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The 'Typical Particle' Approach to Learning Rigid Body Dynamics

Prof. Keith D. Hjelmstad, Arizona State University

Keith D. Hjelmstad is President's Professor of Civil Engineering in the School of Sustainable Engineering and the Built Environment at Arizona State University.

Amie Baisley, University of Florida

Amie Baisley is a lecturer in the Department of Engineering Education at the University of Florida. Her teaching and research interests are centered around the sophomore level courses that engineering students take and how changes in those courses can impact student learning and retention.

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Abstract

Rigid body dynamics is a foundational course in all engineering curricula based upon the mechanical sciences. It is one of three courses that make up *The Mechanics Project*, an effort at a large R1 university in the southwest, to reimagine the learning experience in the sophomore-level engineering mechanics courses (statics, dynamics, and deformable solids). The conversion of these courses to an objective-based system to assess mastery launched a reconsideration of the fundamental strands—the DNA—of the courses. The design objective of focusing learning as much on ‘why’ as on ‘how’ suggested that students should learn how to derive equations of motion from first principles. This approach led to a set of objectives that are a framework to solve any rigid body dynamics problem. The resulting approach differs from the more traditional approach with special equations, already derived, to solve certain types of problems (which can promote plug-and-chug problem solving). Our approach is built around the description of the position vector of a *typical particle* in the system. From there, students sum forces and moments over all the particles to get the equations of motion, essentially leading them through the steps that Euler took to generalize Newton’s laws of motion. Each problem requires the student to visualize and mathematically describe the motion of the system at hand. This approach allows the students to see where the equations of motion come from, it provides a unique opportunity to master vector notation, and it reinforces and improves skills in calculus and differential equations. This paper describes our approach to learning dynamics with an example to show the key role of the position vector in the setup of every dynamics problem.

Introduction

Rigid body dynamics has been a core foundational course for engineering students for decades. Along with statics and deformable solids, this course provides a pivot from the basic math and science courses to engineering courses. The rigid body dynamics course provides an opportunity for engineering students to develop a facility for applying Newtonian mechanics to problems of interest in engineering. Understanding the relationship between force and motion is vital to accessing more advanced knowledge in civil, mechanical, and aerospace engineering.

With the emergence of computational mechanics and the ubiquity of commercial codes in industry, a shift has occurred in what students need to know from their foundational mechanics courses. While the basic concepts are unchanged, the nature of what students need to do in the process of learning those concepts has changed, especially for dynamics. A close look at the problems in standard textbooks on dynamics reveals what might be termed ‘snapshot’ dynamics. In essence, dynamics has traditionally been taught as an extension of statics. For example, a question might ask to evaluate a reaction force or the acceleration at time zero, or the velocity at the state where the kinetic energy is maximum. Dynamics is the study of the evolution of a system in time and that has not been a centerpiece of traditional courses on the subject.

There has also been a dominant tendency to try to simplify problems by lumping distributed effects (e.g., by using resultants for forces and mass moment of inertia and total mass for inertial terms). In essence, this approach attempts to reduce the dynamics of bodies of finite size to an approach similar to particle mechanics. In fact, one can view the lumped versions of balance of linear and angular momentum as being a parallel to the particle $\mathbf{F} = m\mathbf{a}$ by adding, for planar motion, for example, the equation $M = J\alpha$, where M is the net moment, J is the mass moment of inertia, and α is the angular acceleration. One casualty of the process of lumping distributed effects is that students have a hard time developing a sense that the location of mass matters, as does the nature of distribution of forces. The distribution of mass manifests in J, but its computation is separated from balance of moment and is often just extracted from a table.

The Mechanics Project

Textbooks and most traditional learning environments have yet to embrace what is known about student learning [1]. The way information is organized in the traditional layout promotes grouping problems based on the special equations that can be used to solve them [2]. This approach fosters a plug-and-chug mentality wherein the student seeks an equation that appears to have the right inputs and outputs as a means of solving the problem. Some resources are even organized around the idea of providing an equation that suits each different situation even within the same general problem type. Associating an equation with a problem does not allow the students to develop a sense of how the equations come about.

As part of the redesign to the sophomore mechanics courses—an effort we refer to as *The Mechanics Project*—the course on *Engineering Dynamics* was completely redesigned. At the start of the project, we imagined the redesign to be only pedagogical, implementing flipped classroom environments, mastery-based grading, and other strategies. However, as we put those ideas into place, we realized that there were significant content issues as well. We discuss some of the pedagogical strategies elsewhere [3], [4].

In this paper we focus on the content issues in the course, particularly as they relate to the task of getting students to master fundamental concepts in dynamics. In our design, we wanted to address the following observations that we have made about students in traditional learning environments in *Dynamics*:

- Students are typically introduced to vectors but fail to develop facility using vector notation (especially direct notation) as a tool to derive or process equations. In some texts, vector notation is viewed as a power tool to break out only when absolutely necessary.
- While dynamics always results in a differential equation and the derivation of equations of motion relies heavily on the calculus, students are often schooled in workarounds (like lumping distributed effects). These workarounds rob the student of opportunity to advance and perfect their understanding of calculus and differential equations.
- Almost all rigid body problems in textbooks have constraints but the traditional approaches to these problems often do not force the students to grapple with the constraints in a way that allows them to master how to handle them.
- Many students do not see the connection between particle dynamics and rigid body dynamics, treating them as different systems governed by different theories.
- Perhaps because most curricula introduce statics first, students struggle to develop an understanding of how to characterize motion of a system. If they are unable to characterize motion mathematically, they will be hard pressed to make appropriate connections between force and motion.

Our main goal was to find an approach that would put students on a more reliable path toward expert-like thinking about mechanics and discourage the propagation of the plug-and-chug mentality. With the advent of computational mechanics, it is more important for students to learn how to derive equations than it is to learn techniques to solve special cases that happen to be amenable to classical solution. A focus on derivation also keeps students well away from plug-and-chug thinking because they are forced to start from fundamental concepts and build up equations of motion from first principles.

This shift in learning strategy does not require the creation of problems substantially different from those found in traditional textbooks, but it does require laying out a different approach to setting up and solving those problems. It is fairly simple to break dynamics down into key strands that are a part of virtually every dynamics problem [4]. We call these strands the *mastery objectives*. If a student can master these things, then they will be able to solve any problem in rigid body dynamics. Furthermore, they will have a solid understanding of how the pieces of the theory fit together and how the tools of mathematics support the problem-solving process. The mastery objectives for dynamics are given in Table 1.

Table 1. *Dynamics Mastery Objectives*. This table give a brief description of the 16 mastery objectives for *Dynamics*. The objectives are the common strands that form the problem-solving approach for all dynamics problems.

A.1. Geometry and problem setup	F.1. Vector algebra and calculus
A.2. Initial conditions	F.2. Integrate over spatial domain
A.3. Modeling and constraints	G. Conservation of momentum
B. Describe position vector	H.1. Classical solution to differential eqns.
C. Compute velocity and acceleration	H.2. Natural frequencies and mode shapes
D. Free body diagrams	J. Compute dynamic response
E.1. Balance of linear momentum	K. Compute energy and work
E.2. Balance of angular momentum	L. Apply work/energy principles

The objectives labeled A (i.e., A.1, A.2, and A.3) are really about parsing the problem and coming to grips with the geometric features of that particular problem. Objective B is the representation of the position vector in mathematical terms (as translated from the geometry sketch of Objective A). This step is the crux of our approach and is where the *typical particle* makes its first appearance. Objective C is a simple act of differentiation of the position vector to get expressions for velocity and acceleration. Objective D is a sketch of the free body diagram, where the force associated with the *typical particle* is explicitly indicated. In Objective E, balance of momentum is established from the FBD and the acceleration derived in Objective C. These objectives represent the core derivation tasks in dynamics.

In Objective F we distill the mathematical operations needed to solve the problem. In particular we identify two key mathematical steps that come up in every rigid body problem: vector algebra and integration over the domain of the body. We only call out the mathematical pieces that are relatively new to the students at this level. Mathematical operations happen throughout the derivation and solution process. We separate out execution of mathematical steps as distinct objectives so that students will focus on improving on those tasks. Also, we want to track how students progress in their physical understanding apart from some of the more tedious mathematical operations.

Objective K involves the computation of kinetic and potential energy. Apart from applied loads, the kinetic and potential energy of the rigid body is the sum of the kinetic and potential energies possessed by the particles. Hence, we formulate the energy of the *typical particle* and sum (i.e., integrate) over the domain of the body.

Objectives G, H, J, and L do not explicitly relate to the *typical particle* construction. So, we will not describe them further here.

These 16 objectives comprise a comprehensive set of concepts in dynamics. Most problems activate many of the objectives (but usually not all of them). Every problem that the students see in the course (including the mastery assessments) are organized around these objectives.

This approach to learning is to have students solve problems, using the context of the mastery objectives to organize their approach, which they do in a flipped classroom facing new problems each class period [4]. The course is organized in two-week modules, as shown in Table 2. The mastery objectives are generally available in all problems across the modules, so students encounter the objectives repeatedly with steadily increasing complexity as the semester progresses.

Table 2. Module topics for *Dynamics*. These topics are similar to courses on this subject taught in a traditional lecture-based format. The mastery objectives span all of the topics in the course.

Module	General Topic
1	Particles: kinematics and kinetics
2	Particles: work and energy
3	Rigid bodies: mass distributed along a line
4	Rigid bodies: mass distributed over an area or volume
5	Generalizing the concepts of planar motion for rigid bodies
6	Vibration of linear systems
7	The Euler-Lagrangian equations and energy methods

Since the mastery objectives contain components of derivation, students learn how to derive. Because the mastery objectives require the application of calculus tools, they improve their ability at doing the requisite mathematics. This latter benefit allows students to make a connection between the mathematics they are required to learn as freshmen and the problems they will solve in their engineering major, thereby minimizing a classic disconnect among subjects that hampers the student's ability to scaffold concepts.

The typical particle construction

Our approach to organizing the subject of dynamics centers on the notion of the *typical particle*. The idea is to take them down the path that Euler went when extending Newton's laws to rigid bodies. We build rigid body dynamics from particle dynamics, noting that a body can be broken up into pieces for which the laws of particle mechanics hold and then summing the contributions from the particles noting that the internal forces cancel out in the summation process. Most textbooks do this in one form or another, but we have the students take these steps repeatedly to form the derivation. The *typical particle* emerges from this derivation in a manner that says, "If you can do the dynamics for one particle, then all we need to do is sum the particles to find the equations of motion of the rigid body."

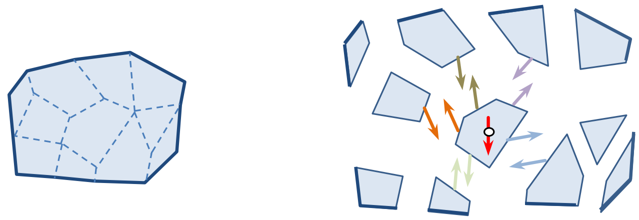

Fig. 1. *The body as a collection of particles*. To build the equations of motion for a rigid body we note that each particle obeys Newton's second law **F**=m**a** and the third law (which creates the equal and opposite forces at the cuts in the FBD). When we sum the particle equations, the internal forces cancel out. What remains is the sum of external forces equal to the sum of the mass times acceleration. Balance of angular momentum can be derived in the same way.

Figure 1 illustrates the representation of the body as a collection of particles, showing the internal forces of interaction between them (on one of the particles and its neighbors). Each particle obeys **F**=m**a**. If the particle equations are summed, the internal forces cancel out (due to Newton's third law). What remains is the sum of external forces and that is equal to the sum of mass times acceleration of the particles. In the limit the sum is an integral over the domain of the body. A similar approach yields the equations of balance of angular momentum. Most textbooks have some variation of this approach.

The process of deriving the equations of motion using the typical particle construction is best illustrated through an example. To describe the kinematics and kinetics, we identify a *typical particle* A as shown in Fig. 2. This problem has a disk of radius R rolling without slipping on a circular surface of radius $2R$. The task is to find the equations of motion.

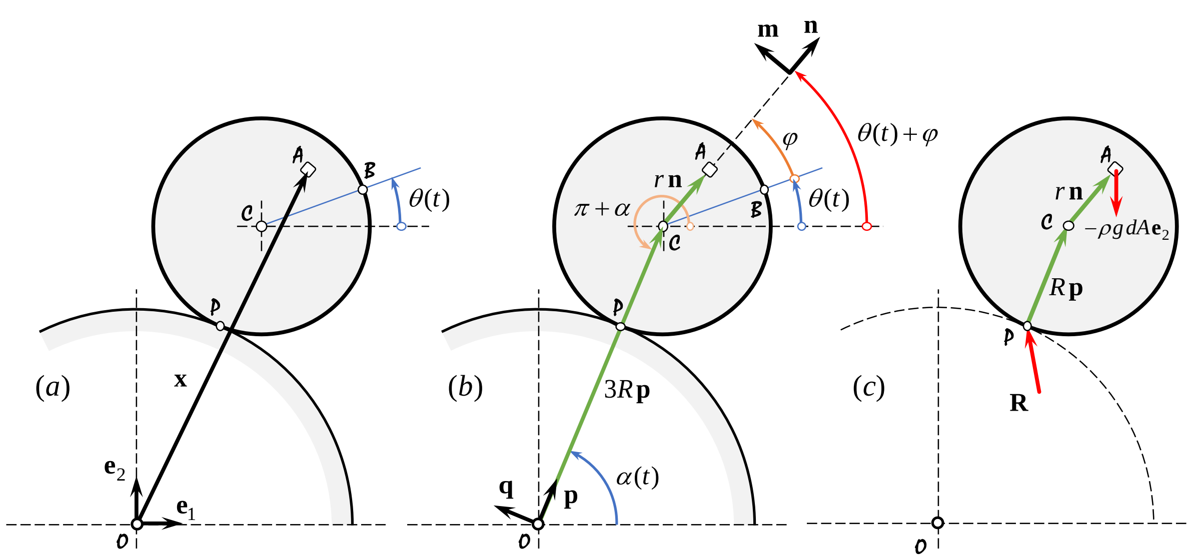

Fig. 2. *The typical particle*. Example of a disk rolling without slipping on a circular surface. The particle A is referred to as the *typical particle* because it is representative of any particle in the disk. To write the position vector of particle A is to write the position vector of all particles in that body. Sketch (a) shows the position vector **x**. Sketch (b) refines the geometry of the motion and shows how to construct **x**. Sketch (c) is a free body diagram of the disk (showing only the force associated with the typical particle and the contact force at P). The geometry sketch is part of Objective A.1 and the free body diagram is Objective D.

The first task is to describe the position of *A*. In order to do that, we must establish an origin *O*, which we take at the center of the larger circle. We need to write the equation of motion at time t, so the sketch 2(b) shows the disk in a general position, which is characterized by the time-dependent angle $\alpha(t)$ measured from the horizontal. The angle $\theta(t)$ measures the rotation of the smaller disk relative to the horizontal. At this point we think of $\alpha(t)$ and $\theta(t)$ as being independent of each other. We can enforce the no-slip condition later.

Rather than rely on the label *A* to keep track of the particle, we describe its location with *spatial* coordinates, in this case we use polar coordinates (r, φ) because of the circular geometry. The physical location of *A* is measured relative to the center of the disk and the line passing through *C* and *B*, which rotates with the disk.

To help with the mathematical characterization, we define the unit vector **p**, which points along the line joining the centers of the two circles, and **q** which is perpendicular to **p**. We also identify unit vector **n**, which points along the line joining *C* and *A*, and **m** which is perpendicular to **n**. These vectors can be characterized mathematically in terms of the fixed base vectors $\{\mathbf{e}_1, \mathbf{e}_2\}$, and the various angles defined, as

$$\begin{aligned} \mathbf{p}(t) &= \cos\alpha(t)\mathbf{e}_1 + \sin\alpha(t)\mathbf{e}_2 \qquad & \mathbf{n}(\varphi,t) &= \cos\big(\theta(t)+\varphi\big)\mathbf{e}_1 + \sin\big(\theta(t)+\varphi\big)\mathbf{e}_2 \\ \mathbf{q}(t) &= -\sin\alpha(t)\mathbf{e}_1 + \cos\alpha(t)\mathbf{e}_2 \qquad & \mathbf{m}(\varphi,t) &= -\sin\big(\theta(t)+\varphi\big)\mathbf{e}_1 + \cos\big(\theta(t)+\varphi\big)\mathbf{e}_2 \end{aligned} \tag{1}$$

The 'helper' unit vectors are all functions of time because their arguments—the angles $\alpha(t)$ and $\theta(t)$—are functions of time. Note that the vectors **n** and **m** are also functions of the spatial variable φ. It is straightforward to compute the time derivatives of these vectors by applying the chain rule of differentiation:

$$\dot{\mathbf{p}} = \dot{\alpha}\,\mathbf{q}, \qquad \dot{\mathbf{q}} = -\dot{\alpha}\,\mathbf{p}, \qquad \dot{\mathbf{n}} = \dot{\theta}\,\mathbf{m}, \qquad \dot{\mathbf{m}} = -\dot{\theta}\,\mathbf{n} \tag{2}$$

This part of the development is Objective A.1, which centers on the sketch in Fig. 2(b). With this setup, we can characterize the position vector of typical particle *A* (relative to the origin) as

$$\mathbf{x}(r,\varphi,t) = 3R\,\mathbf{p}(t) + r\,\mathbf{n}(\varphi,t) \tag{3}$$

The mathematical description of the position vector represents the achievement of Objective B. We construct this vector, which must go from the origin to the particle, by the head-to-tail rule for vector addition. Note how the work on the geometry of the problem aids in describing the position in a simple and compact way. This position vector represents *all* particles in the disk because each particle has its own unique values of r and φ.

With this expression, it is simple to compute the velocity and acceleration of particle *A* by differentiation of the position vector. To wit,

$$\begin{aligned} \dot{\mathbf{x}}(r,\varphi,t) &= 3R\,\dot{\alpha}\mathbf{q} + r\,\dot{\theta}\mathbf{m} \\ \ddot{\mathbf{x}}(r,\varphi,t) &= 3R\big(\ddot{\alpha}\mathbf{q} - \dot{\alpha}^2\mathbf{p}\big) + r\big(\ddot{\theta}\mathbf{m} - \dot{\theta}^2\mathbf{n}\big) \end{aligned} \tag{4}$$

This is the achievement of Objective C. While it is difficult to go from acceleration to position, it is simple to go from position to acceleration. It is also easier to visualize position than it is to visualize velocity or acceleration. We can carry out these steps without yet knowing the unknown functions of time $\theta(t)$ and $\alpha(t)$. With this approach, there is little need to develop ideas like normal and tangential coordinates, etc. Because the geometry of the position vector and subsequent differentiation takes care of all that automatically. You can still clearly identify features like centripetal acceleration from the resulting expression for the acceleration vector.

In this problem we have a no-slip constraint. No-slip implies that the velocity at point P must be zero. Since we have an expression for velocity for the typical particle, it certainly applies to the specific particle P. At that point of not slip we have $r = R$ and $\mathbf{m} = -\mathbf{q}$ (just follow the unit vectors all the way around through an angle such that $\theta + \varphi = \pi + \alpha$). The velocity at P is $\left(3R\dot{\alpha} - R\dot{\theta}\right)\mathbf{q} = \mathbf{0}$. That tells us that $\dot{\theta} = 3\dot{\alpha}$, thereby establishing the no-slip constraint condition. At this point, either θ or α can be algebraically eliminated from the equations by substitution.

To get the equations of motion we need to balance linear and angular momentum on the free body diagram shown in Fig. 2(*c*). Notice that the FBD must be at exactly the same state as the kinematic picture (it does not make sense to say that the net force at one time is equal to the mass times acceleration at another time). All of the geometric parts of this picture are already completed in earlier objectives. The equations of motion come from summing the forces and moments associated with all of the particles and setting them equal to the sum of the mass times acceleration (or moment of mass times acceleration) of all of the particles. For the example in Fig. 2, balance of linear momentum implies

$$\mathbf{R} - \int_A \rho g \mathbf{e}_2 \, dA = \int_A \rho \ddot{\mathbf{x}}(r,\varphi,t)\, dA \tag{5}$$

and balance of angular momentum (about point P) implies

$$-\int_A \left(R\mathbf{p} + r\mathbf{n}\right)\times\left(\rho g \mathbf{e}_2\right) dA = \int_A \left(R\mathbf{p} + r\mathbf{n}\right)\times\left(\rho \ddot{\mathbf{x}}(r,\varphi,t)\right) dA \tag{6}$$

There are a few things to notice about this stage of the derivation. First, the integrals are over the region of the disk. The best way to set this up for this problem is in polar coordinates. Note that there are many parts of the integrands that do not depend upon r or φ. So, the integrals are fairly easy to compute. This task brings students in contact with the integral as a summation (rather than as anti-derivative, which is usually emphasized in calculus). Second, the role of typical particle A is evident at every point in the equation. The acceleration on the right side of both equations is the acceleration of the typical particle A, which can be obtained directly from Eq. (4). You cannot evaluate the integrals without substituting the explicit expression for acceleration. Finally, the cross products that appear in balance of angular momentum involve only the unit vectors and are simple to evaluate from their definitions in Eq. (1). Through these calculations, students cement their understanding of vector operations.

It is also possible to compute the energy in the disk as being the sum of the energies of the particles that comprise the disk. Hence, we can write kinetic energy T and potential energy U as

$$T = \int_A \tfrac{1}{2}\rho\left(\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}\right) dA, \qquad U = \int_A \rho g\left(\mathbf{x}\cdot\mathbf{e}_2\right) dA \tag{7}$$

Again, all we need is the velocity vector $\dot{\mathbf{x}}$ from Eq. (4) and the position $\mathbf{x}$ from Eq. (3). Simply substitute those expressions into Eq. (7) and turn the calculus crank. This approach highlights the role of the *typical particle* in the formulation of the energy of the system. Once the energy is computed, energy principles can be applied (e.g., the Euler-Lagrange equations or conservation of energy).

It is important to note that we have not appealed to the notion of the *center of mass* or the *mass moment of inertia* in this approach. We prefer that the students discover these interesting features of dynamics problems, rather than learning first how to compute these geometric features without dynamic context, and then using them in pre-derived equations that they do not understand at a basic level. Traditional treatments often start with the computation of center of mass and mass moment of inertia before they introduce the kinetics of rigid bodies (e.g., [5]). We bring these ideas together in Module 5 (see Table 2) when students are better positioned to embrace them.

Discussion

As teachers, we hope to help students develop their cognitive ability in our courses and guide them on their way to becoming experts in their field. Yet, traditional pedagogical classrooms, specifically in mechanics, often do not provide the environment and strategies to promote this cognitive growth [6], [7]. Experts are known to categorize problems by the major principles of the problem, but novices categorize by "the superficial attributes of the problems" [8]. Students of a traditional dynamics course fall prey to this idea that there are many problem types and fail to see the similarity amongst them. The problem-solving strategy presented here explicitly rehearses the principles for each problem, it does not hide any part of the calculation, and it creates a set of common strands for all problems in the course.

The approach to teaching *Dynamics* advocated here was motivated by a few basic ideas. First, many of the problem-formulation and problem-solving strategies that have been in service for many decades were formulated in an era when hand computations were the only feasible approach to doing engineering computations. The approach to learning engineering mechanics that grew up around that constrained context pushed the focus toward simple calculation approaches and away from formulation and derivation. It also pushed the focus toward statics-like thinking. This bias has had the effect of reducing the amount of mathematics used in undergraduate mechanics with a resulting decline in the ability of students to do the mathematics. The failure to alter course has likely contributed to the dismal outcomes in sophomore-level mechanics courses lamented by instructors almost everywhere today.

We have reversed course and intentionally *increased* the level of mathematics used routinely in problem formation and solution. We build more directly from the prerequisite math courses that students take in their freshman year and reinforce those ideas almost immediately in the curriculum. The students, then, strengthen their math abilities while they learn mechanics. Direct observation and interaction with over 800 students in 12 semesters during problem-solving recitations has confirmed this shift.

The creation of the mastery objectives in *Dynamics* were the result of spending a year solving problems from a first-principles approach and contrasting the approach with the common traditional approaches to formulating problems as done in the most popular textbooks. The change to the problem-solving style was motivated by the desire to create students who are better problem solvers and who understand dynamics at a more fundamental level.

Student outcomes

Figure 3 gives some indication of the success of this effort. The mastery results shown in this figure indicate that students generally succeed with the formulation parts of dynamics. Mastery Objectives A.1, A.2, B, C, D, E.1, E.2, K, and L all center on the notion of the *typical particle* and its role in the formulation of a dynamics problem. Objectives F.1 and F.2 give an indication of student success in doing the mathematics associated with the typical particle (e.g., vector operations and integration over the domain of the body).

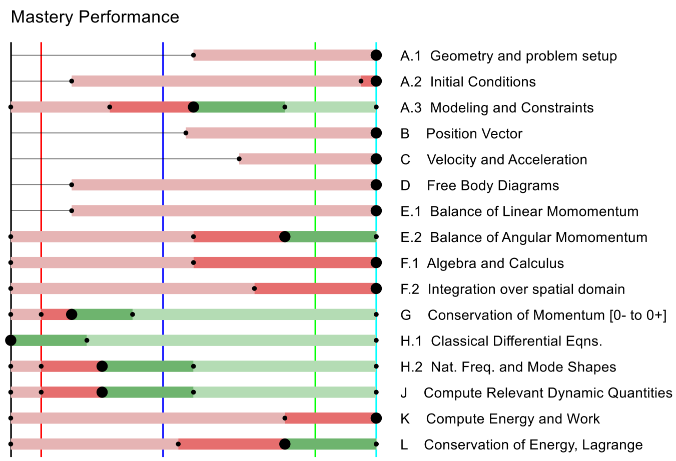

Fig. 3. *Student outcomes*. This chart shows the mastery performance for students who passed *Dynamics* from the SP14 through FA19 semesters (n=829). The mastery performance is shown in quartiles for each objective (with different colors and a small dot at the ends of the quartile). Light red represents the fourth quartile, dark red the third, dark green the second, and light green the first. The large dot is the mean (and separates the second and third quartile). The vertical lines distinguish level of mastery. Below the red line no mastery is evident. The blue line indicates emerging mastery. The green line is considered 'good' mastery. The cyan line is 'strong' mastery.

It is clear that all but some of those students in the lower quartile master the idea of the position vector and its relationship to velocity and acceleration. These students can successfully write a mathematical expression for the position vector of the typical particle, based upon a sketch of the body in motion. In that sense, they understand the motion of all points on the body and can distinguish that each point can have a different velocity and acceleration and, therefore, contribute differently to the balance of momentum.

The ability to execute free body diagrams has a longer tail, but on average, students show strong mastery of this task (only students in the fourth quartile fail to get to strong mastery). Balance of linear momentum has better mastery than balance of angular momentum, but students have more opportunity to master balance of linear momentum because the course starts with a month-long review of particle dynamics (which we do primarily to cement the notation and approach while they are working on problems familiar to them from Physics I).

The good performance in Objectives F.1 and F.2 demonstrate that if you focus on the development of math skills, the students can get to mastery. We employ a full array of mathematical tools (from algebra through calculus and differential equations) throughout the semester. In the beginning, the students struggle with vector notation (both how to do it and what it means physically) and they have little confidence in their calculus skills. Regular practice moves them toward better self-efficacy. Students can generally do much more than we initially expect, but they need the support and encouragement to get there. They are also in a period of transition from rote knowledge of concepts in mathematics to the chunked knowledge associated with mastery.

The mastery objectives related to solving the problems (i.e., Objectives G through J) come in a bit lower than the ones associated with formulation [4]. In part, that is caused by shifting the focus to formulation. Solving a problem always comes *after* formulating it, and students don't always have time to finish the longer problems that fit this pedagogy in the mastery assessments. We also devote nearly half of the course to completing computing projects, which absorb a lot of student time outside of class.

Lessons learned

Because we changed so many things in our approach to teaching *Dynamics*, it is difficult to make direct comparisons to verify that the approach yields the outcomes of the design. In some fundamental ways, the goals of the course—what we really want students to know—are different from the traditional goals (even though we solve many of the same kinds of problems). However, because we teach in a flipped environment, we have the opportunity to watch students solve problems and talk with them about their thought process in every class period. Although this amounts to anecdotal evidence, we have this evidence on *every* student who has passed through the classes (i.e., the 829 students mentioned in Fig. 3 plus the 62 students who failed the course during that time [4]). This approach has given us a very clear view of what students can do and how they progress in this system.

The following observations constitute some of the lessons that we have learned about student response to the *typical particle* construction approach.

- Our understanding of the math abilities of the student entering the course was often wrong and some students came in less prepared than we had anticipated. At our institution we have a wide range of preparation and the playing field is far from leveled by the time they get to the sophomore-level mechanics courses.
- The general math abilities of students steadily increase throughout the semester of *Dynamics*. They go from having a very loose grasp of mathematical notation (beyond scalar algebra and basic differential calculus) to much more reliable use of notation, especially for vectors.
- At the start of the semester, almost all students fail to make sound connections between their geometry sketch and the equations that they write down, indicating a very weak connection between visualization of motion and mathematical characterization of motion. This connection gets much stronger as the semester progresses, partly because their facility with the mathematical notation has improved, opening up more opportunity to focus on the physical aspects of the problems.
- Over the course of the semester, students dramatically increase their tolerance for mathematical complexity. Even fairly simple problems in *Dynamics* get mathematically complex. Most students have been trained to expect a low level of complexity. Doing the requisite derivations over and over again changes that expectation.

When we started *The Mechanics Project*, we implemented *Dynamics* first. A year later we implemented *Statics* [3]. One of the biggest lessons that we have learned is the importance of continuity between courses. Once we had *Statics* fully implemented, students came to *Dynamics* far better prepared. For example, we foster the same notation and approach to distributed effects in *Statics* that we use in *Dynamics*. While some might think that such an approach is overkill in *Statics*, it pays multiple benefits downstream as students encounter strategies that they can scaffold. It also suggests that the commonly exercised 'style and approach' differences among faculty who teach these courses serves to confuse the students and ultimately slows down their progress toward mastery.

Another important lesson that we have (re)learned is that patience is a virtue. At the start of the project, the vast majority of students were unprepared for the changes we made—especially those associated with the mathematical approach of the *typical particle*. Students are very heavily influenced by their immediate predecessors as they form their opinions about how the learning process should go. Initial resistance from students was high but dissipated quickly. Now, the culture of our school has shifted to view this approach as the norm, with almost no pushback from students.

Finally, we note that faculty colleagues generally fall into two categories: those who see virtue in the philosophical shift and those who appear willing to fight it to the death. Those with the largest investment in the traditional approaches most commonly fall into the latter group and those new to teaching most commonly fall into the former group. All faculty at our institution who have been trained in this pedagogy have become strong advocates of it.

Conclusion

Mechanics has long served as a staple of foundational education of engineers who study in fields associated with the mechanical sciences. The approach to teaching these subjects has not changed substantially over time, in part because they have been commoditized as lower division required courses that are often the subject of transfer articulations. This stasis has ignored the question of whether or not students need to *know* the same things about the subject or whether or not they need to be able to *do* the same things as an outcome of learning these subjects compared with earlier generations of engineering students. The *typical particle* approach represents a simple shift in emphasis on what students should know and be able to do.

The *typical particle* construction has provided students with a problem-solving strategy that is consistent for all rigid body dynamics problems which provides a roadmap for the students to follow as they begin to develop their expertise in dynamics [9]. The students emerge from this course with a more expert-like problem-solving technique and clarity on dynamics principles rather than relief that they made it through a required, but traditionally confusing course. This approach also puts students in a better position to pursue an advanced study in dynamics.

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