# AC 2010-130: THE MILITARY TANK – AN EXAMPLE FOR RIGID BODY KINEMATICS

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# The Military Tank – An Example for Rigid Body Kinematics


## Abstract

Rigid body kinematics in an undergraduate dynamics course is typically a challenging area for undergraduate students to master. Much of this difficulty stems from the inability to "see" or physically comprehend the motion of multiple rigid bodies. Couple this rigid body motion with the context of reference frames, and the students "sight" and understanding of the motion becomes even more clouded. Numerous examples and demonstrations exist to aid in this understanding of motion and rigid body kinetics, however, the military tank is one of the best examples for many aspects of rigid body kinematics covered in an undergraduate dynamics course. Nearly every student can picture, in his or her mind, a military tank and the motion of the chassis and the independent motion of the turret atop this chassis. It is this easy vision which allows the military tank to become such a powerful model for student understanding of rigid body kinematics in an undergraduate engineering course. The military tank is useful to show kinematic concepts of relative velocity, rotating reference frames, relative motion, and instantaneous centers of rotations. The military tank is a single example that an instructor can thread through two-dimensional kinematics, as well as, three-dimensional kinematics.


## 1. Introduction

One of the most difficult concepts for students in an undergraduate dynamics course is that of rigid body kinematics. The geometry of rigid body motion, a topic most students are familiar with from undergraduate physics, takes on additional complexity as one introduces angular velocities and accelerations of rigid bodies. Couple this with the fact that the majority of students in an undergraduate dynamics course just completed an undergraduate statics course free of motion, and the result is clouded confusion and inability to understand what is really happening to the rigid body.

Dynamics is a course best taught with demonstrations and videos of the motion of rigid bodies. Unfortunately, textbook pictures require the student to imagine the motion of these rigid bodies from a still picture with limited depth perspective. For new students just entering the engineering discipline, the ability to imagine this motion can be quite difficult depending upon the student's limited engineering experience and intuition. This topic of the "novice" college student has been investigated by numerous researchers, but Wankat and Oreovicz comment that when solving problems, students in general are not proficient at strategy, interpretation, and generation.[1] It is this interpretation that a "good" model or demonstration can assist with.

Most engineers tend to be primarily left-brain-oriented, which is mainly involved in verbal analytical thinking.[2] The right hemisphere of the brain mainly processes visual and perceptive thought, and its mode of processing involves intuition and leaps of insight. Since engineering education is predominantly left-hemisphere oriented, fostering the student's use of the right side of the brain becomes an integral part of successful learning. Adams identified a perceptual block that students encounter where they have difficulty seeing various aspects or ramifications of a problem.[3] It is important to get past this block because visual learning techniques increase the student's comprehension and learning.[4] Engineering educators facilitate such visual learning through pictures, images, and demonstrations. Educators desire to get the right brain involved, thus increasing conceptual understanding and perceptive abilities.

The "good" model that Wankat and Oreovicz comment on for rigid body kinematics is the military tank (Figure 1).   Its motion of chassis, turret, and barrel is easily recognized by the students.  Their interpretation of the motion becomes much easier than a random linkage or connection of rigid bodies never seen before, that must be scrutinized to even understand the motion.  Through easy recognition, the military tank becomes a powerful image and demonstration for the visual learner.  The right hemisphere of the brain is engaged in a manner that increases the students understanding and education of the kinematics inherent in the motion. The educator needs only to tie the analytic equations of the kinematics to the already-known motion of the military tank.


**Figure 1. Example of Toy Military Tank (with attached body-fixed coordinate systems)**

## 2.   The Military Tank Example – Rigid Body Planar (2D) Kinematics

The military tank's usefulness first becomes apparent when introducing angular velocity and rotation about a fixed axis.  Describing angular velocity as a spin rate about a spin axis, students can easily picture the turret motion on the tank chassis and thus understand the vector relation (magnitude and directionality) of angular velocity.  Figure 1 illustrates angular velocity using the military tank.

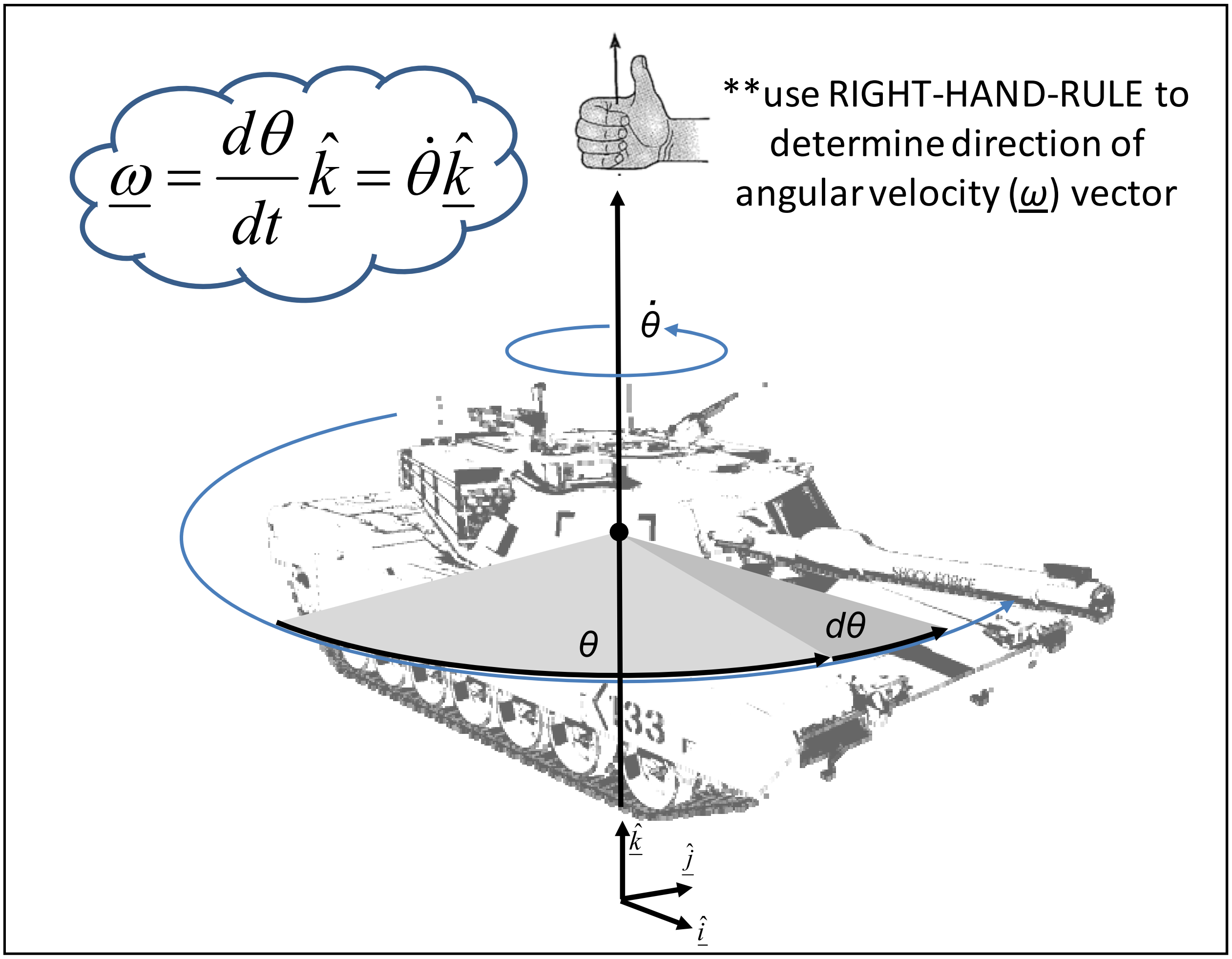


**Figure 2. Angular velocity (spin rate about a spin axis) illustrated using the military tank**

By marking a point on the barrel (i.e. the end of the barrel), the motion of point *P* can further be illustrated (Figure 2). It is easy to show that the velocity of point *P* (the end of the barrel) is the cross product of the angular velocity and the position vector from the point of rotation out to *P*, $\underline{v}_P = \underline{\omega} \times \underline{r}_{P/O}$.

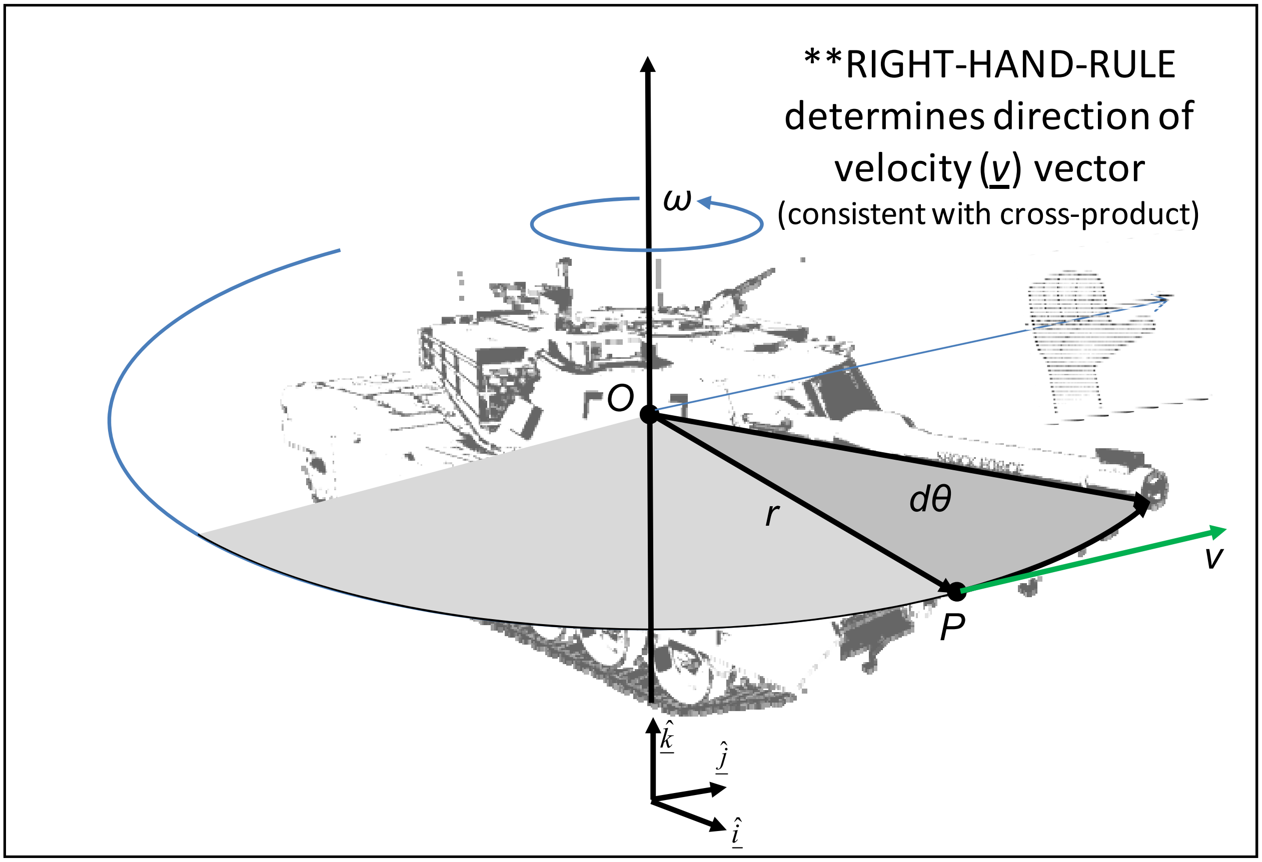


**Figure 3. Motion of Point P (Velocity)**

Similarly, the acceleration of point P can be illustrated in terms of its normal and tangential components (Figure 3). The total acceleration of point *P* expressed in vector form is $\underline{a}_P = \underline{\alpha} \times \underline{r}_{P/O} + \underline{\omega} \times \left(\underline{\omega} \times \underline{r}_{P/O}\right)$. The tank illustrates these concepts better than other generic rigid bodies because the barrel extends out to a point (point *P*) that the students easily identify its location away from the rotation point (axis) and the angular motion associated with this point due to the turret rotation. Again, it is this familiarity with the military tank motion, which aids the synthesis of understanding of these aspects of rigid body kinematics.

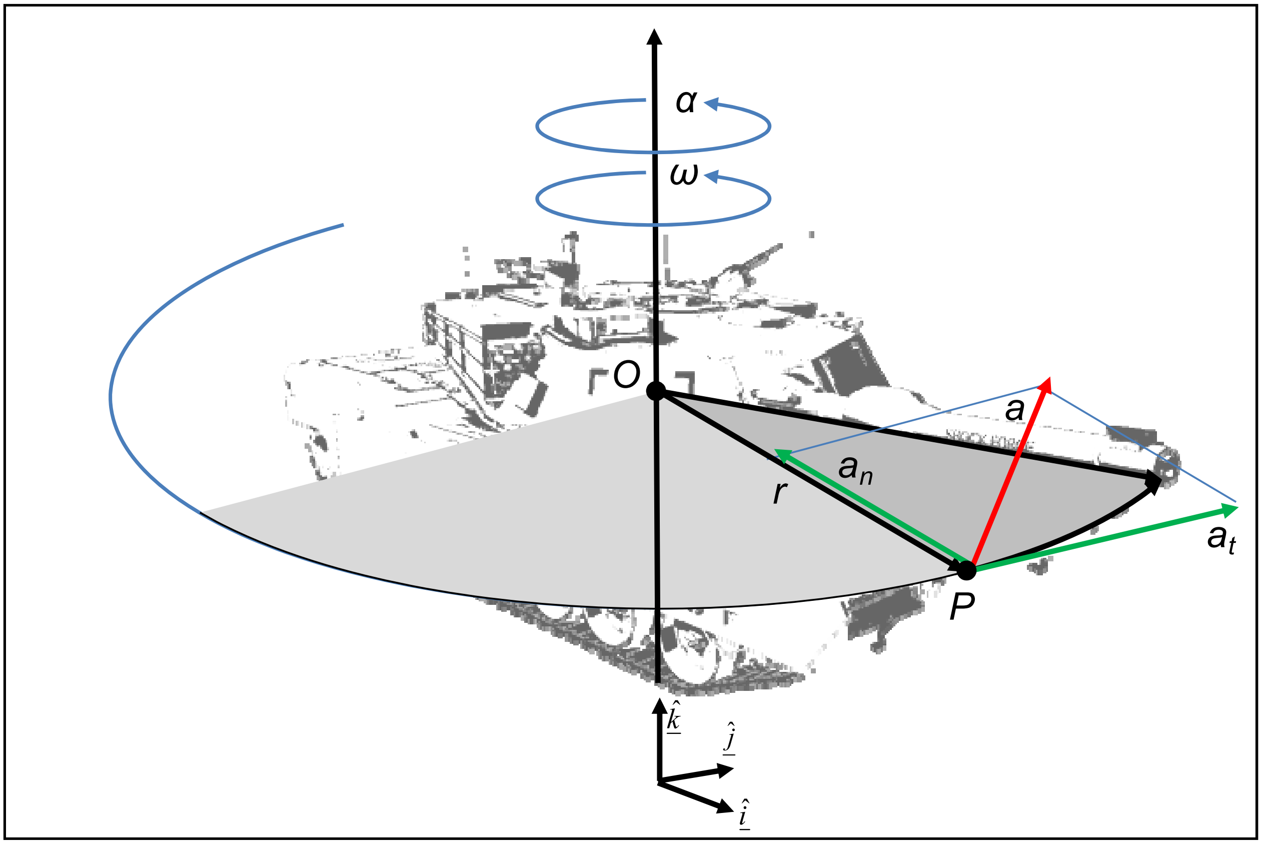

**Figure 4. Motion of Point P (Acceleration)**

With the students' thorough understanding of angular velocity and rotation about a fixed axis, the military tank shows even more value illustrating rotating reference frames and the concepts of relative motion. For planar motion, the military tank can be thought of as two distinct reference frames – the chassis reference frame and the turret/barrel reference frame. Students can relate to these two distinct reference frames because they already know the independent motion of the turret with respect to the chassis. If the chassis (reference frame N) remains fixed to the ground and does not rotate but the turret (reference frame T) is free to rotate, the relative motion of two points, *P* and *Q* on the turret can be discussed (Figure 4). By fixing the chassis of the tank to the ground, we can establish a fixed inertial point, *O*, and establish a Newtonian (inertial) reference frame, N.

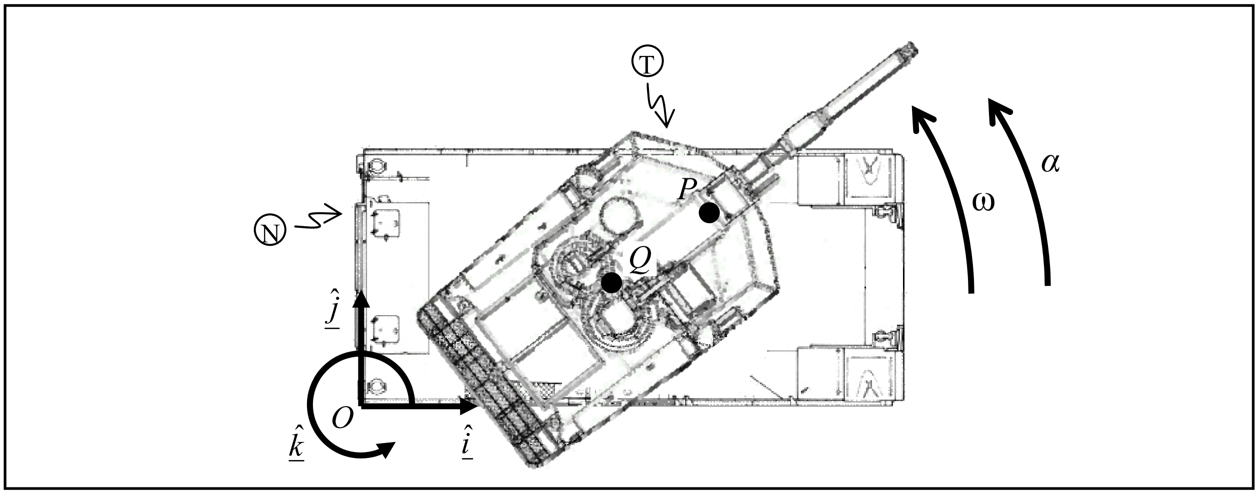


**Figure 5. Two Points Fixed on a Rotating Reference Frame**

Point $Q$ is fixed to the turret in reference frame T, at the location of the turret rotation point. Point $P$, the bullet in the barrel, is also fixed to the turret in reference frame T, at the location of the tank round prior to firing. The turret rotates at a rate of ω in a counter-clockwise direction. Given the position of point $Q$ with respect to fixed inertial point $O$, $\underline{r}_{Q/O}$ and the relative position of point $P$ with respect to point $Q$, $\underline{r}_{P/Q}$ and using relative motion equations, one can find the position of point $P$ with respect to fixed inertial point $O$, $\underline{r}_{P/O}$. Differentiating this relative position with respect to the Newtonian reference frame results in the relative velocity equation for two points fixed in the same reference frame: $\underline{v}_P = \underline{v}_Q + \underline{\omega} \times \underline{r}_{P/Q}$. Differentiating again with respect to the Newtonian reference frame results in the relative acceleration equation for two points fixed in the same reference frame: $\underline{a}_P = \underline{a}_Q + \underline{\alpha} \times \underline{r}_{P/Q} + \underline{\omega} \times \left(\underline{\omega} \times \underline{r}_{P/Q}\right)$. Conceptually, this can be difficult for the student to understand, but using a familiar example with motion that is easily visualized, the student's understanding is easier and faster.

The military tank example can be extended to include relative motion on a rotating reference frame. Now the bullet that was initially fixed in the reference frame of the turret is fired from the tank and is free to move down the rotating barrel (Figure 5).

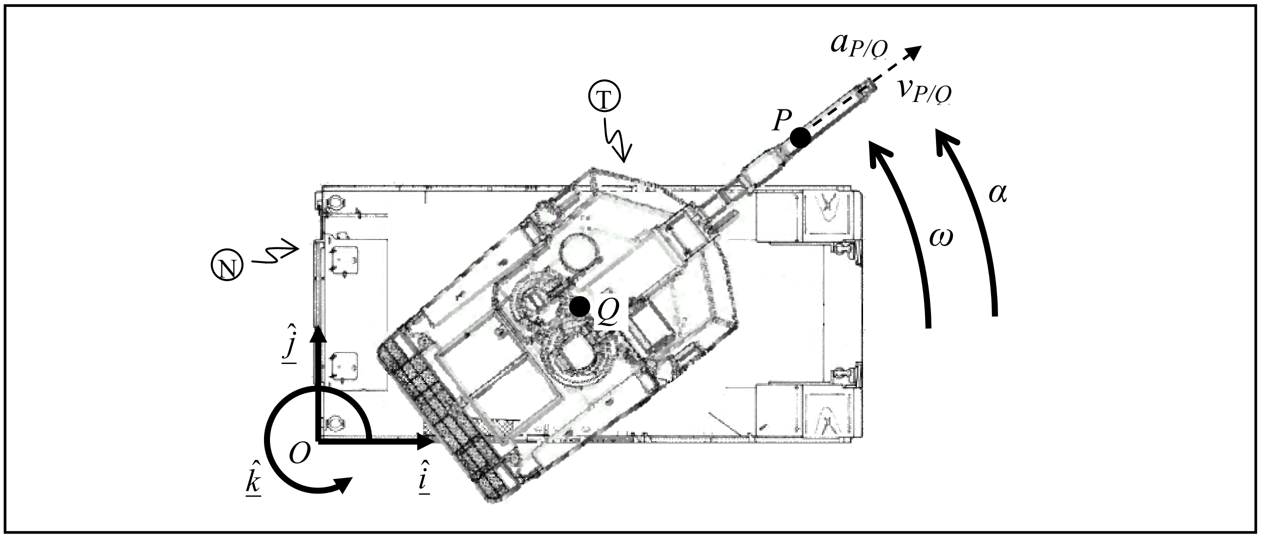


**Figure 6. Relative Motion on a Rotating Reference Frame**

Using the equation for the position of point *P* with respect to fixed inertial point *O*, $\underline{r}_{P/O}$, and differentiating with respect to the Newtonian reference frame, the result is the most general form of the relative velocity equation for two points in the same reference frame:

$$\underline{v}_P = \underline{v}_Q + {}^T\underline{v}_{P/Q} + \underline{\omega} \times \underline{r}_{P/Q}$$

Differentiating again with respect to the Newtonian reference frame results in the relative acceleration equation for two points in the same reference frame:

$$\underline{a}_P = \underline{a}_Q + {}^T\underline{a}_{P/Q} + \underline{\alpha} \times \underline{r}_{P/Q} + 2\left(\underline{\omega} \times {}^T\underline{v}_{P/Q}\right) + \underline{\omega} \times \left(\underline{\omega} \times \underline{r}_{P/Q}\right)$$

Again, while this most general form of the relative acceleration equation is conceptually difficult for many students, the fact that they understand the bullet moving down the barrel as it is fired while the turret is rotating assists in understanding the kinematic concept of relative motion in a rotating reference frame. By threading the military tank model through consecutive topics that build upon one another, the overall student learning of rigid body kinematics is increased

One additional concept in planar kinematics, the concept of *instantaneous center of rotation (ICR)* or *instantaneous center of zero velocity (IC)*, can easily be shown with the military tank. Particularly when discussing a non-slip wheel. The tread on the tank is an excellent example of an instantaneous center of zero velocity. If you take a circular shape of foam material (easily deformable, yet elastic) you can easily show that at the contact point at the ground (foam wheel / ground interface) the velocity of the contact point is zero velocity (it does not translate relative to the ground). If you squish it down like a tank tread, so that a large portion is flat against the ground, the students can observe that a marked portion of the foam trend remains in contact (and zero velocity) with the ground as the tank translates over it (Figure 8). As you allow the foam to reform to its circular shape, that single point continues to remain in contact with the ground and is a point of zero velocity on the wheel. The rigid body of the wheel can now be seen to rotate about this zero velocity rotation point (or axis).

**Figure 7. Foam Tank Tread / Non-slip Wheel Demonstration of Instantaneous Center of Rotation**

## 3. The Military Tank Example – Rigid Body (3D) Kinematics

As the complexity of rigid body kinematics continues into three dimension, the military tank model continues to be valuable to the student because of the student's ability to picture the motion. If one analyzes the military tank in terms of all three dimensions, the motion of the chassis, turret, and barrel set the stage for excellent examples of consecutive rotations and relative motion in three dimensions. While there is extensive set-up for a problem such as this, it is consistent with previous examples in the course and simple builds upon the two-dimensional representations seen earlier.

The military tank depicted in Figure 8 can be described using three reference frames, each with the ability to rotate: the chassis is reference frame *C*, the turret is reference frame *T*, the barrel is reference frame *B*. Let Newtonian reference frame *N*, have an inertial coordinate system with right-handed unit vectors $\underline{\hat{i}}, \underline{\hat{j}}, \underline{\hat{k}}$ and origin *O*. Let the chassis have a body-fixed coordinate system with right-handed unit vectors $\underline{\hat{c}}_1, \underline{\hat{c}}_2, \underline{\hat{c}}_3$ and origin $O_C$. Let the gun turret have a body-fixed coordinate system with right-handed unit vectors $\underline{\hat{t}}_1, \underline{\hat{t}}_2, \underline{\hat{t}}_3$ and origin $O_T$. Let the gun barrel have a body-fixed coordinate system with right-handed unit vectors $\underline{\hat{b}}_1, \underline{\hat{b}}_2, \underline{\hat{b}}_3$ and origin $O_B$.

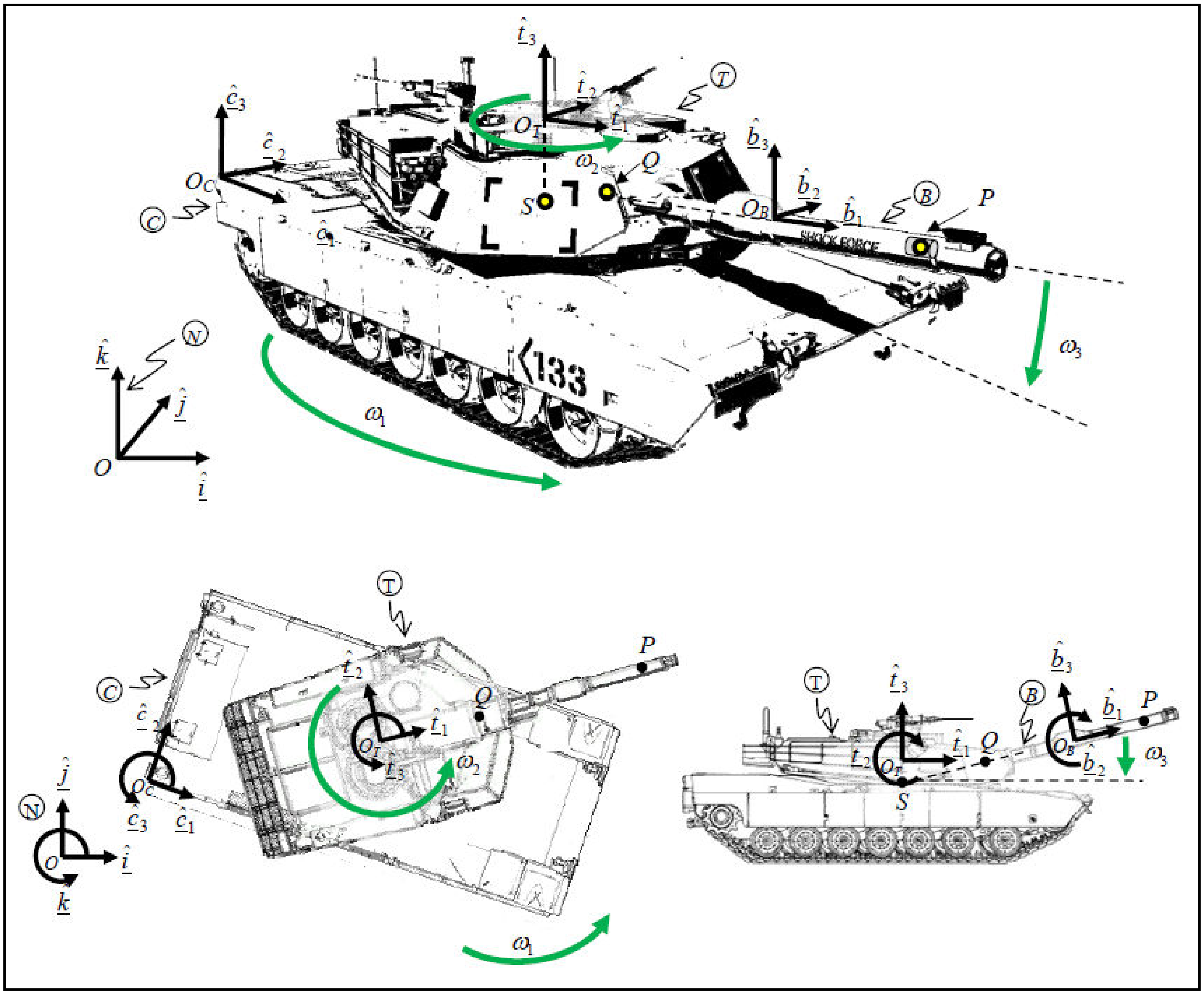


**Figure 8. Relative Motion in Three Dimensions**

While engaging the enemy tank, the chassis $C$, translates relative to the ground at a constant speed $v$ ($\underline{v}_{O_C} = v\hat{\underline{c}}_1$) and turns with a time varying spin rate $\omega_1$ with respect to the ground, the turret $T$, turns with a time varying spin rate $\omega_2$ with respect to the chassis, and the barrel $B$, depresses with a time varying spin rate $\omega_3$ with respect to the turret.

Let point $S$ be along a line drawn from $O_T$ in the $-\hat{\underline{t}}_3$ direction at the intersection of the chassis and turret (fixed in $C$ and $T$), such that $\underline{r}_{S/O_C} = s_1\hat{\underline{c}}_1 + s_2\hat{\underline{c}}_2 + s_3\hat{\underline{c}}_3$. Let point $Q$ be along a line drawn from $O_B$ in the $-\hat{\underline{b}}_1$ direction at the rotation point of the barrel (fixed in $T$ and $B$) such that $\underline{r}_{P/S} = q_1\hat{\underline{t}}_1 + q_2\hat{\underline{t}}_2 + q_3\hat{\underline{t}}_3$. Let the tank round be modeled simply as particle $P$ and that $P$ is moving relative to $Q$ in the $\hat{\underline{b}}_1$ direction, such that $\underline{r}_{P/Q} = p\hat{\underline{b}}_1$. While in the gun barrel, the round, $P$, has a speed $\dot{p}$ and acceleration $\ddot{p}$, both measured relative to the main gun barrel.

Again, the set-up and given information is extensive, but the student is synthesizing what they are reading with the three-dimensional motion of the military tank that they already know.

The student understands the problem set-up with much more clarity simply because they can envision the military tank in action and are only describing this motion in a kinematic sense that is now logical to them.

This scenario easily lends itself to an example of the addition theorem of angular velocity. The addition theorem tells us we can simply add the angular velocities measured with respect to different reference frames to find the total angular velocity. That is, the angular velocity of a rigid body *B* in a reference frame *A* can be expressed in the following form involving *n* auxiliary reference frames:

$$ {}^{A}\underline{\omega}^{B} = {}^{A}\underline{\omega}^{A_1} + {}^{A_1}\underline{\omega}^{A_2} + \cdots + {}^{A_{n-1}}\underline{\omega}^{A_n} + {}^{A_n}\underline{\omega}^{B} \qquad 5 $$

In the military tank example, to find the angular velocity of the barrel (reference frame *B*) with respect to the Newtonian reference frame *N*, the angular velocities of the intermediate reference frames (chassis *C* and turret *T*) must be considered. Using the addition theorem the angular velocity of the barrel reference frame *B* with respect to the Newtonian reference frame *N* is:

$$ {}^{N}\underline{\omega}^{B} = {}^{N}\underline{\omega}^{C} + {}^{C}\underline{\omega}^{T} + {}^{T}\underline{\omega}^{B} $$

This makes sense to the student because he or she knows that the total angular velocity of the barrel depends upon how fast the chassis is turning on the ground plus how fast the turret is turning on the chassis plus how fast the barrel is depressing on the turret. This is intuitive knowledge that the student knows and now understands mathematically as well.

The same three reference frames of the military tank (chassis *C*, turret *T*, and barrel *B*), along with the Newtonian reference frame *N*, can be used in continuation of the scenario, to assist in the understanding of angular acceleration ($\alpha$). By differentiating the angular velocity equation above and using the addition theorem again, one can determine the angular acceleration of the barrel reference frame *B* with respect to the Newtonian reference frame *N*:

$$ {}^{N}\underline{\alpha}^{B} = {}^{N}\underline{\alpha}^{C} + {}^{C}\underline{\alpha}^{T} + \left({}^{N}\underline{\omega}^{C} \times {}^{C}\underline{\omega}^{T}\right) + {}^{T}\underline{\alpha}^{B} + \left(\left({}^{N}\underline{\omega}^{C} + {}^{C}\underline{\omega}^{T}\right) \times {}^{T}\underline{\omega}^{B}\right) $$

It should become apparent to the student, that there is no addition theorem for angular accelerations. There are cross products of the angular velocities of the intermediate reference frames, as well as, the angular accelerations of the intermediate reference frames when determining the angular acceleration of the barrel reference frame *B* with respect to the Newtonian reference frame *N*.

With this scenario set of the tank maneuvering and engaging an enemy target and the equations for angular velocity and angular acceleration above, one can conduct the kinematic analysis for the relative motion of point *P* (tank round as it moves down the barrel):

$$ \underline{v}_{P} = \underline{v}_{O_C} + \left({}^{N}\underline{\omega}^{C} \times {}^{C}\underline{r}_{S/O_C}\right) + \left({}^{N}\underline{\omega}^{T} \times {}^{T}\underline{r}_{Q/S}\right) + {}^{B}\underline{v}_{P/Q} + \left({}^{N}\underline{\omega}^{B} \times {}^{B}\underline{r}_{P/Q}\right) $$

$$\underline{a}_P = \underline{a}_{O_C} + \left({}^N\underline{\alpha}^C \times {}^C\underline{r}_{S/O_C}\right) + \left({}^N\underline{\omega}^C \times \left({}^N\underline{\omega}^C \times {}^C\underline{r}_{S/O_C}\right)\right) + \left({}^N\underline{\alpha}^T \times {}^T\underline{r}_{Q/S}\right) + \left({}^N\underline{\omega}^T \times \left({}^N\underline{\omega}^T \times {}^T\underline{r}_{Q/S}\right)\right)$$
$$+ \left({}^N\underline{\alpha}^B \times {}^B\underline{r}_{P/Q}\right) + \left({}^N\underline{\omega}^B \times \left({}^N\underline{\omega}^B \times {}^B\underline{r}_{P/Q}\right)\right) + {}^B\underline{a}_{P/Q} + 2\left({}^N\underline{\omega}^B \times {}^B\underline{v}_{P/Q}\right)$$

These equations are typically challenging for the student to understand, but generally become clearer using the military tank scenario, than using other (less imaginable) three-dimensional rigid body systems. The instructor can explain each component of the equations above, talking through the military tank reference frames and showing the students (with the toy tank model) how each part of the equation has physical meaning.

Lastly, by using just the barrel of the tank (rigid body *B*), one can use the military tank scenario to determine the angular momentum of the barrel in three dimensions as it depresses and rotates (Figure 9).

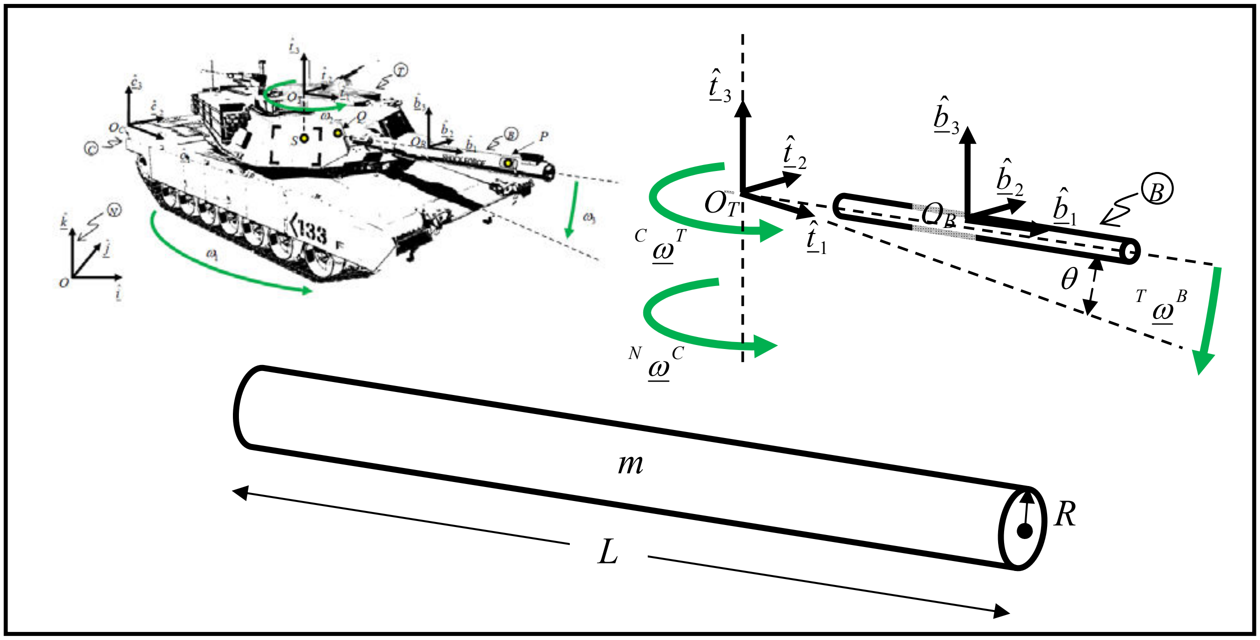


**Figure 9. Angular Momentum of Barrel (Rigid Body *B*) in Three Dimensions**

Using principal axes for the barrel and approximating the barrel as a circular cylinder of known mass $m$, length $L$ and radius $R$, one can determine the principal inertial matrix $[\bar{I}]$. Then using the three given angular velocities of the chassis, turret, and barrel above (and that the rotation angle between the barrel $B$ and the turret $T$ reference frames is $\theta$), one can determine the angular momentum of the barrel about its center of mass, $O_B$:

$${}^B\underline{H}_{O_B} = \frac{mR^2}{2}\left({}^N\underline{\omega}^C + {}^C\underline{\omega}^T\right)\sin\theta\,\hat{\underline{b}}_1 + \frac{m\left(3R^2 + L^2\right)}{12}\left({}^T\underline{\omega}^B\right)\hat{\underline{b}}_2 + \frac{m\left(3R^2 + L^2\right)}{12}\left({}^N\underline{\omega}^C + {}^C\underline{\omega}^T\right)\cos\theta\,\hat{\underline{b}}_3$$

While this is certainly a more advanced topic of kinematics, the military tank scenario makes it easier for the student to digest and comprehend. The student knows that the barrel rotates in multiple dimensions and thus, the equation above seems logical that there is angular momentum in all three unit vector directions.

## 4. Conclusions

While this paper does not offer any experimental results or evidence that using the military tank as a model for rigid body kinematics earns students higher marks in dynamics or increases their conceptual knowledge of kinematics, it seems logical that utilizing a recognizable system, in terms of motion, is advantageous for the student learner. The fact that most students can quickly envision the motion of a military tank as it translates across the landscape with both the turret and barrel traversing and elevating, respectively, increases the student's comprehension in the subject of rigid body kinematics. The military tank example, while complex in the three dimensional sense, can be threaded throughout the entire rigid body kinematics block starting with planar two dimensional motion through to three dimensional angular momentum. This consistent example thread can only increase the students conceptually understanding of a traditionally difficult concept of rigid body kinematics.

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