THE VISUALIZATION OF BOUNDARY VALUE PROBLEMS

Abstract

In this paper the author will demonstrate how using Maple software, and physical models, in an introductory Boundary Value Problems course, helps students learn the concepts presented. By using Maple software and simple demonstrations done in class, the instructor and students were not only able to solve partial differential equations analytically, but were able to see how the solutions visually compared with the classroom demonstrations. Demonstrations will include the heat equation, the one dimensional wave equation, and the beating drum problem. The paper will also discuss why engineers and physicists can use only the first couple of terms of the solution to a boundary value problem. This course helps prepare engineering students to take courses such as Heat Transfer, Waves, Thermodynamics, and Electromagnetic Fields.

Introduction

When the author began teaching the Boundary Value Course it was decided to do a few things differently. The first change was to make the course more applied and less theoretical. To do this a new text was needed. While many texts were considered, including ones that were based on the use of a computer algebra system such as Maple, it was finally decided to pick a text that as one publisher stated "this text is the one the engineers would use." Thus, "Boundary Value Problems" by Powers was the text selected. For our purposes it has turned out to be a good choice.

The second change was to use Maple extensively in the course. The idea was that since we were going to rely heavily on the technique of separation of variables which leads to solutions obtained from Fourier Series, the students would be expected to actually calculate specific terms of the Fourier Series solution rather than simply writing down the general Fourier series.

The third change was the easiest to make. We simply advertised the course as a mathematics course that the engineering students might actually find useful.

This paper is going to deal mainly with the issues of using the Maple software and demonstrations and how they were used in the Boundary Values Course to help make the material "more interesting" and hopefully easier to understand. Our goal was to make sure
students were aware of how the heat, wave, and Laplace’s equations were derived and how they are solved using the technique of separation of variables. We also wanted to have the ability to "check" their solutions by using the Maple software to not only aid in the calculation of the solution of the boundary value problems, but also to see if the solution made sense, and that the solution really does model reality.

Calculations and Demonstrations

A) The Heat Equation:

Our first calculations and demonstration deal with the heat equation of an insulated rod where the ends of the rod are kept at a constant temperature of 0. If we let \( U(x,t) \) represent the temperature of an insulated rod at any position \( 0 \leq x \leq L \) and time any time \( t \geq 0 \), the boundary value problem is given by, where \( \kappa \) is the diffusivity constant:

\[
\frac{\partial U}{\partial t} = \kappa \frac{\partial^2 U}{\partial x^2}
\]

\( U(0,t) = 0 \), temperature at left hand end of rod is 0

\( U(L,t) = 0 \), temperature at the right hand end of rod is 0

\( U(x,0) = f(x) \), the initial temperature of the rod is \( f(x) \)

Suppose we let the length of the rod be \( L = 6 \), with diffusivity constant, \( \kappa = 4 \), and initial temperature in the rod be given by \( f(x) = 6x - x^2 \). While these values may not be very realistic, they serve the purpose for our calculations and demonstrations. Our boundary value problem now becomes:

\[
\frac{\partial U}{\partial t} = 4 \frac{\partial^2 U}{\partial x^2}
\]

\( U(0,t) = 0 \)

\( U(6,t) = 0 \)

\( U(x,0) = 6x - x^2 \)

By letting \( U(x,t) = X(x)T(t) \) and separating variables we arrive at the solution:

\[
U(x,t) = \sum_{n=1}^{\infty} A_n e^{-\frac{n^2 \pi^2 2t}{9}} C_n \sin\left(\frac{n \pi x}{6}\right) = \sum_{n=1}^{\infty} B_n e^{-\frac{n^2 \pi^2 2t}{9}} \sin\left(\frac{n \pi x}{6}\right)
\]

Setting \( t = 0 \) and \( U(x,t) = 6x - x^2 \) we arrive at
\[ 6x - x^2 = \sum_{n=1}^{\infty} B_n e^{-\frac{n^2 \pi^2}{9}} \sin\left(\frac{n \pi x}{6}\right) \text{ or} \]

\[ 6x - x^2 = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n \pi x}{6}\right) \Rightarrow \]

\[ B_n = \frac{2}{3} \int_{0}^{6} (6x - x^2) \sin\left(\frac{n \pi x}{6}\right) dx. \]

Thus, the solution given by many texts and courses is

\[ U(x, t) = \sum_{n=1}^{\infty} \left( \frac{2}{3} \int_{0}^{6} (6x - x^2) \sin\left(\frac{n \pi x}{6}\right) dx \right) e^{-\frac{n^2 \pi^2}{9} t} \sin\left(\frac{n \pi x}{6}\right). \]

For a lot of students the feeling is "Well, that's nice, but so what?" We now use Maple to calculate

\[ B_n = \frac{2}{3} \int_{0}^{6} (6x - x^2) \sin\left(\frac{n \pi x}{6}\right) dx = \left( -\frac{72(-2 + 2(-1)^n)}{n^3 \pi^3} \right) \]

This implies that our solution is now given by:

\[ U(x, t) = \sum_{n=1}^{\infty} \left( -\frac{72(-2 + 2(-1)^n)}{n^3 \pi^3} \right) e^{-\frac{n^2 \pi^2}{9} t} \sin\left(\frac{n \pi x}{6}\right) \]

Using Maple's "animate" command we can now see what the solution looks like.
Figure 1. Plot of $U(x,0)$

Figure 2. Plot of $U(x,3)$
Figure 3. Plot of $U(x,6)$

Figure 4. Plot of $U(x,10)$
While the figures above are impressive, they are not nearly as impressive as seeing the animation which shows the heat flow out of the rod.

A mathematician may well take five or more terms, which is what I did for the graphs above. An engineer, on the other hand, may simply say the answer is given by

\[ U_1(x, t) = \frac{288}{\pi^3} e^{-\frac{x^2 t}{9}} \sin\left(\frac{\pi x}{6}\right). \]

which is only the first term of the "answer". To explain why the engineers can get away with this we show the students the function

\[ U(x, t) - U_1(x, t) = \sum_{n=2}^{\infty} \left( \frac{-72(-2 + 2(-1)^n)}{n^3 \pi^3} \right) e^{-\frac{n^2 \pi^2 t}{9}} \sin\left(\frac{n \pi x}{6}\right) - \left( \frac{288}{\pi^3} e^{-\frac{x^2 t}{9}} \sin\left(\frac{\pi x}{6}\right) \right) \]

We can now animate the function

\[ U(x, t) - U_1(x, t) = \sum_{n=2}^{\infty} \left( \frac{-72(-2 + 2(-1)^n)}{n^3 \pi^3} \right) e^{-\frac{n^2 \pi^2 t}{9}} \sin\left(\frac{n \pi x}{6}\right). \]

We only need to show values for \( t = 0, 1, 2, \) and 3, before the students realized that after
$t = 3$, the two answers are virtually the same. The graphs of $U(x, t) - U_1(x, t)$ for $t = 0, 1, 2,$ and 3 are shown below.

Figure 6. Plot of $U(x, 0) - U_1(x, 0)$

Figure 7. Plot of $U(x, 1) - U_1(x, 1)$
For the reader who is interested, the Maple code for the Fourier Series and the animation plots are listed below. For formatting purposes we'll let $U(x,t)$ be defined by $U$ and $U_1(x,t)$ be defined by $U_1$.

Figure 8. Plot of $U(x,2)-U_1(x,2)$

Figure 9. Plot of $U(x,3)-U_1(x,3)$
By Hand

\[ B_n = \frac{2}{6} \int_0^6 (6x-x^2) \sin\left(\frac{n\pi x}{6}\right) dx \]

\[ U = \sum_{n=1}^{\infty} \left( -\frac{72(-2+2(-1)^n)}{n^3\pi^3} \right) e^{-\frac{n^2\pi^2 t}{9}} \sin\left(\frac{n\pi x}{6}\right) \]

Good luck plotting by hand.

\[ U_1 = \frac{288}{\pi^3} e^{-\frac{\pi^2 t}{9}} \sin\left(\frac{\pi x}{6}\right) \]

\[ U - U_1 = \sum_{n=2}^{\infty} \left( \frac{72(-2+2(-1)^n)}{n^3\pi^3} \right) e^{-\frac{n^2\pi^2 t}{9}} \sin\left(\frac{n\pi x}{6}\right) \]

Again, plotting \( U - U_1 \) is difficult.

Using Maple

\[ > B_n := (2/6)*\text{int}((6*x-x^2) \cdot \sin(n^*\pi^*x/6), x=0..6); \]

\[ > U := \text{sum}(B_n \cdot \sin(n^*\pi^*x/6), n=1..10); \]

\[ > \text{with(plots)}; \]

\[ > \text{animate}(U, x=0..6, t=0..20, \text{frames}=100); \]

\[ > U_1 := \text{sum}(B_n \cdot \sin(n^*\pi^*x/6), n=1..1); \]

\[ > U - U_1 := \text{U-U1}; \]

\[ > \text{animate}(U - U_1, x=0..6, t=0..20, \text{frames}=100); \]

B) The Wave Equation:

Our second example will be the one dimensional wave equation. The example we will use is a "traveling" wave. This is a demonstration we do in class with a cord. Two people hold a cord which is several feet long. One person plucks the cord near one endpoint of the cord and a traveling wave results. Unfortunately the wave is so small that a picture of the demonstration only shows two people holding the cord. This is not a very interesting picture and is not included in the paper. Once the students see the traveling wave we go to the mathematics and show that the mathematical model agrees with reality. If we have a cord of length 6 we can let \( u(x,t) \) be the displacement of the cord at any position \( 0 \leq x \leq 6 \) and any time \( t \geq 0 \).

The wave equation now becomes:

\[ \frac{\partial^2 u}{\partial^2 x} = \frac{1}{4} \frac{\partial^2 u}{\partial^2 t} \quad \text{for } 0 < x < 6, \ 0 < t \]

\[ u(0,t) = 0, \quad u(6,t) = 0, \quad \text{no motion at the end points of the cord} \]

\[ u(x,0) = \begin{cases} 
  x & \text{for } 0 < x < 1 \\
  2 - x & \text{for } 1 < x < 2 \\
  0 & \text{for } 2 \leq x \leq 6 
\end{cases}, \quad \text{initial displacement of the cord} \]

\[ \frac{\partial u(x,0)}{\partial t} = 0, \quad \text{initial velocity of the cord}. \]
Using the method of separation of variables, we assume \( u(x, t) = X(x)T(t) \). This leads us to find

\[
X(x) = X(x) = \sin\left(\frac{n\pi}{6}x\right), \quad \text{and}
\]
\[
T(t) = c_1 \cos\left(\frac{n\pi}{3}t\right) + c_2 \sin\left(\frac{n\pi}{3}t\right).
\]
Thus,

\[
u(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{6}x\right) \left( a_n \cos\left(\frac{n\pi}{3}t\right) + b_n \sin\left(\frac{n\pi}{3}t\right) \right).
\]

Using the initial conditions we arrive at the final solution:

\[
u(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{6}x\right) \left( \frac{12}{n^2 \pi^2} \left( 2 \sin\left(\frac{n\pi}{6}\right) - \sin\left(\frac{n\pi}{3}\right) \right) \right) \cos\left(\frac{n\pi}{3}t\right).
\]

Plots of the wave for \( 0 \leq t \leq 3 \) are shown below. We first note that that wave inverts itself and then "travels" down the cord. At \( t = 3 \) the wave hits the end of the cord and is ready to make its return journey.

Figure 10. Plot of \( u(x,0) \)
Figure 11. Plot of $u(x,0.3)$

Figure 12. Plot of $u(x,0.4)$
Figure 13. Plot of $u(x,0.5)$

Figure 14. Plot of $u(0.056)$
Figure 15. Plot of $u(x,0.6)$

Figure 16. Plot of $u(x,1)$
Figure 17. Plot of $u(x,1.5)$

Figure 18. Plot of $u(x,2)$
Figure 19. Plot of $u(x,2.4)$

Figure 20. Plot of $u(x,2.6)$
Here again, the still plots do not have the same impact as the animation. With Maple's "animation" command the wave comes alive and the students see that the mathematics does model reality reasonably well.
Again for the reader who is interested we will go through the mathematics and Maple commands which allow us to calculate the solution and the animation.

We recall that our general solution with \( u(0, t) = 0 \) and \( u(6, t) = 0 \) was given by:

\[
u(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{6} x\right) \left( a_n \cos\left(\frac{n\pi}{3} t\right) + b_n \sin\left(\frac{n\pi}{3} t\right)\right)\]

Now since

\[
u(x, 0) = f(x) \begin{cases} x & \text{for } 0 < x < 1 \\ 2-x & \text{for } 1 < x < 2 \\ 0 & \text{for } 2 \leq x \leq 6 \end{cases}
\]

we are now able to find the values of \( a_n \) and \( b_n \). Solving for \( a_n \) we get:

\[
a_n = \frac{2}{6} \int_{0}^{1} x \sin\left(\frac{n\pi x}{6}\right) dx + \frac{2}{6} \int_{1}^{2} (2-x) \sin\left(\frac{n\pi x}{6}\right) dx
\]

Since \( \frac{\partial u(x, 0)}{\partial t} = 0 \) this implies that \( b_n = 0 \). Thus,

\[
u(x, t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{6} x\right) \left( \left( \frac{2}{6} \int_{0}^{1} x \sin\left(\frac{n\pi x}{6}\right) dx + \frac{2}{6} \int_{1}^{2} (2-x) \sin\left(\frac{n\pi x}{6}\right) dx \right) \cos\left(\frac{n\pi}{3} t\right) + 0 \sin\left(\frac{n\pi}{3} t\right) \right)
\]

The Maple commands to calculate the solution

**By Hand**

- \( a_n \) done above
- \( u(x, t) \) as above

**Using Maple**

- \( a1 := \frac{2}{6} \int_{0}^{1} x \sin(n \pi x / 6), x = 0..1; \)
- \( a2 := \frac{2}{6} \int_{1}^{2} (2-x) \sin(n \pi x / 6), x = 1..2; \)
- \( an := a1 + a2; \)

- \( u := \text{sum}(\text{sin}(n \pi x / 6) * an \text{cos}(n \pi t / 3), n = 1..10); \)

Plotting is difficult

- \( \text{with(plots):} \)
- \( \text{animate}(u, x = 0..6, t = 0..24, \text{frames} = 1000); \)

C) 2-D Wave Equation:

For this example we will use the example of the beating circular membrane (a drum). Since we have waves moving in two directions (x and y). If we let \( u(x, y) \) be the
displacement of the membrane at any point \((x, y)\), at any time \(t\), we get the two dimensional wave equation given by:

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}.
\]

The boundary conditions are:

A) \(u(x, y, t) = 0\) no movement along the boundary

B) \(u(x, y, 0) = f(x, y)\) initial displacement

C) \(\frac{\partial u(x, y, 0)}{\partial t} = g(x, y)\) initial velocity

But, since we are on a circular drum the wave equation becomes a function two variables, \(r\) and \(t\) only. Note that \(r\) is the radial distance from the center of the drum to its edge and \(t\) represents time. In this example we are assuming that the wave is independent of the variable \(\theta\). We now let displacement, in polar form, be given by \(v(r, t)\) where \(r\) is the distance from the center of the membrane, and \(t\) is time. The wave equation for a drum of radius \(a\) now becomes:

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v}{\partial r} \right) + \frac{1}{c^2} \frac{\partial^2 v}{\partial t^2} = 0 \quad \text{for } 0 \leq r \leq a, \text{ and } 0 \leq t
\]

\(v(a, t) = 0\) for \(0 \leq t\), no movement along the boundary

\(v(r, 0) = f(r)\), initial displacement

\(\frac{\partial v(r, 0)}{\partial r} = g(r)\), initial velocity.

By letting \(v(r, t) = R(r)T(t)\) and separating variables we get:

\[
\frac{1}{rR} \left( rR' \right)' = -\lambda^2 c^2 T'' \quad \text{for } 0 \leq t
\]

\(rR' + r\lambda^2 = 0\) we get \(R(r) = AJ_0(\lambda r) + BY_0(\lambda r)\). Since \(r \to 0\) \(\lambda r \to 0\) this means that \(B = 0\). Thus \(R(r) = AJ_0(\lambda r)\). Now \(R(a) = 0\) means that \(AJ_0(\lambda a) = 0 \Rightarrow \lambda a = \alpha_n\) where \(\alpha_n\) is a root of \(J_0\). Thus, \(\lambda_n = \frac{\alpha_n}{a}\), and \(R(r) = AJ_0(\lambda_n r)\). Note that \(J_0\) is the Bessel function of the first kind of order zero and \(Y_0\) is the Bessel function of the second kind of order zero.

Now \(\frac{1}{c^2} \frac{\partial^2 T}{\partial t^2} = -\lambda^2 = \left( \frac{\alpha_n}{a} \right)^2\). Thus \(T(t) = a_n \cos(c\lambda_n t) + b_n \sin(c\lambda_n t)\) for \(n = 1, 2, 3, \ldots\)

Now we have
\[ v(r, t) = \sum_{n=1}^{\infty} (a_n \cos(c \lambda_n t) + b_n \sin(c \lambda_n t))J_0(\lambda_n r). \]

Now \( v(r, 0) = f(r) = \sum_{n=1}^{\infty} (a_n)J_0(\lambda_n r) \) which means that \( a_n = \frac{\int_0^a f(r)J_0(\lambda_n r)rdr}{\int_0^a J_0^2(\lambda_n r)rdr} \).

Now
\[
\frac{\partial v}{\partial t} = \sum_{n=1}^{\infty} (-a_n c \lambda_n \sin(c \lambda_n t) + b_n c \lambda_n \cos(c \lambda_n t))J_0(\lambda_n r) \Rightarrow \\
\frac{\partial v(r, 0)}{\partial t} = g(r) = \sum_{n=1}^{\infty} (b_n c \lambda_n)J_0(\lambda_n r).
\]

Thus,
\[
b_n = \frac{1}{c \lambda_n} \frac{\int_0^a g(r)J_0(\lambda_n r)rdr}{\int_0^a J_0^2(\lambda_n r)rdr}.
\]

We will consider the example where the radius of the drum is \( \frac{\pi}{2} \), the value of \( c \) is one, and there is no movement on the edge of the drum. We will let the initial displacement be \( \cos(r) \) and the initial velocity be \( \sin(r) \). This leads to the following boundary value problem.

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v}{\partial r} \right) = \frac{1}{1^2} \frac{\partial^2 v}{\partial t^2} \quad \text{for } 0 \leq r \leq \frac{\pi}{2}, \text{ and } 0 \leq t \\
v(a, t) = 0 \quad \text{for } 0 \leq t \\
v(r, 0) = \cos(r) \\
\frac{\partial v(r, 0)}{\partial r} = \sin(r)
\]

We now need a few zeros of the Bessel function of order zero. For our example we will
take the first four zeros. We also need to find the values for \( \lambda_n \) and \( J_0(\lambda_n r) \). The first four zeros of the Bessel function of order zero are given by:

**By Hand (Tables)**

\[
\begin{align*}
a_1 &= 2.404825558 \\
a_2 &= 5.520078110 \\
a_3 &= 8.653727913 \\
a_4 &= 11.79153444
\end{align*}
\]

We now calculate the \( \lambda_n \)s:

**By Hand (Tables)**

\[
\begin{align*}
\lambda_1 &= \frac{2.404825558}{\frac{\pi}{2}} = 1.530959499 \\
\lambda_2 &= \frac{5.520078110}{\frac{\pi}{2}} = 3.514190869 \\
\lambda_3 &= \frac{8.653727913}{\frac{\pi}{2}} = 5.509134294 \\
\lambda_4 &= \frac{11.79153444}{\frac{\pi}{2}} = 7.506723969
\end{align*}
\]

We now find \( J_0(\lambda_n r) \) which we call \( R_n \), for \( n = 1, 2, 3, 4 \).

**By Hand (Tables)**

\[
\begin{align*}
R_1 &= J_0(1.530959499r) \\
R_2 &= J_0(3.514190869r) \\
R_3 &= J_0(5.509134294r) \\
R_4 &= J_0(7.506723969r)
\end{align*}
\]

**Using Maple**

\[
\begin{align*}
&> \text{BZ} := \text{evalf}([\text{BesselJZeros}(0, 1 .. 4)]); \\
&> \text{L1} := \text{evalf}(\text{BZ}[1]/(\pi/2)); \\
&> \text{L2} := \text{evalf}(\text{BZ}[2]/(\pi/2)); \\
&> \text{L3} := \text{evalf}(\text{BZ}[3]/(\pi/2)); \\
&> \text{L4} := \text{evalf}(\text{BZ}[4]/(\pi/2));
\end{align*}
\]

We now find \( J_0(\lambda_n r) \) which we call \( R_n \), for \( n = 1, 2, 3, 4 \).

**By Hand (Tables)**

\[
\begin{align*}
R_1 &= J_0(1.530959499r) \\
R_2 &= J_0(3.514190869r) \\
R_3 &= J_0(5.509134294r) \\
R_4 &= J_0(7.506723969r)
\end{align*}
\]

**Using Maple**

\[
\begin{align*}
&> \text{R1} := \text{BesselJ}(0, \text{L1} \cdot r); \\
&> \text{R2} := \text{BesselJ}(0, \text{L2} \cdot r); \\
&> \text{R3} := \text{BesselJ}(0, \text{L3} \cdot r); \\
&> \text{R4} := \text{BesselJ}(0, \text{L4} \cdot r)
\end{align*}
\]

We now find the \( a_n \)s given by:

\[
a_n = \frac{\int_0^a f(r)J_0(\lambda_n r)rdr}{\int_0^a J_0^2(\lambda_n r)rdr}
\]
By Hand (Tables)

\[
a_1 = \frac{\pi}{2} \int_0^{\lambda_1} \cos(r)J_0(\lambda_1 r)rdr
\]

Using Maple

\[ > a_1 := (\text{Int}(fr^{R1^2} r, \ r = 0 .. (1/2)\pi))/(\text{Int}(R1^2 r, \ r = 0 .. (1/2)\pi)); \]

\[
a_2 = \frac{\pi}{2} \int_0^{\lambda_2} \cos(r)J_0(\lambda_2 r)rdr
\]

\[ > a_2 := (\text{Int}(fr^{R2^2} r, \ r = 0 .. (1/2)\pi))/(\text{Int}(R2^2 r, \ r = 0 .. (1/2)\pi)); \]

\[
a_3 = \frac{\pi}{2} \int_0^{\lambda_3} \cos(r)J_0(\lambda_3 r)rdr
\]

\[ > a_3 := (\text{Int}(fr^{R3^2} r, \ r = 0 .. (1/2)\pi))/(\text{Int}(R3^2 r, \ r = 0 .. (1/2)\pi)); \]

\[
a_4 = \frac{\pi}{2} \int_0^{\lambda_4} \cos(r)J_0(\lambda_4 r)rdr
\]

\[ > a_4 := (\text{Int}(fr^{R4^2} r, \ r = 0 .. (1/2)\pi))/(\text{Int}(R4^2 r, \ r = 0 .. (1/2)\pi)); \]

The next step is to calculate the \( b_n \)'s which are given by 

\[
b_n = \frac{1}{c\lambda_n} \frac{\int_0^a g(r)J_0(\lambda_n r)rdr}{\int_0^a J_0^2(\lambda_n r)rdr} \]  

that in our example \( c = 1 \) and \( g(r) = \sin(r) \), so we have:
By Hand (Tables) Using Maple

\[ b_1 = \frac{\pi}{2} \int_0^\infty \sin(r) J_0(\lambda_1 r) r \, dr \]

\[ \lambda_1 \int_0^\infty J_0(\lambda_1 r) r \, dr > b1 := \left( \int (r R1^* r, r = 0 .. (1/2)^*Pi) / ((\int (R1^2 r, r = 0 .. (1/2)^*Pi))^*L1) \right) \]

\[ b_2 = \frac{\pi}{2} \int_0^\infty \sin(r) J_0(\lambda_2 r) r \, dr \]

\[ \lambda_2 \int_0^\infty J_0(\lambda_2 r) r \, dr > b2 := \left( \int (r R2^* r, r = 0 .. (1/2)^*Pi) / ((\int (R2^2 r, r = 0 .. (1/2)^*Pi))^*L2) \right) \]

\[ b_3 = \frac{\pi}{2} \int_0^\infty \sin(r) J_0(\lambda_3 r) r \, dr \]

\[ \lambda_3 \int_0^\infty J_0(\lambda_3 r) r \, dr > b3 := \left( \int (r R3^* r, r = 0 .. (1/2)^*Pi) / ((\int (R3^2 r, r = 0 .. (1/2)^*Pi))^*L3) \right) \]

\[ b_4 = \frac{\pi}{2} \int_0^\infty \sin(r) J_0(\lambda_4 r) r \, dr \]

\[ \lambda_4 \int_0^\infty J_0(\lambda_4 r) r \, dr > b4 := \left( \int (r R4^* r, r = 0 .. (1/2)^*Pi) / ((\int (R4^2 r, r = 0 .. (1/2)^*Pi))^*L4) \right) \]

Our answer now becomes

\[ v(r, t) = \sum_{n=1}^{\infty} (a_n \cos(c\lambda_n t) + b_n \sin(c\lambda_n t)) J_0(\lambda_n r) \]

\[ v(r, t) = J_0(1.530959499r) (1.049523433cos(1.530959499t) + 0.7084088513sin(1.530959499t)) + J_0(3.514190869r)(-0.6279292521e - 1cos(3.514190869t) - 3826204356sin(3.514190869t)) + J_0(5.509134294r)(0.1883819963e - 1cos(5.509134294r) + 1473189850sin(5.509134294r)) + J_0(7.506723969r)(-0.8481812025e - 2cos(7.506723969t) - 1038196726sin(7.506723969t)) \]

In Maple we’d have:
\[
BesselJ(0., 1.530959499*r)*(1.049523433*cos(1.530959499*t) + 0.7084088513*sin(1.530959499*t)) + \\
BesselJ(0., 3.514190869*r)*(-0.6279292521e-1*cos(3.514190869*t) - 0.3826204356*sin(3.514190869*t)) + \\
BesselJ(0., 5.509134294*r)*(0.1883819963e-1*cos(5.509134294*t) + 0.1473189850*sin(5.509134294*t)) + \\
BesselJ(0., 7.506723969*r)*(-0.8481812025e-2*cos(7.506723969*t) - 0.1038196726*sin(7.506723969*t)).
\]

Now assuming that we could actually get this answer by hand what does it look like?
Again Maple comes to the rescue with the "animate" command.

Using Maple

>with(plots);
>animate(plot3d,[[r*cos(theta),r*sin(theta),urt],
    r=0..(1/2)*Pi,theta=-Pi..Pi,t=0 .. 20,frames=100);

Graphs of \( U(r, \theta, t) \), for various values of \( t \), follow.

Figure 23. Plot of \( U(r, \theta, 0) \)
Figure 24. Plot of $U(r, \theta, 1.0101)$

Figure 25. Plot of $U(r, \theta, 1.2121)$
Figure 26. Plot of $U(r, \theta, 1.4141)$

Figure 27. Plot of $U(r, \theta, 1.6162)$
Figure 28. Plot of $U(r, \theta, 1.8182)$

Figure 29. Plot of $U(r, \theta, 2.0202)$
Figure 30. Plot of $U(r, \theta, 2.2222)$

Figure 31. Plot of $U(r, \theta, 2.6263)$
Figure 32. Plot of $U(r, \theta, 3.2323)$

Figure 33. Plot of $U(r, \theta, 3.6364)$
Results

The question, of course, is does using Maple and having in class demonstrations actually make any difference to students and is there anything “new” presented in this paper.
We'll start with is there anything "new" in this paper. The answer is probably "no" since we have been doing these demonstrations since the author began teaching this course in 2002. Thus, texts have been written which incorporate the use of computer algebra systems into a boundary value course, such as "Partial Differential Equations & Boundary Value Problems with Maple V", written by G.A. Articolo. While the use of a computer algebra system in the course may not be "new" in 2010, but it was a fairly new idea when the author started teaching the course.

The real question is: Does doing actual demonstrations in class, using a computer algebra system to arrive at actual solutions, and having animations of the solutions make a difference to the students? Here we can answer the question with a definite "yes".

The boundary value course had been taught for several years before the author volunteered to start teaching the course in the spring of 2002. The course was an elective course and was not required of any major in the institute. The course enrollment during the spring of 2000 was six students. In 2001 the enrollment was also six students. In 2002 the author decided that one of the problems with enrollment was that the students didn’t really know about the course and that the students also didn’t understand that the course would be useful in their engineering careers. Thus, the following e-mail was sent to all students.

"MA 336 Boundary Value Problems will be offered during the spring term. If you are a ChemE, EE, ME, Math, or Physics major this course may be of interest to you. MA 336 picks up where MA 222 leaves off with Fourier series. (Anybody who has passed MA 222 (DEII) is well prepared for this course.)

This is a very applied course. If you plan to take Heat Transfer, Thermodynamics, or E-Mag Fields this is a good prerequisite course as we will derive and show you how to solve the partial differential equations that you will use in these courses.

In this course we will discuss the solution to the heat equation (how heat dissipates in a bar), the wave equation (how strings and drum heads vibrate) and the potential equation."

The enrollment in the spring of 2002 was twenty-two students. Gee, maybe it’s a good thing to advertise. It’s one thing to advertise, but it is another to have a course that students want to take, especially an elective higher level mathematics course. In 2003 when the enrollment was twenty-nine students, the author was told by some senior faculty members that the only reason the course had such high enrollment was that the author had "watered down the course" and that students were not getting a "real" mathematics course. It was also suggested that the author had high enrollments in the course simply because he required all his advisees (all of who where mathematically advanced freshman students) to
take the course. While the author did suggest to his advisees that the course was a good
course to take, only a few (less that ten) could fit the course into their schedules. There had
to be a different reason for the high enrollments.

The answer, the author believes, is in the use of in-class demonstrations, and the use of
Maple to do calculations to see the results. One student said after seeing the animation of
the heat equation, "When I saw the animation, I thought that was the coolest thing I'd ever
seen." One year the author intended to spend the last class showing the animation of the
"beating drum." Unfortunately the entire school lost electrical power about three hours
before class. Being either optimistic or very naive, the author went to class in spite of the
blackout. After all, the classroom did have windows and so there was some light. To the
author's great surprise, the entire class was in the room. When it came time to show the
animation of the beating drum the author started up his laptop and loaded the Maple code.
Just about the time the animation was to start the laptop's battery died. One of the students
immediately said "I didn't walk all the way over here not to see the animation." He pulled the
battery out of his laptop and gave it to the author. The batteries were switched and thirty
people stood around a laptop watching the "beating drum" in the dark.

Comments on the students' course evaluations tend to say things such as: "The course
helped me to understand many of the problems I have seen in my engineering courses.; or
I liked "...Emphasis on process and meaning, not memorization. Integration of Maple to see
what solutions look like.; and we "Learn a lot of insights into things other courses take for
granted like the heat equation and the wave equation."

There are also unsolicited comments from students who have written the author after
having taken the course. One alumnus wrote:

"I took your Boundary Value Problems course the first year you taught it, two springs
ago. I just started at Georgia Tech studying for a Ph.D. in Aerospace Engineering after
finishing up in November, and thought you'd like to hear that had I not taken that course, I
would be very lost in my Structural Dynamics course here. I actually have a test tomorrow,
largely on the Euler-Bernoulli beam, which I remember from your class as a Sturm-Liouville
problem. Maybe this e-mail will help a student take your class and save some heartache in
grad school...

Another alumnus wrote, "Everything is a separable PDE"; another wrote, "BVP is the
most useful course ever." and another, "I wish I had taken MA 336 before I took E-Mag
fields."
Enrollment in the class continues to do well. In the spring of 2006 we started offering two sections of the course taught by two different faculty members. The enrollments continued to stay at reasonable levels with total enrollments of forty-eight students in 2006, forty-six students in 2007, forty-seven students in 2008, thirty-nine students in 2009, and forty-eight students in the spring of 2010. We will also be adding a section of the course to be taught in the Fall of 2010. Thus, it is not the instructor of the course that makes the difference, but the way the course is taught. Our course is very applied, with demonstrations, and Maple is incorporated into the course.

The course is now "highly recommended" by the Mechanical Engineering Department for students in their aerospace concentration, and just this past year the Physics Department has made the course a required course for their majors.

Conclusion

The purpose of this paper is to demonstrate how live demonstrations, and the use of a computer algebra system can help improve the learning or, and the interest in taking an upper level boundary values course. We believe the enrollment numbers and the change in view of the Mechanical Engineering, Physics, and Mathematics, departments clearly prove our point.

Our students, all of which are science, mathematics, and engineering majors, typically are very goal oriented and very practical. They like courses that apply theory to "real world" situations, and they like to get concrete answers that you can see. In our Boundary Value Problems course we explain how the models are built mathematically, we have physical demonstrations of the situations, and we teach the students how to solve the mathematical models. By using Maple the students can do the necessary calculations required to arrive at answers which they can graph on their computers and literally see that they have the correct solution.

Thus, by incorporating classroom demonstrations, and the integration of Maple into our Boundary Value Problems course we have taken a course that was on the verge of being cancelled, to one that is popular with students and valued by our engineering and science colleagues. We are no longer accused of "coercing" students to take the course or of having a "watered down" course. Now the question is "How did you do it?" The answer is basically very simple.

As mathematicians we need to know that it is important that we meet the needs of our students and prepare them not only for their future engineering, mathematics, and science courses, but that we also prepare them to solve problems once they leave school and enter the fields of engineering and science. A boundary value course, we believe, is an excellent step to help students become better engineers and scientists. The answer to the question,
"How did you do it?" is, "Seeing is believing."

References