The Use of Spreadsheets in Teaching
Boundary-Value Problems in Electromagnetics

Abstract

Electromagnetics is arguably one of the most challenging courses in any electrical engineering curriculum. A solid foundation in vector calculus and a good intuition based on physical grounds are the normal requirements for a student to successfully complete this course. This paper presents a simple, yet powerful approach to introducing boundary-value problems arising in electrostatics. The principles of electrostatics find numerous applications such as electrostatic machines, lightning rods, gas purification, food purification, laser printers, and crop spraying, to name a few.

This paper focuses on the use of spreadsheets for solving electrostatic boundary-value problems. Sample problems that introduce the finite difference and the finite element methods are presented. The geometries included in the problems are sufficiently nontrivial for hand calculation or analytical solution, but reasonably manageable using spreadsheets. Although specialized software is available for this purpose, oftentimes such sophistication tends to obscure the mathematical underpinnings of the solution methods. Spreadsheets offer a transparent alternative – perhaps proximate to hand calculation – for students to better appreciate the numerical methods for solving boundary-value problems.

1. Introduction

Many phenomena arising in science and engineering are modeled by partial differential equations (PDEs). In such cases the quantity of interest (e.g., temperature, potential, or displacement) is a function that depends on more than one variable (typically, space variables x, y, z and the temporal variable t). The heat equation, wave equation, and Laplace’s equation are among the most common PDEs that undergraduate engineering students will encounter. The usual practice is to introduce the student to the analytical solution of these equations via the method of separation of variables. Under the assumption of linearity, the method naturally leads to the formulation of solutions as Fourier series expansions.

Treatment of PDEs and boundary-value problems (BVPs) may be found in many standard books. Reference 1 provides a very accessible presentation of the topic, while references 2 through 4 provide a more concise presentation geared toward compendium courses in engineering mathematics. This paper will not expound the theories that provide the mathematical underpinnings of PDEs; instead, the paper emphasizes on numerical solutions of PDEs and suggests implementations through spreadsheets.

This paper focuses on some numerical methods for solving PDEs; in particular, the finite difference and the finite element methods are presented in the context of problems arising in electrostatics. Much of the development of these methods will follow those found in electromagnetics books. The examples presented in this paper include geometries that are
sufficiently nontrivial for hand calculation or analytical solution, but reasonably manageable using spreadsheets. Although specialized software is available for this purpose, oftentimes such sophistication tends to obscure the inner workings of the numerical methods employed in the solution of PDEs. Spreadsheets offer a transparent alternative – perhaps proximate to hand calculation – for students to better appreciate the numerical methods for solving PDEs and BVPs.

The use of spreadsheets in teaching finite element analysis has been reported in the literature. Reference 6 presents finite element analysis in the context of a plane truss structure, wherein the objective is the determination of stresses given certain strains; the successful use of spreadsheets for introducing finite element analysis in an undergraduate mechanical engineering class is also discussed therein.

To the best of the authors’ knowledge, spreadsheets are more prevalent in mechanical and civil engineering. Not much has been reported on uses of spreadsheets in electrical engineering – perhaps in this branch of engineering the use of specialized software is more consonant with the dynamic nature of the field. However, educators and students who wish to stress the importance of fundamentals, rather than effortless and expedite solutions, may want to consider using spreadsheets as a tool. Spreadsheets offer a reasonable tradeoff between user-defined programming and specific-purpose software. It is in this spirit that this paper is presented, continuing the efforts initiated by the authors in a power systems course.

This paper is organized as follows. Section 2 presents the finite difference method for solving an electrostatic problem and includes the corresponding spreadsheet implementation. Section 3 presents the finite element method along with a spreadsheet implementation of the method. Section 4 compares the results obtained by each method. Section 5 discusses the pedagogical advantages of the spreadsheet implementations. Finally, Section 6 gives concluding remarks.

2. The finite difference method

The finite difference method (FDM) is conceptually simple. The problems to which the method applies are specified by a PDE, a solution region (geometry), and boundary conditions. Only a brief outline of the method is given in this paper; for more detailed derivations the reader may consult Reference 5. The finite difference method entails three basic steps:

(1) Divide the solution region into a grid of nodes. Grid points are typically arranged in a rectangular array of nodes.

(2) Approximate the PDE and boundary conditions by a set of linear algebraic equations (the finite difference equations) on grid points within the solution region.

(3) Solve this set of linear algebraic equations.

The method is illustrated with an example arising in electrostatics. Consider the charge-free region depicted in Figure 1. The region has prescribed potentials along its boundaries. The region
Figure 1. Charge-free region showing prescribed potentials at the boundaries and rectangular grid of free nodes to illustrate the finite difference method.

is divided into a rectangular grid of nodes, with the numbering of free nodes as indicated in the figure. The potential \( V = V(x,y) \) at an interior point \( (x,y) \) within the region is governed by the two-dimensional Laplace equation

\[
\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0. \tag{1}
\]

Let the location of an interior grid point be identified by a pair of integers \( (i,j) \), where \( i \) and \( j \) represent the position along the horizontal and vertical directions, respectively. For a grid having equal horizontal and vertical step sizes, the potential is given by the finite difference equation

\[
V_{i,j} = \frac{1}{4} (V_{i+1,j} + V_{i-1,j} + V_{i,j+1} + V_{i,j-1}). \tag{2}
\]

In essence, the potential at an interior grid point is the average of its four closest neighboring grid points that lie along the horizontal and vertical grid lines that intersect at the point. For example, the finite difference equation for node 4 in Figure 1 is

\[
V_4 = \frac{1}{4} (V_5 + V_3 + V_7 + V_2). \tag{3}
\]

For node 6, which neighbors boundaries with prescribed potentials, the corresponding finite difference equation is
\[ V_6 = \frac{1}{4}(V_7 + 0 + 0 + V_3). \]  \hspace{1cm} (4)

Similar equations are formulated for the remaining free nodes leading to a system of linear algebraic equations. This system of equations may be solved by a variety of methods. In this section the Gauss-Seidel method is implemented in a spreadsheet to solve this system of equations. The Gauss-Seidel method is a relatively simple iterative method for solving systems such as those encountered in the finite difference formulation.

The steps to implementing the spreadsheet for FDM are given below:

1. The first step in the spreadsheet implementation is to input the prescribed potentials at the boundaries of the solution region. This step is shown in Figure 2. The prescribed potentials occupy the cell range C6:C10 and correspond to the potentials (0 V and 50 V) given in Figure 1. The user may change the numerical values of these potentials to accommodate different boundary conditions.

2. The next step is the implementation of the Gauss-Seidel method for solving the finite difference equations. There are 7 potentials at interior grid points that need to be determined (nodes 1 through 7 in Figure 1). These have been labeled \( V_1 \) through \( V_7 \) and occupy the cell range B14:H14 in Figure 3. To start the iterations initial estimates are given in B15:H15; in this case all initial estimates have been set to zero.

The cell range B16:H16 implements the difference equations which are then solved using the Gauss-Seidel method. For example, for node 4, cell E16 contains the Microsoft Excel formula that implements Equation (3)

\[ =\text{SUM(D16,F15,C16,H15)}/4; \]

while for node 6, cell G16 contains the formula that implements Equation (4)

\[ =\text{SUM(C8,H15,D16,C6)}/4. \]

Observe that the formula in cell G16 makes absolute references to nodes with prescribed potentials; these prescribed potentials must remain constant throughout the Gauss-Seidel iterations. Making the appropriate absolute references will prevent spurious results from occurring when formulas are copied to other cells.

3. The final step is to reproduce the formulas in cell range B16:H16 by using the Copy command of Microsoft Excel. The formulas have been copied over the cell range B17:H27. It can be seen from Figure 3 that convergence is reached after 12 iterations for a precision index of 10^{-2}. 

Figure 2. Input section of spreadsheet implementation of the finite difference method.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
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</tbody>
</table>

**Boundary conditions**

6 Top side | 0 V
7 Bottom side | 0 V
8 Left side | 0 V
9 Top slanted side | 50 V
10 Bottom slanted side | 50 V

Figure 3. Screenshot showing the Gauss-Seidel iterations of the difference equations.

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
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<tr>
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<td>Iteration</td>
<td>V1</td>
<td>V2</td>
<td>V3</td>
<td>V4</td>
<td>V5</td>
<td>V6</td>
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<td>0.000</td>
<td>6.250</td>
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<td>0.000</td>
</tr>
<tr>
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<td>3</td>
<td>7.813</td>
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<td>25.537</td>
<td>43.884</td>
<td>6.787</td>
</tr>
<tr>
<td>19</td>
<td>4</td>
<td>10.132</td>
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<td>27.249</td>
<td>44.312</td>
<td>7.799</td>
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<td>34.595</td>
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<td>27.929</td>
<td>44.482</td>
<td>8.202</td>
</tr>
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<td>34.974</td>
<td>12.064</td>
<td>28.307</td>
<td>44.577</td>
<td>8.426</td>
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<td>11.760</td>
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<td>12.123</td>
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<td>44.587</td>
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<td>35.041</td>
<td>12.156</td>
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</tr>
<tr>
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<td>12</td>
<td>11.801</td>
<td>35.044</td>
<td>12.161</td>
<td>28.378</td>
<td>44.594</td>
<td>8.468</td>
</tr>
</tbody>
</table>

**Potentials at interior grid points**

**Method 1: Finite difference, Gauss-Seidel iteration**
3. The finite element method

The finite element method (FEM) is a numerical technique for solving PDEs. FEM was originally applied to problems in structural mechanics. Unlike FDM, FEM is better suited for solution regions having irregularly shaped boundaries. The finite element analysis involves four basic steps:

1. Divide the solution region into a finite number of elements. The most common elements have triangular or quadrilateral shapes. The collection of all elements should resemble the original region as closely as possible.

2. Derive governing equations for a typical element. This step will determine the element coefficient matrix.

3. Assemble all elements in the solution region to obtain the global coefficient matrix.

4. Solve the resulting system of equations.

Again, the method is illustrated with an example. Consider the same charge-free region shown in Figure 1. The region is divided into 24 equal triangular elements as indicated in Figure 4. The elements are identified by encircled numbers 1 through 24. In this discretization there are 19 global nodes numbered 1 through 19. In the paragraphs that follow a brief outline of FEM is provided; for detailed derivations the reader may consult Reference 5.

For each element $e$ the following quantities are computed

$$
\begin{align*}
P_1 &= y_2 - y_3, & P_2 &= y_3 - y_1, & P_3 &= y_1 - y_2, \\
Q_1 &= x_3 - x_2, & Q_2 &= x_1 - x_3, & Q_3 &= x_2 - x_1,
\end{align*}
$$

(5)

where the subscripts refer to the local node numbers 1, 2, and 3 of element $e$. For example, in Figure 4, element 20 has global nodes 13, 14, and 17 which correspond, respectively, to local nodes 1, 2, and 3. Local node numbering is arbitrary; however, local node numbers must be assigned so that global nodes associated with an element are traversed in a counterclockwise sense. The coordinates (in meters) of nodes 13, 14, and 17 are $(x_1, y_1) = (0.0, 0.75)$, $(x_2, y_2) = (0.25, 0.75)$, and $(x_3, y_3) = (0.0, 1.0)$, respectively. These sets of coordinates yield $P_1 = -0.25$, $P_2 = 0.25$, $P_3 = 0$, $Q_1 = -0.25$, $Q_2 = 0$, and $Q_3 = 0.25$.

With $P_i$ and $Q_i (i = 1, 2, 3)$ for element $e$ thus computed, the entries of the $3 \times 3$ element coefficient matrix are then given by

$$
C_{ij}^{(e)} = \frac{1}{4A} \left[ P_i P_j + Q_i Q_j \right] \quad (i, j = 1, 2, 3)
$$

(6)

where

$$
A = \frac{1}{2} \left[ P_2 Q_3 - P_3 Q_2 \right].
$$

(7)
As an example, for elements 20 and 21, the element coefficient matrices computed according to Equations (6) and (7) are

\[
C^{(20)} = \begin{bmatrix}
1 & -0.5 & -0.5 \\
-0.5 & 0.5 & 0 \\
-0.5 & 0 & 0.5 \\
\end{bmatrix} \quad \text{and} \quad C^{(21)} = \begin{bmatrix}
0.5 & -0.5 & 0 \\
-0.5 & 1 & -0.5 \\
0 & -0.5 & 0.5 \\
\end{bmatrix}.
\] (8)

The global coefficient matrix is then assembled from the element coefficient matrices. Since there are 19 nodes, the global coefficient matrix will be a 19 x 19 matrix. In the following, the computations of one diagonal and one off-diagonal entries are shown. For example, node 17, which corresponds to the $C_{17,17}$ entry in the global coefficient matrix $C$, belongs to elements 20 and 21; since node 17 is assigned local node number 3 in both elements (as seen in the middle table of Figure 5), the corresponding global coefficient is

\[
C_{17,17} = C^{(20)}_{3,3} + C^{(21)}_{3,3} = 0.5 + 0.5 = 1.
\] (9)

For the off-diagonal entry $C_{14,17}$, global link 14–17 corresponds to local link 2–3 of element 20 and local link 1–3 of element 21 and hence

\[
C_{14,17} = C^{(20)}_{2,3} + C^{(21)}_{1,3} = 0 + 0 = 0.
\] (10)

Defining the vector of potentials $v_f$ and $v_p$, where the subscripts $f$ and $p$ refer to nodes with free (unknown) and prescribed potentials, respectively, the global coefficient matrix is then partitioned accordingly and the unknown potentials are obtained from
The spreadsheet implementation of the finite element solution involves the following steps:

(1) Generate the input data section as shown in Figure 5. The input data consists of three tables: global node x and y coordinates; global and local node correspondence for each element; and list of nodes with prescribed (fixed) potentials.

(2) For each element, compute the values of $P_i$ and $Q_i$ from Equation (5) and obtain the element coefficient matrix from Equation (6) as shown in Figure 6. Rows for elements 4 through 21 were hidden because of space limitations. The table in the figure is constructed as follows:

a. For element 1, global node information is linked to local node numbers. The link is implemented in the cell range C78:C80 via the Microsoft Excel VLOOKUP function.
More precisely, cells C78, C79, and C80 contain, respectively, the formulas

\[ \begin{align*}
    & = \text{VLOOKUP}($A78,$F$47:$I$71,2) \\
    & = \text{VLOOKUP}($A79,$F$47:$I$71,3) \\
    & = \text{VLOOKUP}($A80,$F$47:$I$71,4)
\end{align*} \]

which search the item in column A (element number) and return the global nodes from the lookup table $F$47:$I$71.

b. The VLOOKUP function is invoked once again to retrieve global node coordinates. In the cell range D78:E80 the formulas implement this task:

\[ \begin{align*}
    & = \text{VLOOKUP}($C78,$A$47:$C$67,2) \\
    & = \text{VLOOKUP}($C78,$A$47:$C$67,3) \\
    & = \text{VLOOKUP}($C78,$A$47:$C$67,3)
\end{align*} \]

which search the item in column A (element number) and return the global nodes from the lookup table $F$47:$I$71.

c. Cell ranges F79:H79 and J79:L79 compute \( P_i \) and \( Q_i \) (\( i = 1, 2, 3 \)) from Equation (5) with the Excel formulas

\[ \begin{align*}
    & = E79-E80 \\
    & = E80-E78 \\
    & = E78-E79 \\
    & = D80-D79 \\
    & = D78-D80 \\
    & = D79-D78
\end{align*} \]
d. Cells I79 and M79 are optional. These cells simply compute $P_1 + P_2 + P_3$ and $Q_1 + Q_2 + Q_3$, as both sums must equal zero.

e. Cell N79 implements Equation (7):

$$=(G79*L79-H79*K79)/2$$

f. The cell range O78:Q80 computes the entries of the coefficient matrix for element 1 from Equation (6). For example, the entry $C_{12}^{(1)}$ is computed by the following Excel formula:

$$=(F79*G79+J79*K79)/(4*N79)$$

g. The formulas in cell range B78:Q80 are copied over the cell range B81:Q149 to complete the computation of the remaining element coefficient matrices.

(3) The assembly of the global coefficient matrix is shown in Figure 7. The numbers in B159:T159 and A160:A178 refer to global node numbers. The coefficients of the global matrix occupy cell range B160:T178, which contain formulas similar to those in Equations (9) and (10). For example, the coefficient $C_{17,17}$ in cell R176 contains the formula

$$=Q137+Q140,$$

while that of $C_{14,17}$ in cell R173 is

$$=Q136+Q138.$$  

(4) The matrices $C_f$ and $C_p$ are formed by extracting the appropriate rows and columns from the global coefficient matrix $C$. In this case, nodes 5, 6, 9, 10, 11, 14, 15 are the free nodes, while 1, 2, 3, 4, 7, 8, 12, 13, 16, 17, 18, and 19 are the nodes with prescribed potentials. The result is shown in Figure 8.

(5) The final solution is obtained by using the matrix capabilities of Microsoft Excel. In particular, the spreadsheet functions MINVERSE (matrix inverse) and MMULT (matrix multiplication) are used in the implementation of Equation (11). This step is shown in Figure 9.
Figure 7. Screenshot showing global coefficient matrix C.

Figure 8. Matrices $C_{ff}$ and $C_{fp}$ obtained from global coefficient matrix C.
4. Discussion of results

In Sections 2 and 3 spreadsheet implementations of FDM and FEM were presented. As indicated in Table 1, the potentials at the free nodes computed by both methods compared fairly well. The node numbers in the table for FDM correspond to those in Figure 1, while those for FEM correspond to the node numbers shown in Figure 4. A better agreement can be obtained if more iterations of the Gauss-Seidel method are performed; this can be easily accommodated by copying a complete row of existing formulas, say cell range B27:H27 in Figure 3, to new rows until a desired level of agreement is achieved.

Table 1. Comparison of results obtained by FDM and FEM.

<table>
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<tr>
<th>Node</th>
<th>Potential (V)</th>
<th>Node</th>
<th>Potential (V)</th>
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<tbody>
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<td>1</td>
<td>11.801</td>
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<td>11.802</td>
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<td>2</td>
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<td>11</td>
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<tr>
<td>6</td>
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</table>
In general, FEM can better handle complex geometries and boundary conditions, while FDM is more suitable for solution regions having a certain degree of regularity. The tradeoff between the simplicity and the generality of the methods is evidenced by the amount of programming required by each method.

5. Pedagogical advantages of spreadsheet implementations

In light of the spreadsheet implementations presented in this paper, the following remarks could be made:

(i) Setting up spreadsheets demands precise attention to detail from the user. Incorrect use of the Copy command or improper referencing of cells will lead to erroneous results. Attention to detail is a desirable skill that students should hone. Spreadsheets offer an environment in which such skill can be honed; the use of highly specialized software without proper understanding of the underlying methods may at times impede development of the skill in the student.

(ii) Spreadsheets offer a reasonable compromise between the sophistication of specific-purpose software and programming. The amount of programming in a typical spreadsheet is minimal, often reduced to formula editing and copying. This approach allows students to concentrate on analysis and interpretation of results rather than on time-consuming code debugging.

(iii) The spreadsheet implementations mimic hand calculations. The notepad-like interface of spreadsheets allows the student to keep track of results and ascertain convergence. Because of the resemblance to hand calculations, the spreadsheet approach may provide the student with a deeper understanding of the numerical methods, which could be obscured if specific-purpose software is used without proper knowledge of such methods.

(iv) The spreadsheets implementations may be presented to students to introduce numerical methods for solving BVPs. Students may be asked to implement similar spreadsheets to solve other types of PDEs and BVPs. The ambitious students may even improve upon the spreadsheets presented in this paper by macro programming or creative use of other Microsoft Excel functions.

(v) It would be interesting to investigate the impact of the use of spreadsheets in electrical engineering courses. At present, the authors do not have formal assessment data other than positive reactions from students.

In fairness to specific-purpose software designed for BVPs modeled by PDEs, it can be argued that such programs serve other purposeful needs, namely, handling large-scale systems, accommodating highly irregular geometries, and handling complex boundary conditions. Problems of considerable size and high complexity may not be handled efficiently by spreadsheets.
6. Conclusions

This paper presented spreadsheet implementations of two numerical methods for solving electrostatics problems. The spreadsheet approach is ideal if the emphasis is on understanding of numerical techniques. Spreadsheets may be considered as a viable alternative to enhancing education in other subjects and engineering fields.

The interested reader may obtain a copy of the Microsoft Excel file that implements the FDM and FEM solutions by sending an e-mail to mlau@suagm.edu.

References