The Use of Numerical Propagation of Error Analyses in Experimental Design

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Abstract

The importance of statistics in engineering is not disputed, but how to effectively take it from a meaningless ritual to a truly interesting and integral part of a laboratory is disputed, especially at the introductory level. The current paper describes a simplified statistical procedure that is used in a sophomore level laboratory course that permits students to easily couple a propagation of error analysis to a system's theoretical model. This is accomplished through the use of the *function* capability that modern spreadsheets possess. In this case, a *Visual Basic* function macro is written to calculate the desired experimental result in terms of the mean values of its measured parameters. This function is then used to numerically estimate the variance of the result with respect to each of its measured properties and, therefore, its respective sensitivity to errors in each of the measurements, as well as the experiment's maximum probable error. This technique permits the investigation of more complex and realistic systems in a beginning laboratory. It also permits the use of experimental design both to determine what instrumentation should be used and how to configure the apparatus to minimize the resulting error. Embedding this uncomplicated technique in a spreadsheet environment is very helpful to the student since spreadsheets are the natural experimental platform for data presentation and reduction, and this software already possesses various statistical packages. The details of an example with four degrees of freedom are documented.

I. Introduction

In 1992, University of Wyoming's College of Engineering completed an internal review in which a questionnaire was sent to alumni who graduated in the last decade[1]. Most reported that they were adequately prepared to compete with their colleagues but recommended that more "real world" engineering tasks be incorporated in future curriculums. These tasks included the development of better oral and written communication skills plus more exposure to computer tools (engineering graphics, computer programming, spreadsheets and word processing), engineering statistics, teamwork, and general management. The Mechanical Engineering Department (ME) immediately initiated a major effort to integrate these constructive criticisms throughout its curriculum. Many of these suggestions are also delineated in the new Accreditation Board of Engineering and Technology (ABET) 2000 criteria[2]. The UW's

Engineering Task Force on Undergraduate Education also recommended students should be made to rely more on their own resources and less on formal lectures[3].

A curriculum reform integrating many of the above recommendations was undertaken. The first one credit-hour ME laboratory was restructured from basically a demonstration format to a class/laboratory mix based on a "discovery" approach. The primary goal of this sophomore level course is to help students develop the logical thought process that is required in engineering practice. In this case, it is how to integrate their fragmented knowledge base in mathematics, dynamics, statistics, programming, and communications to solve authentic, hands-on technical challenges and then to clearly and concisely report the results. Other educators also have emphasized the use of hands-on experiments for similar reasons[4-10] but many of these are so uncomplicated that they can be performed at home.

In the past, the inclusion of experimental design and basic statistical analysis of the results in elementary courses limited investigations to fairly simple and often boring systems. In this type of environment, statistics becomes almost a meaningless ritual to the students[11]. Fortunately, the dramatic changes in computer software now make it easier for students in introductory courses to incorporate simplified statistical considerations into their experimental work and design problems. This permits the use of more engaging and realistic systems in which students can truly comprehend the critical importance of statistics to the engineering profession.

This paper provides an overview of a simplified but concise statistical procedure to combine a propagation of error calculation with a theoretical analysis. This procedure is used to both optimize experimental designs and to help evaluate the experimental results. An application of this procedure to determine the optimum configuration of a reasonably complex multi-degree of freedom system is detailed here. This example uses a spreadsheet-computing environment, which is the natural platform for recording data from simple experiments, and its accompanying *Visual Basic* function capability.

II. Statistical Procedure

The revised ME class/laboratory emphasizes the *application* of the participants' existing knowledge base, rather than the introduction of new information. The experiments build from a purely direct measurement, statistical lab to optimizing the configuration of an involved dynamics system. In this case the pertinent prerequisite courses and material include:

1. *Introduction to Engineering and Computing*[12], which introduces freshman students to engineering problem solving and computer productivity software. This contains instruction in the utilization of Excel spreadsheets, including the application of its Solver tool to problems involving non-linear regression analysis. A chapter on statistics is also included which introduces the following basic concepts: sample size, population, central tendency descriptors, normal distributions, residuals, and histograms.
2. *Math sequence* which exposes students to partial differentiation and the construction of mathematical modeling of physical phenomena.
3. *Calculus-based physics course*. This freshman course introduces classical mechanics and the associated physics laboratory also utilizes the basic statistical concepts listed above and introduces a simplistic procedure to estimate experimental errors.

4. *Engineering Dynamics Course*.

The simple statistical rules that are utilized in the above physics laboratory were adopted for the introductory ME class/laboratory course; subsequent ME laboratories refine the statistical methods. The equations and rules specified in this procedure are as follows[13]:

Rule 1. All direct experimental measurements are reported in terms of their *mean value* $\pm$ the *standard deviation of its mean*, $X = \overline{X} \pm \sigma_{\overline{X}}$, where

$$\overline{X} = \frac{\sum\limits_{i=1}^{n} X_i}{n}, \tag{1}$$

n is the sample size and X_i is an individual measurement. The normal distribution is used to demonstrate that a sample's *mean value* is the best estimate of a quantity's true value. This relationship is assumed to be reasonable for all the laboratory measurements without any formal justification. The specified relationship for the *standard deviation of the mean, $\sigma_{\overline{X}}$*, is defined in terms of the sample's standard deviation, S, or measurement accuracy where

$$\sigma_{\overline{X}} \approx Maximum\left[\frac{S}{\sqrt{n}}, \frac{1}{\sqrt{12}} (smallest\ measuring\ instrument\ division)\right] \tag{2}$$

and

$$S = \sqrt{\frac{1}{n-1} \sum\limits_{i=1}^{n} \left(X_i - \overline{X}\right)^2} \tag{3}$$

No theoretical arguments for the above definition of S are presented other than to note that the standard deviation of a sample consisting of a single measurement should be undefined. The use of the quantity $\left(S/\sqrt{n}\right)$ to calculate $\sigma_{\overline{X}}$ is also stated without proof but again assumes a normal distribution.

The second quantity in the $\sigma_{\overline{X}}$ formula gives an estimate of a measurement's accuracy with regards to a measuring scale's resolution and the round-off error of a digital output. It is based on the theoretical assumption of a uniform probability within the minimum scale markings (output). This result is also given without proof but justified with a physical argument that one should be able to estimate a reading to approximately one-third ($\cong 1/\sqrt{12}$) the minimum scale.

Rule 2. A result property $R(x,y,...)$ that is computed from an explicit function of the measured parameters $(x,y...)$ is reported in terms of its mean value $\pm$ its maximum probable error,

$$R = \overline{R}(\overline{x}, \overline{y},...) \pm \sigma_{\overline{R}} \tag{4}$$

where the maximum probable error $\sigma_{\overline{R}}$ is defined as[11, 13, 14]

$$\sigma_{\bar{R}}^2 \approx \left(\frac{\partial R}{\partial x}\bigg|_{\bar{x},\,\bar{y},\ldots} \right)^2 \bullet \sigma_{\bar{x}}^2 + \left(\frac{\partial R}{\partial y}\bigg|_{\bar{x},\,\bar{y},\ldots} \right)^2 \bullet \sigma_{\bar{y}}^2 + \cdots \qquad (5)$$

This maximum probable error expression reduces to the exact expression for the variance of R if R is a linear function of the independent variables $(x,y\ldots)$, which in turn all have normal distributions, and their respective errors are independent of each other. The linear function requirement is met if the variances of the independent variables are small enough so that a first order Taylor series expansion of R is valid in this error domain. Researchers have shown that the assumption of normality is reasonable except in situations concerned with the extreme values of the variables, i.e. the tails of a distribution[15].

This overly simplified procedure gives the students a methodology that they can easily understand to calculate the gross statistical characteristics of their experimental design and resulting data. Of course this assumes that the desired experimental result R can be written as a simple explicit function of the measured variables. This is the case for all the demonstration experiments that are performed in the introductory physics laboratory; each apparatus has a given configuration, the experimental procedure and the instrumentation to be used are specified, and the relevant relationships are delineated. The above statistical procedure is only used to calculate the maximum probable error and perform a sensitivity analysis upon completion of the experiment. Therefore, no experimental design is done before an experiment is performed in an attempt to minimize the error in the results.

Unfortunately, the numerical evaluation of the partial derivatives in the maximum probable error relationship (i.e. *Equation 5*) is often quite difficult to perform analytically for a majority of the more realistic and intriguing problems. This can be true even if a symbolic logic program is utilized since these programs are frequently quite cryptic and difficult to use in parametric studies, such as experimental design problems.

III. Numerical Error Evaluation

If the variances of the measured variables are small enough, the maximum probable error relationship, *Equation 5*, can be approximated by the first order relationship

$$\sigma_{\bar{R}}^2 \approx \left[R\left(\bar{x} + \sigma_{\bar{x}}, \bar{y}, \cdots \right) - R\left(\bar{x}, \bar{y}, \cdots \right) \right]^2 + \left[R\left(\bar{x}, \bar{y} + \sigma_{\bar{y}}, \cdots \right) - R\left(\bar{x}, \bar{y}, \cdots \right) \right]^2 + \cdots \qquad (6)$$

The above calculation can be readily and efficiently executed in a worksheet environment through the use of a *Visual Basic function macro* in *Excel,* independent of the complexity of the explicit result function R. Students are given brief instructions on how to write a *function macro* along with an illustration of *Equation 6* that they can download to help them learn by example. This illustration and corresponding homework assignments are all elementary enough to permit comparisons between analytical and numerical results. These homework assignments present a practical method for estimating the relative sensitivity of a dependent variable's variance to errors in each of its directly measured independent variables. Furthermore, this approach permits

parametric studies and corrections to be readily implemented. Consequently, it helps eliminate most of the drudgery associated with these types of calculations.

It should also be noted that the students enter this class/lab with no formal programming background. Besides the brief introduction to *Visual Basic* function programming, the use of "proper programming practices" is also demonstrated and enforced. This includes such things as well commented spreadsheets and the use of *Named Variables* rather than cell references. The point is made that a little time spent initially can aid *substantially* in debugging spreadsheets as well as effectively documenting one's work for future references.

These concepts are reinforced the work performed in the inaugural laboratory and subsequent homework exercises. The first objective of this laboratory is to get acquainted with all the measuring instruments that will be available for use in every experiment and to determine their respective accuracies. This information is important in the student's experimental design of subsequent experiments and data reduction. Some of the available instruments are quite crude to intentionally introduce significant errors in certain measurements. This forces the students to visually recognize that there must be uncertainty in their results. Other educators have noted sophisticated equipment can give students the illusion of accuracy[4]. Further, sophisticated gadgetry often obscures the studied phenomena and may confuse some students[16].

Simple experimental measurements are then performed during this initial laboratory that involves obtaining a statistical sample of some directly measurable property or properties. The students are asked to perform some basic statistical analysis on these data that includes a sample's histogram and standard deviation of its mean according to *Equation 2*. These measured data are also used to determine a simple calculated property such as density or moment of inertia and its respective maximum probable error. The students are again asked to compare analytically (*equation 5*) and numerically (*equation 6*) calculated maximum probable error.

The students are asked to produce a brief but comprehensive memo with attachments that reports their experimental results. Since this is the first time that many of these sophomores have been required to produce a narration that integrates various mathematical, statistical, numerical, programming, graphical and experimental concepts, it is very important that this initial experimental assignment does not overwhelm them. The homework assignments must adequately prepare them for this work and the first experiment must be so basic that it also helps to foster a non-intimidating environment.

IV. Details of an Example Laboratory Design Project

As previously mentioned, the real power in the proposed maximum probable error numerical approach is the ability to perform a basic error analysis on more realistic and complicated systems. Since engineers seek superior if not optimal designs within well-defined constraints, the last laboratory experiment mimics this objective by challenging the students to determine an optimum experimental configuration for a multi-degree of freedom, dynamics apparatus. This final laboratory assignment is intended to enhance the students' design competence via a hands-on and open-ended laboratory experience. It is always linked to at least one previous experiment in which pertinent empirical information is obtained. *Figure 1* illustrates the type of experimental design problem that is presented. In this case the design objective is to determine the pendulum

launcher configuration that minimizes the maximum probable error for a specified range L_0. The string length R, the launch pin pivot angle ϕ and the initial ball-string displacement angle θ that is used to launch the ball are all completely arbitrary while the pin height H is confined by our particular apparatus to a range between 0.8 to 1.6 m.
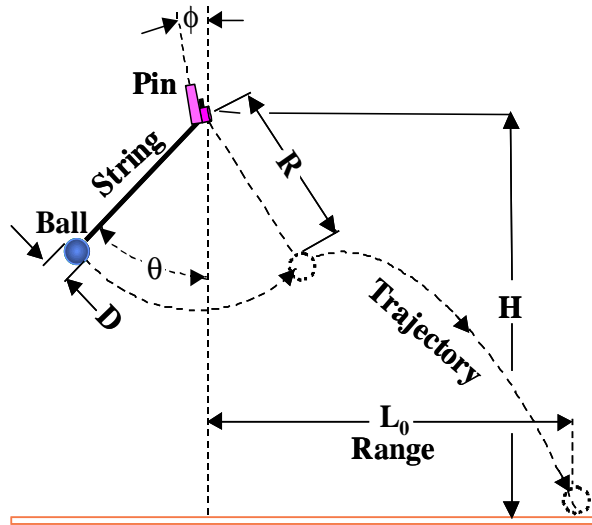
For this particular design experiment, the slip angle for various string/wire combinations is measured in the linked experiment. The experimental procedure calls for the suspension of a plumb bob that utilizes a designated string on a specified straight wire. Each two-person team makes multiple measurements of the string/wire slip angle (ϕ_μ) by rotating the wire next to a small protractor, both of which are handheld. In this case, both the measuring instrument and experimental procedure are intentionally specified to be crude but simple. Each team also investigates if there appears to be any functional dependence on the plum bulb's weight and/or on the wire's diameter. It should be noted that these diameter measurements represent the first time many students have utilized micrometers and calipers. Each team investigates only a single string/wire material combination, but several material combinations are utilized among all the laboratory teams. The class' accumulative data set presents the class with some quantitative information about how the string/wire slip angle varies with some possible key

Figure1. Pendulum Launcher

physical parameters. One of these string/wire combinations is specified for use in the pendulum launcher design experiment, i.e. the chosen wire will be used to construct the pin. Sample class results are shown in Figure 2.

Figure 2. Slip Angle for Nylon Twine on Steel Pins with Varying Diameters

For the design problem, each student must come up with what s/he believes to be an optimum launcher configuration. To model this launcher, students must recognize on their own that:

1. the classical pendulum and free trajectory analyses are joined by the empirical slip angle which was measured in the linked laboratory experiment;
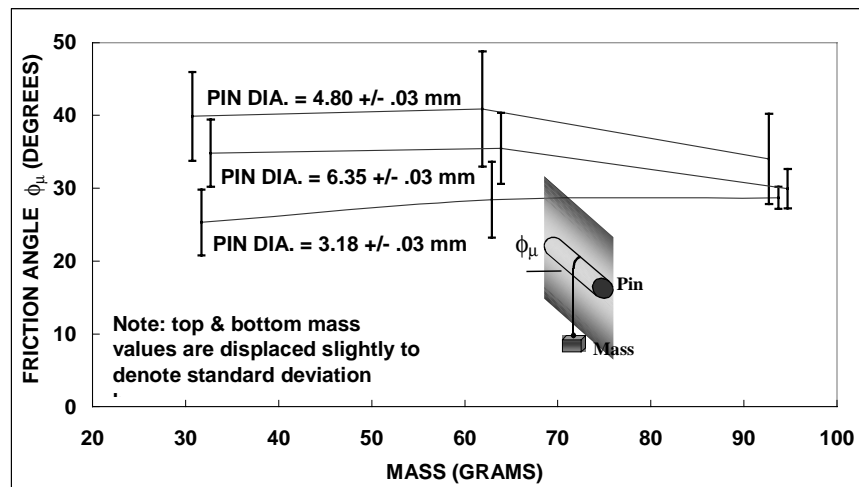
2. there are constraints on R, θ and ϕ in addition to the before-mentioned pin height (H) constraint, and that these constraints must be imposed to obtained a valid configuration.
3. for a given range (L_0), ball diameter (D) and slip angle (ϕ_μ), only three of the four geometric variables in the set (R, ϕ, θ, H) are independent.

H was arbitrarily chosen as the dependent variable in this example and the *Excel Visual Basic* function that performs the corresponding dynamic calculations to produce $H(L_0, R, \theta, \phi, \phi_\mu, D)$ is presented in *Appendix I*. The analogous range function $L(H, R, \theta, \phi, \phi_\mu, D)$ is also presented in this appendix. This range function permits the numerical calculation of the variation of range for a given uncertainty in any of the independent variables. This in turn allows the numerical calculation of the maximum probable error for a given configuration. The *Excel* worksheet in *Appendix II.A* illustrates this for the case with the following specified variables and their respective standard deviations: $L_0 = 0.50\ m$, $D = 4.76 \pm 0.08\ cm$ and $\phi_\mu = 40° \pm 7°$ which corresponds to the nylon twine/steel pin of 4.80 ± 0.03 mm diameter combination (see *Figure 2*). This *Appendix* also indicates that the initial guesses and their respective standard deviation for the independent parameters are: $R = 10.0 \pm 0.5$ cm, $\theta = 65° \pm 3°$ and $\phi = 0 \pm 2°$.

The worksheet results shown in *Appendix II.A* indicate that the required pin height is $H = 1.54 \pm 0.02\ m$. This assumed configuration is within the constraint envelope, and it results in a predicted *maximum probable error* of $\pm 9.3\ cm$ in the range, L_0 . The results also imply that the uncertainty in ϕ_μ (i.e. spreadsheet variable *dLdphif^2*) produces most of the error in the range. Note that meaningful functional relationships are easily expressed through the use of cell names.

Excel's Solver tool, which uses a generalized reduced gradient nonlinear optimization algorithm, can then be used to determine an optimum configuration. *Figure3* indicates that cell *MPE*, the *maximum probable error* location, is to be minimized by varying the independent variables R, θ and ϕ within their respective constraints. Most of the launcher constraints are listed in *Figure 3*. Running *Solver* results in the following configuration: $R = 6.4 \pm 0.5\ cm$, $\theta = 84° \pm 3°$, $\phi = -27° \pm 2°$ and $H = 0.80 \pm 0.02\ m$ which is at its lower constraint. The range's *maximum probable error* drops from $\pm 9.3\ cm$ for the initial guessed configuration to $\pm 2.2\ cm$
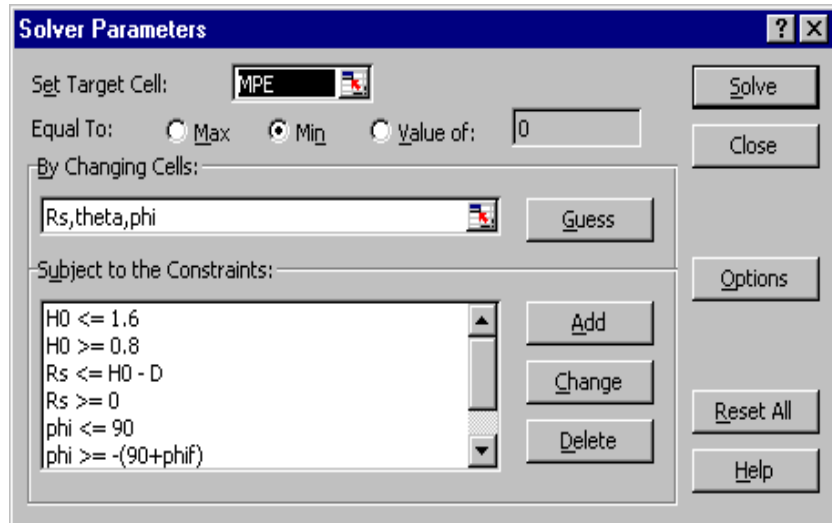

Figure 3 Excel's Solver Window

for this optimum configuration (see *Appendix II.B*). For this new configuration, the uncertainties in θ and R instead of ϕ_μ are now responsible for most of the maximum probable error in the range. The optimum configuration reduced this error by 76% in this case.

When the students physically perform this experiment they can determine the validity of their experimental design. If the experimentally determined pattern of impact points lie outside the predicted maximum probable error band, they must try to identify possible causes and corrective actions. For instance, closer examination of the apparatus may indicate that the somewhat a priori estimation of measurement errors made in the pre-lab must be improved if they were underestimated for the actual experiment. They may discover errors in their assumptions, analysis, programming, or that the model must be refined. Any changes that students decide to make to their pre-lab model and/or input values can be rapidly implemented by editing the spreadsheet and/or its associated *Visual Basic* macro function if these were initially well done.

Overall, the experiment is quite successful demonstrating that error analysis is essential in design. Though the students use the Solver optimization tool as a black box, they are able to observe and demonstrate via parameter studies that an optimum solution has been found. While the experiment is intentionally designed to use inexpensive and unsophisticated equipment and instrumentation, it is still able to mimic the "real world" where modeling of a system with some complexity is required. Further, the experiment is designed with a flexible configuration such that 1) different experimental objectives may be posed, with different constraints becoming active and 2) a particular design variable may be measured via alternative methods. This versatility helps prevent the problem with "dry labs." The parameter study found in *Appendix II.C* illustrates that changing the desired range yields a very different solution with different constraints becoming active.

V. Conclusions

With the dramatic changes in computational tools, from slide rules to calculators to spreadsheets, it is now possible to perform a basic propagation of error analysis efficiently and effectively through the use of numerical differentiation. Through repeated implementation of a given statistical procedure, the instructors have tried to provide the students with a deeper understanding of the importance and power of performing an error analysis in experimental design. Namely, the sensitivity of the probable error of a calculated result to the accuracy of the various measurements that are to be made can be investigated *ahead of time*. Though the statistical concepts are oversimplified, the authors feel that they have been successful in giving students 1) a sense of the importance of error analysis in experimental design and 2) some introductory tools which will assist them in performing these types of analyses in their engineering practice. Generally, the students become quite knowledgeable in using *function modules* by the end of the semester, and many of them continue to utilize the spreadsheet's *function modules* and optimization technique throughout their educational careers. More importantly, students leave the course with a sense of accomplishment that they have independently integrated various components of experimental design (modeling, numerical uncertainty analysis, experiment execution, and evaluation of results).

VI. References

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Appendix I

```
Function H (L, R, theta, phi, phif, D)
'This function determines the required launcher pivot point
'height given the other geometric parameters
    'D = ball diameter (m)
    'L = range (m)
    'H = launcher's pivot height (m)
    'phi = pivot angle (deg)
    'phif = string/pivot wire slip angle (deg)
    'R = string length (m)
    't = free trajectory time of flight (s)
    'theta = launch angle (deg)
    'V0 = exit velocity from launcher (m/s)
    'y0 = vertical free fall distance (m)

    g = 9.807  'gravitational constant (m/^2)
    Pi = Application.Pi()
    R = R + D / 2  'pivot point to ball's c.g. radius

    'change all angles to radians
    phi = phi * Pi / 180
    phif = phif * Pi / 180
    theta = theta * Pi / 180

    'mechanical energy balance applied to launcher:
    'ball's initial energy = its exit energy
    V0 = Sqr(2 * g * R * (Cos(phi + phif) - Cos(theta)))
    V0x = V0 * Cos(phi + phif)  'horizontal exit velocity (m/s)
    V0y = V0 * Sin(phi + phif)  'vertical exit velocity (m/s)

    'free trajectory analysis
    t = (L - R * Sin(phi + phif)) / V0x
    y0 = g * t ^ 2 / 2 - V0y * t
    H = y0 + D / 2 + R * Cos(phi + phif)
End Function
```

```
Function L(H, R, theta, phi, phif, D)
'This function determines the range given the other geometric
    parameters
    'D = ball diameter (m)
    'L = range (m)
    'H = launcher's pivot height (m)
    'phi = pivot angle (deg)
    'phif = string/pivot wire slip angle (deg)
    'R = string length (m)
    't = free trajectory time of flight (s)
    'theta = launch angle (deg)
    'V0 = exit velocity from launcher (m/s)
    'y0 = vertical free fall distance (m)

    g = 9.807  'gravitational constant (m/^2)
    Pi = Application.Pi()
    R = R + D / 2  'pivot point to ball's c.g. radius

    'change all angles to radians
    phi = phi * Pi / 180
    phif = phif * Pi / 180
    theta = theta * Pi / 180

    H = H - D / 2  'pivot's effective height (m)
    'mechanical energy balance applied to launcher:
    'ball's initial energy = its exit energy
    V0 = Sqr(2 * g * R * (Cos(phi + phif) - Cos(theta)))
    V0x = V0 * Cos(phi + phif)  'horizontal exit velocity (m/s)
    V0y = V0 * Sin(phi + phif)  'vertical exit velocity (m/s)
    y0 = H - R * Cos(phi + phif)

    'free trajectory analysis
    t = (V0y + Sqr(V0y ^ 2 + 2 * g * y0)) / g
    L = R * Sin(phi + phif) + V0x * t
End Function
```

Appendix II

A. Initial Guesses

	Given Geometric Parameters					Mean Standard Error					
L0 =	0.500	(m)	Desired Target Range			or Measured Error					
D =	4.76E-02	(m)	Handball Outer Diameter		dD =	8E-04	(m) Given in handball rule handbook				
phif =	40	(deg)	String/Pivot Wire Slip Angle		dphif =	7	(deg) Based on Experiment #1 results				
	Independent Geometric Parameters								Constraints in Solver		
Rs =	0.100	(m)	String Length		dRs =	0.005	(m) assumed		0 <= Rs <= H0 - D	(m)	
theta =	65	(deg)	launch angle		dtheta =	3	(deg) assumed		0 <= theta <= 90	(deg)	
phi =	0	(deg)	pivot angle		dphi =	2	(deg) assumed		-(90 + phif) <= phi <= 90	(deg)	
	Geometry Dependent Variable										
H0 =	1.54	(m)	= H(L0,Rs,theta,phi,phif,D)		dH =	0.02	(m) assumed		0.8 <= H <= 1.6	(m)	
	= Required height for above Geometric Paramters										
	Numerical Differentials								Numerical Variances		
dLdH =	0.003	(m)	=L(H0 + dH,Rs,theta,phi,phif,D) - L0					(dLdH)^2 =	6.90E-06	(m^2)	
dLdRs =	0.012	(m)	=L(H0,Rs+dRs,theta,phi,phif,D) - L0					(dLdRs)^2 =	1.45E-04	(m^2)	
dLdtheta =	0.032	(m)	=L(H0,Rs,theta+dtheta,phi,phif,D) - L0					(dLdtheta)^2 =	1.01E-03	(m^2)	
dLdphi =	-0.023	(m)	=L(H0,Rs,theta,phi+dphi,phif,D) - L0					(dLdphi)^2 =	5.08E-04	(m^2)	
dLdphif =	-0.084	(m)	=L(H0,Rs,theta,phi,phif+dphif,D) - L0					(dLdphif)^2 =	6.97E-03	(m^2)	
dLdD =	9.11E-04	(m)	=L(H0,Rs,theta,phi,phif,D+dD) - L0					(dLdD)^2 =	8.29E-07	(m^2)	
								Variance =	8.65E-03	(m^2)	
							Max Prob Error =	MPE =	0.093	(m)	

B. *Excel* Solver's Optimum Solution

	Propagation of Error Analysis for a Pendulum Launcher										
	Given Geometric Parameters					Mean Standard Error					
L0 =	0.500	(m)	Desired Target Range			or Measured Error					
D =	4.76E-02	(m)	Handball Outer Diameter		dD =	8E-04	(m) Given in handball rule handbook				
phif =	40	(deg)	String/Pivot Wire Slip Angle		dphif =	7	(deg) Based on Experiment #1 results				
	Independent Geometric Parameters								Constraints in Solver		
Rs =	0.064	(m)	String Length		dRs =	0.005	(m) assumed		0 <= Rs <= H0 - D	(m)	
theta =	84	(deg)	launch angle		dtheta =	3	(deg) assumed		0 <= theta <= 90	(deg)	
phi =	-27	(deg)	pivot angle		dphi =	2	(deg) assumed		-(90 + phif) <= phi <= 90	(deg)	
	Geometry Dependent Variable										
H0 =	0.80	(m)	= H(L0,Rs,theta,phi,phif,D)		dH =	0.02	(m) assumed		0.8 <= H <= 1.6	(m)	
	= Required height for above Geometric Paramters										
	Numerical Differentials								Numerical Variances		
dLdH =	0.006	(m)	=L(H0 + dH,Rs,theta,phi,phif,D) - L0					(dLdH)^2 =	4.08E-05	(m^2)	
dLdRs =	0.014	(m)	=L(H0,Rs+dRs,theta,phi,phif,D) - L0					(dLdRs)^2 =	1.96E-04	(m^2)	
dLdtheta =	0.015	(m)	=L(H0,Rs,theta+dtheta,phi,phif,D) - L0					(dLdtheta)^2 =	2.37E-04	(m^2)	
dLdphi =	0.002	(m)	=L(H0,Rs,theta,phi+dphi,phif,D) - L0					(dLdphi)^2 =	3.34E-06	(m^2)	
dLdphif =	0.001	(m)	=L(H0,Rs,theta,phi,phif+dphif,D) - L0					(dLdphif)^2 =	2.11E-06	(m^2)	
dLdD =	1.00E-03	(m)	=L(H0,Rs,theta,phi,phif,D+dD) - L0					(dLdD)^2 =	1.00E-06	(m^2)	
								Variance =	4.81E-04	(m^2)	
							Max Prob Error =	MPE =	0.022	(m)	

C. Parameter Study

	Parametric Study					
L0 (m) =	0.330	0.500	0.750	1.000	2.000	
Rs (m) =	0.014	0.064	0.100	0.150	0.525	
theta (deg) =	84	84	**90**	**90**	**90**	
phi (deg) =	-27	-27	-23	-23	-8	
H (m) =	**0.80**	**0.80**	1.10	1.38	**1.60**	
Max Prob Error (m) =	0.025	0.022	0.028	0.033	0.101	
	* **Shaded Cells** indicate variable is at a constraint boundary					

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