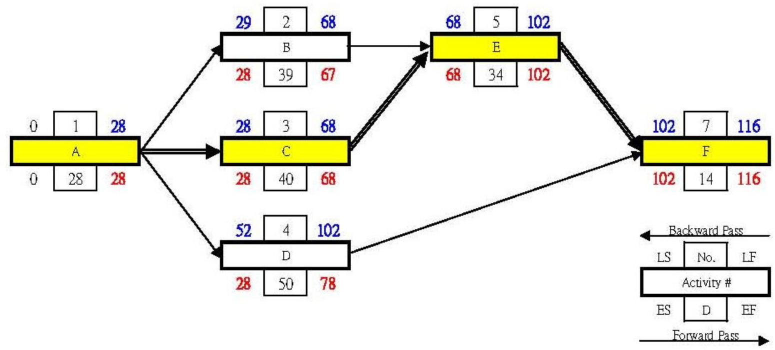

Figure 7: T=116→115; the case of crashing activity-B

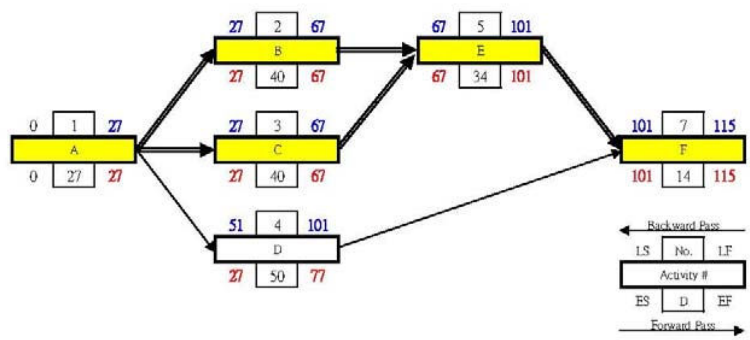

Figure 8: T=116→115; the case of crashing activity-A

Concluding Remarks

Prompted by the present emphasis on time-based competition in industry, there are more and more issues focus on the problem of time-cost tradeoffs. A potential project management involving time used of a project can always be tradeoff by additional resources input. Such a tradeoff may come from different options of the activity on the critical path of the project. The available technology of shortening the duration of each activity is often the sources of the time-cost tradeoffs problem. And the problem solving processes always rely upon the techniques of CPM calculation. After advanced evaluation, one can use the CPM to calculate each possible combination of project time/cost one by one, and finally obtain a project's time-cost tradeoffs curve. But it will be more efficient for the problem solving if one includes mathematical programming now. Reading the significant meaning of linear segments of time-cost tradeoff curve, it will be helpful if including the diagrammatic CPM and the solutions of LP together. This paper presents a diagrammatic approach in spreadsheet form, which can provide an easy-to-use tool and calculate the critical path in a more easy way. How a time-cost trade-off problem can be represented as a spreadsheet form, then use mathematical programming to obtain the time-cost tradeoff curve. The diagrammatic expression of critical path method and mathematical method will be combined with interaction way, by which a more clear and efficient exposition of solving the time-cost tradeoffs problem.

Bibliography

1. Burns, S-A, Liu, L., and Feng, C-W., 1996, LP/IP hybrid Method for Construction Time-Cost Trade-off Analysis, *Construction Management and Economics*, 14: 265276.
2. Elmaghraby, S.E., Pulat, P.S., 1979, Optimal Project Compression. with Due Dated Events, *Nay. Research Logistics* Q., 26 (2), 331-348.
3. Fulkerson, D. R., 1961. A Network Flow Computation for Project Cost Curves, *Management Science*, Vol. 7, No. 2. (Jan., 1961), pp. 167-178.
4. Kelley Jr., James E., 1961 Critical-Path Planning and Scheduling: Mathematical Basis, *Operations Research*, Vol. 9, No. 3. (May - Jun., 1961), pp. 296-320.
5. Hillier, F. S. and Lieberman, G. J., 2007, *Introduction to Operation Research*, (8th ed.), McGraw-Hill Company.
6. Hillier, F. S. and Hillier, M., 2008, *Introduction to Management Science: a modeling and case studies approach with spreadsheets*, (2th ed.), McGraw-Hill Company.
7. Liu, L., Burns, S-A and Feng, C-W., 1995, "Construction Time-Cost Trade-off Analysis Using LP/IP hybrid Method. " *J. Constr. Engrg. and Mgmt.*, 121(4) 446-54.
8. Moder, J. J., Philips, C. R., and Davis, Edward W., 1983, *Project Management with CPM, PERT and Precedence Diagramming*, 3d ed., Van Nostrand Reinhold, New York.
9. Oberlender, G. D., 2002, *Project Management for Engineering and Construction*, 2e, The Mc.Graw-Hill Conyianies, Inc.,
10. Pinedo, C., 2002, *Scheduling Theory, Algorithms, and Systems*, Prentice-Hall, Inc.
11. Stevens, J. D., 1990, *Techniques for Construction Network Scheduling*, Stanford University.