Towards Removing Barriers Against learning Control Systems Design: A Comprehensive Review of the Math Required for Reaching Milestones in Control Systems Design

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Abstract

This paper focuses on the importance of math knowledge for learning Control Systems Design course, and outlines the necessary concepts that must be grasped by students prior to enrollment in this course. The Control Systems Design is a senior-level course and its learning outcomes include developing mathematical models of control components and systems, simplifying block diagrams, calculating transfer functions and state space models of linear, time-invariant systems, formulating and/or utilizing specifications needed to design system controllers, and more. A study in the author's Controls class of 39 students showed that 49 problems (out of 195 total problems for all students in the exam) were solved incorrectly due to students' knowledge deficiency in math aspects. This indicates that 25% of the students' knowledge was not rigorous towards solving the problems correctly. Review of the above learning outcomes and the assessed statistics elaborates the significance of mathematical background as a building block of the required skills for learning the concepts of Control Systems Design course. To address the solution, this paper sheds light into necessary math and algebra that students need before taking Control Systems Design so that they can fully focus on learning the controls concepts during the course. This paper will go through the course outline, discuss each topic and its prerequisite math knowledge and worked examples with references to the relevant math courses. Finally, authors will provide suggestions for improving STEM students' knowledge and confidence in becoming adept in the Control Systems Design course and similar courses in their engineering curriculum.

Introduction

The reason for not being sufficiently strong in math when students enter college/ university can be either not having had four years of math in high school or having completed four years but not having mastered the materials [1]. Colleges and universities have made a lot of efforts to help students to be sufficiently proficient in math for their higher education studies. Many of them set placement assessments. As an example, at CSU Chico, the Entry Level Math test (ELM) assesses the student's level of ability in math through the beginning and intermediate algebra and plane geometry. The test emphasizes the understanding of mathematical concepts and problem-solving skills. Students will not be allowed to use a calculator during the test, and the score determines placement into an appropriate level math course. The ELM is not used for admission purposes, and can be taken more than once. There have been many programs such as bridge programs in which students practice and learn math on their own by using online tools (such as ALEKS), with available mentors to answer their questions [2]. At Los Medanos College, one of the authors developed a one credit math course as a corequisite for electrical circuits. Students would benefit from the math course by reviewing all the math concepts they needed right before the specific topic that was being covered in the circuits course.

The efforts mentioned above all help students to be more prepared when entering a STEM program; However, it has been proved that students find it challenging when it comes to transferring their knowledge between mathematics and engineering concepts. A survey was conveyed to investigate the importance of mathematics courses in different engineering majors, and it was found out that specific subjects of math are more crucial for Electrical and Computer Engineering major students [3]. These subjects include calculus, linear algebra, ODE, complex numbers, and vectors. The complex number was rated of high importance only in the electrical and computer engineering which can imply that there is a possibility of ignoring this significantly important topic in math classes since other majors' curricula do not find it much necessary. The same study shows that students have low competence in the areas of linear algebra and complex numbers, the two topics which are important for understanding Control System Design.

A four-year Electrical Engineering curriculum in the United States ABET universities is designed to equip students with the required math by passing courses such as Calculus, Elementary Differential Equation and Signals and Systems; However, yet math seems a challenge for most of the Engineering students and plays as a barrier to their success. While most of the students who take the Control Systems Design course are Electrical or Computer Engineering major, there are also Mechanical Engineering and Civil Engineering major students. These students do not pass Signals and Systems, and find Control systems Design even more difficult since they do not have proper math background.

The curriculum of the Control System Design that is taught in the Electrical Engineering Departments assumes that students have the Signals and Systems knowledge [4]. A comparison between the grades of students in Signals and Systems and the same students in Control System Design showed that the grades of Electrical Engineering students in these two courses were close, approximately one step higher or lower. This alone shows how important the math background and Signals and Systems are in grasping the control systems concepts. The three exams taken during the Control System Course show the following statistics (Table 1) about problems solved incorrectly due to lack of knowledge in math.

The topics covered in a control system design course typically include [5]:

1. Introduction to Control Systems
2. Mathematical Modeling of Control Systems
3. Mathematical Modeling of Mechanical Systems and Electrical Systems
4. Model Reduction Using Mason's rule
5. Transient and Steady State Response Analysis
6. Control Systems Analysis and Design by the Root-Locus Method
7. Control Systems Analysis and Design by the Frequency-Response Method
8. Control Systems Analysis and Design in State Space

Table 1: Percentage of problems solved incorrectly in Control System Exams due to lack of knowledge in math.

Exam Number	Percentage of problems solved incorrectly due to insufficient math knowledge	Topics covered in the exam
Exam 1	25%	- Mathematical Modeling of Systems (Transfer Function and State Space, Block Diagram Reduction) - Linearization of nonlinear systems
Exam 2	25.6%	- Electronic Controllers - Steady State Errors of Systems - Transient Response of Systems - Routh-Hurwitz Stability Criterion - Root Locus Sketches
Exam 3 (final Exam)	22%	- All of the above - Controller Design by Root Locus - Frequency Response Gain Margin, Phase Margin - Nyquist Stability Theorem - Design of Lag, Lead Controllers by Frequency Response - Pole Placement Method

Table 2 summarizes the math topics that are needed to be reviewed by students for each exam.

In this paper, we intend to go through the math topics that authors believe they play as barriers for students in obtaining correct responses to Control System Design course problems. The purpose of this paper is not to teach control systems concepts. In each topic listed above, based on the experience of the authors, the topics that students usually struggle with will be discussed with references. Worked examples of the discussed topics will be presented in the appendix of this paper. Students are encouraged to further study the topics in the references.

Table 2: Math topics required for each exam

Exam number and content	Required math topics
Exam 1: - Mathematical Modeling of Systems (Transfer Function and State Space, Block Diagram Reduction) - Linearization of nonlinear systems	- Transfer Function definition - Laplace transform of time domain signals - Partial fraction decomposition - Inverse Laplace of transfer functions - Matrix algebra, inverse of a matrix - Correlating transfer function and state space representation - Partial derivative used for linearization of nonlinear systems
Exam 2: - Electronic Controllers - Steady State Errors of Systems - Transient Response of Systems - Routh-Hurwitz Stability Criterion - Root Locus Sketches	- Final value theorem - Taking limits of functions at zero and infinity - Inverse Laplace of transfer functions - Geometry, trigonometry - Complex numbers
Exam 3 (final Exam): - All of the above - Controller Design by Root Locus - Frequency Response Gain Margin, Phase Margin - Nyquist Stability Theorem - Design of Lag, Lead Controllers by Frequency Response - Pole Placement Method - Design of Controllers by State Space	- All the above - Understanding the concept of Frequency Response of a system - Magnitude and phase of fractions with complex numbers in their numerators and denominators - Calculating the output of a system from the frequency response - Eigenvalues and eigenvectors - Linear independence

Important math subjects used in general areas of Control System Design

1- Math required for Mathematical Modeling of Control Systems

While most of the required mathematics calculations for control systems design can be done using software such as Matlab, it is expected that students be able to perform the preliminary math calculations in controls to show their understanding of the systems. We start from the second topic “mathematical modeling of control systems”. Each system can be modelled by a transfer function or by state space representation. Students need to learn both methods. The

Electrical Engineering students learn about transfer functions in Circuits as well as Signals and Systems. Yet they make mistakes due to missing/forgetting some details about this concept. Functions called transfer functions characterize the input-output relationship of Linear Time Invariant (LTI) systems under the assumption that all initial conditions are zero [5].
If x and y are the input and output of the system respectively, and the LTI system is defined by the following differential equation:

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 x^{(m)} + b_1 x^{(m-1)} + \cdots + b_{m-1}\dot{x} + b_m x$$

Where $n \geq m$, and $y^{(n)}$ is the n^th derivative of y, and $x^{(m)}$ is the m^th derivative of x.

The transfer function of this system is the ratio of the Laplace transform of the output $Y(s)$ to the Laplace transform of the input $X(s)$ when all initial conditions are zero.

$$Transfer\ function = G(s) = \frac{Y(s)}{X(s)}|_{zero\ initial\ condition}$$

$$= \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{(n-1)} + \cdots + a_{n-1}s + a_n}$$

The transfer function is the property of the system and does not depend on the inputs applied to the system. Therefore, if the input to the system changes, the transfer function will not change, but the output of the system will change. If the transfer function of a system is known, the output or response can be found for different inputs.

The other concept students must know from Signals and Systems is how to use partial fraction decomposition and inverse Laplace of transfer functions for obtaining the response of the system in time domain [4].

The second method of representation of a system is state space. This is one area that multiple students might make mistakes due to lack of review in matrix algebra. The following topics are among those areas:

a) How to multiply two matrices

If we multiply a $m \times n$ (m is the number of rows and n is the number of columns) matrix by a $n \times p$ matrix, the result will be a $m \times p$ matrix. If the number of columns of the first matrix is not equal to the number of rows of the second matrix, we cannot multiply those two matrices.

In multiplication of two matrices, if the order of the two matrices change, if still the dimensions allow multiplication, the result of multiplication will be different than the previous multiplication unlike the multiplication of two scalars.

b) How to write a set of equations with multiple unknowns in the form of matrices.

If we have a set of q linear equations with q unknowns, the equations can be written in the form of matrix equation and inverse matrix can be used to find the unknowns.

c) How to calculate the inverse of a matrix

The inverse of matrices can be easily found using calculators; but, in control system design it is sometimes required to calculate the inverse of a matrix with parametric elements which some calculators are not capable of doing.

Here is the procedure that must be followed to obtain the inverse of a square matrix A:

- Calculate minors: Minor m_{ij} of element a_{ij} of a square matrix A is the determinant of the array remaining when the i^{th} row and j^{th} column are deleted from A.
- Calculate cofactors: Cofactor c_{ij} of element a_{ij} of the matrix A is defined as $c_{ij} = (-1)^{i+j} m_{ij}$
- Calculate the Adjoint: the matrix of cofactors when transposed is the adjoint of A

$$adj\ A = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}^T$$

The inverse of A is given by:

$$A^{-1} = \frac{adj\ A}{|A|}$$

Where |A| and A^{-1} denote the determinant and the inverse of A respectively [6].

d) Being able to correlate transfer function, and state space representation of a system.

If an LTI system transfer function is given by:

$$\frac{Y(s)}{U(s)} = G(s)$$

The system may be represented in state space by the following equation:

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(\text{t})$$

The transfer function can be obtained by taking the Laplace Transform of the above State equations, and setting the initial conditions to zero ($x(0) = 0$).

$$sX(s) - x(0) = AX(s) + BU(s)$$
$$Y(s) = CX(s) + DU(s)$$

From the two equations, we have:

$$Y(s) = [C(sI - A)^{-1}B + D]U(s)$$
$$G(s) = \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B + D$$

This is the transfer function expression of the system in terms of A, B, C and D matrices [5].

It is important to note that if instead of the transfer function we are looking for the output of the system having the state space representation and the initial conditions, the initial conditions cannot be neglected, and the output will be:

$$Y(s) = C(sI - A)^{-1}[BU(s) + x(0)] + DU(s)$$

e) Partial derivative used for linearization of nonlinear systems

Assume $z = f(x, y)$, where z is the dependent variable and depends on the two independent variables x and y. Since z is a function of two variables, if we want to differentiate we need to decide whether we are differentiating with respect to x or with respect to y.

- $\frac{\partial z}{\partial x}$ is read as “partial derivative of z with respect to x” and means differentiate with respect to x holding y constant.
- $\frac{\partial z}{\partial y}$ means differentiate with respect to y holding x constant [6].

2- Mathematical modeling of Mechanical Systems and Electrical Systems

Mathematical modeling of mechanical systems mainly needs knowledge of physics for writing the Newton’s second law for those systems, deriving the differential equations from them and consequently deriving the transfer function or state space representation. Mathematical modeling of electrical systems needs that students have a comprehensive knowledge of solving circuits by means of nodal and mesh analysis. Both topics are important for learning Control System Design but are out of scope of this paper.

3- Model Reduction using Mason’s rule

Model Reduction using Mason’s rule does not involve much math. It mainly needs students to understand how to draw the signal flow graph equivalent to the system block diagram. Knowledge of circuits analyses is helpful to clarify where to define the nodes and branches of the signal flow graph.

4- Transient and Steady State Response Analyses

Transient response means the response that goes from the initial state to the final state. Steady state response means the way the system output behaves as t approaches infinity [5]. One of the common mistakes that students make is that they tend to forget to calculate the output, and take only the inverse Laplace of the system transfer function. To obtain the response, the Laplace Transform of the input is multiplied by the transfer function of the system; then, the inverse Laplace of the obtained output is calculated. The result includes both the transient and steady state response. The terms that decrease to zero as $t \to \infty$ correspond to the transient response, and the rest of the terms are associated with the steady state response of the system. The steady state error of a unity feedback system (Figure 1) to different inputs is calculated using the following system:

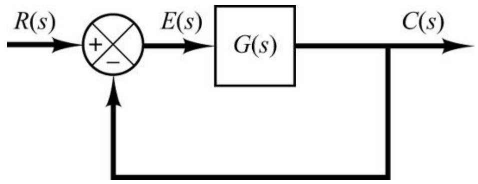

Figure 1: Unity feedback control system

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

$$E(s) = R(s) - C(s)$$

$$\frac{E(s)}{R(s)} = 1 - \frac{C(s)}{R(s)} = \frac{1}{1 + G(s)} \;\; \Rightarrow E(s) = \frac{R(s)}{1 + G(s)}$$

Employing the final value theorem, the steady state error will be:

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)}$$

Therefore, the steady state error is determined by the input and the open loop transfer function G(s).

The students need to know how to take the limit of functions when the ‘s’ variable is approaching zero. The steady state error can be zero, infinity or a finite nonzero number. This depends on not only the input R(s) but also the type of the open loop system transfer function (defined as the number of open loop poles at the origin).

5- Control Systems Analysis and Design by the Root-Locus Method

Using root locus method, the designer of a controller can evaluate the effects of varying a gain value or adding open-loop poles and/or zeros.

The negative feedback system shown in Figure 2 has the closed loop transfer function:

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

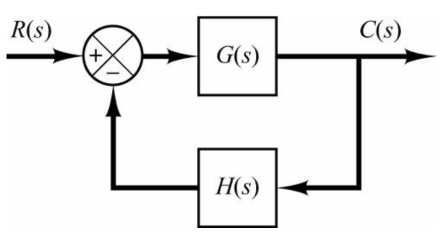

Figure 2: Control system

By setting the denominator of this transfer function to zero, the characteristic equation of the closed loop system is obtained:

$$1 + G(s)H(s) = 0 \quad \Longrightarrow \; G(s)H(s) = -1$$

Students need to understand that the above equation is equivalent to the following angle and gain conditions respectively [5]:

$$\angle G(s)H(s) = \pm 180°(2k+1) \qquad k = 0, 1, 2, \ldots$$

$$|G(s)H(s)| = 1$$

The values of s that fulfill the angle and magnitude conditions are the roots of the characteristic equation, and therefore the closed loop poles. Students must be able to use the two conditions to identify if a point on the s plane is part of the root loci of a closed loop system. This requires geometry and trigonometry, and complex numbers knowledge. The following example shows how the two conditions can be checked.
The angle and magnitude condition can be written for the open loop poles and zeros of the open-loop transfer function G(s)H(s) shown in the following s-plane.

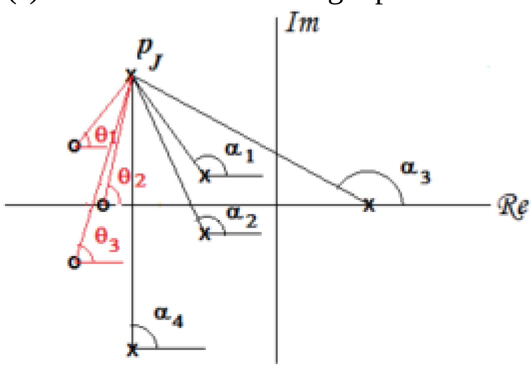

Figure 3: Diagram showing angle measurement from open loop poles and zeros [7]

The purpose of writing the two conditions is to test and see if p_J is on the root locus. The open-loop poles are shown with x's, and open-loop zeros are shown with o's. The angle at a specific point for the system shown in Fig. 3 is:

$$\angle G(s)H(s) = \theta_1 + \theta_2 + \theta_3 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4$$

Each of these angels are the angles made with a line parallel to the positive real axis and the line connecting the corresponding pole or zero to the test point on the s-plane.

Many students make mistakes when calculating the θ_i and α_i angles due to lack of knowledge in geometry and trigonometry.

6- Control Systems Analysis and Design by the Frequency-Response Method

Students must first understand that the Frequency Response of a system means the steady-state response of that system to a sinusoidal input. The frequency of the input signal over a certain range is varied, and the resulting response is studied. The frequency response can be represented in forms of Bode diagrams, Nyquist plots and Nichols chart. Control Systems Analysis and

Design by the Frequency-Response Method needs the knowledge of complex numbers. Students must be able to work with fractions that have complex numbers in their numerators and denominators, and know how to calculate the magnitude and phase of them.

The frequency response can be calculated by replacing s in the transfer function with $j\omega$. If G(s) is the transfer function of the system, then G($j\omega$) can be written as:

$$G(j\omega) = |G(j\omega)|e^{j\emptyset}$$

$$\emptyset = \angle G(j\omega) = \ tan^{-1}[\frac{imaginary\ part\ of\ G(j\omega)}{real\ part\ of\ G(j\omega)}]$$

Where $|G(j\omega)|$ represents the magnitude and $\emptyset$ represents the angle of $G(j\omega)$.

$G(j\omega)$ can also be represented in terms of the input, $X(j\omega)$, and output, $Y(j\omega)$, of the system:

$$G(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

The magnitude of $G(j\omega)$ can be written as the amplitude ratio of the output sinusoid to the input sinusoid.

$$|G(j\omega)| = |\frac{Y(j\omega)}{X(j\omega)}|$$

The angle of $G(j\omega)$ can be written as the phase shift of the output sinusoid with respect to the input sinusoid.

$$\angle G(j\omega) = \angle \frac{Y(j\omega)}{X(j\omega)}$$

7- Control Systems Analysis and Design in State Space

a) This topic includes writing the state space representation of systems in different canonical forms such as observable canonical forms, Diagonal canonical forms and Jordan canonical forms. The background math required for these topics is the same as the concepts covered earlier in this paper for matrix calculations. In addition, it is important to know how to calculate eigenvalues and eigenvectors of a square matrix.

Eigenvalues are important in control systems. One of their applications is that they provide us with information about the stability of control systems. If the eigenvalues of the system matrix A (in state space representation) are real and positive, or if they are complex with positive real parts, the system state and therefore the system output, will approach infinity as time *t* approaches infinity.

Eigenvalues of a square matrix A are the roots of the polynomial equation $|A - \lambda I| = 0$ where |.| denotes the determinant, and λ is a scalar.

Eigenvectors of a square matrix A are the vectors x_i that satisfy the equation $\lambda_i x_i = Ax_i$ where λ_is are the eigenvalues of A.

b) For investigating controllability and observability of continuous time systems it is required to know the concept of linearly independent column vectors.

Given a set of m vectors $a_{(1)}, \ldots, a_{(m)}$ and scalars $c_1, \ldots, c_m$ consider the equation:

$$c_1\, a_{(1)} + c_2\, a_{(2)} + c_m a_{(m)} = 0$$

This vector equation holds if all $c_j\,'s$ are zero. If this is the only m scalars which holds, then the vectors $a_{(1)}, \ldots, a_{(m)}$ are said to form a linearly independent set. Otherwise, if the above equation holds with scalars not all zero, these vectors are called linearly dependent. [6]

One way to test for linear independence of the vectors is using the determinant. A set of m vectors of length m is linearly independent if the matrix with these vectors as rows has a non-zero determinant.

Discussion

The examples and methods discussed here are to help instructors to know of the areas where if reviewed prior to the lectures could help students to proceed easier in the course. The review lecture notes could be distributed to the students, and they can be asked to go through them before each related topic to save time for the Control System concepts in the class. Students can also be referred to the math prerequisites where they learned each topic and review them. Table 3 lists the pre-requisite math courses for the topics presented in table 2. Multiple online resources such as Khan Academy are available for students to practice the necessary math subjects, as well as textbooks such as the one mentioned in the references of this paper.

The worked examples at the end of this paper, cover the math topics discussed in the previous section. Due to limitations in the length of this paper, the worked examples discussed in the appendix can be considered as an important starting point.

Other ways to improve learning Control Systems is to conduct labs in addition to the lectures. By having labs, students will learn better through visualization and hands-on experience, and will be able to relate theory to practice.

Table 3: Courses where each of the math topics are taught

Required math topics	Pre-requisite math courses
- Transfer Function definition - Laplace transform of time domain signals - Partial fraction decomposition - Inverse Laplace of transfer functions - Matrix algebra, inverse of a matrix - Correlating transfer function and state space representation - Partial derivative used for linearization of nonlinear systems	Differential equations Differential equations Differential equations Differential equations Calculus 3 (Vector calculus) Calculus 3 Calculus 3/ Differential equations
- Final value theorem - Taking limits of functions at zero and infinity - Inverse Laplace of transfer functions - Geometry, trigonometry - Complex numbers	Signals and systems Calculus 1 Differential equations, Signals and systems Calculus 1, Calculus 2, Calculus 3 Calculus 3
- Understand the concept of Frequency Response of a system - Magnitude and phase of fractions with complex numbers in their numerators and denominators - Calculating the output of a system from the frequency response - Eigenvalues and eigenvectors - Linear independence	Signals and systems Calculus 3 Signals and systems Differential equations Calculus 3

Conclusion

Control System Design, a senior level course in the Electrical Engineering curriculum, can be very challenging for students if they do not have a rigorous background in the prerequisite math courses and Signals and Systems. Certain areas in math, such as linear algebra and complex numbers, are employed in only Electrical Engineering major, not other engineering majors. Therefore, these topics are sometimes ignored in math courses, and are not usually practiced enough by students. It was shown that up to 25% of the problems in Control Systems course could be solved incorrectly if there is deficiency in students' math knowledge. This paper discussed the areas of math where students need to review and focus before taking the Control System Design course to grasp the materials better and perform better in their exams. It included few worked examples for each topic in the appendix to show students where they can start, and to continue the related areas more in detail at the given references. The purpose of the paper was not to teach Control Systems contents, but to inform students what could be helpful towards learning control and to prepare them more for learning the course contents.

Appendix: Worked Examples

Mathematical Modeling of Control Systems:

Example 1: Obtain the transfer function of a system with *step* input *u(t)* and output

$y(t) = (1 - 2e^{-4t} + e^{-8t})\, u(t)$ in time domain, when the initial condition is zero.

Laplace transform of the step input: U(s) = $\frac{1}{s}$

Laplace transform of the output: Y(s) = $\frac{1}{s} - \frac{2}{s+4} + \frac{1}{s+8}$

$$Transfer\ function = \frac{Y(s)}{U(s)} = \frac{\frac{1}{s} - \frac{2}{s+4} + \frac{1}{s+8}}{\frac{1}{s}} = \frac{32}{s^2 + 12s + 32}$$

Example 2: Obtain the inverse Laplace Transform of X(s).

$$\mathrm{X(s)} = \frac{1}{(s+1)(s+2)}$$

$$\mathrm{X(s)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$\mathrm{A} = [(\mathrm{s}+1)\mathrm{X(s)}]|_{s=-1} = 1$$

$$\mathrm{B} = [(\mathrm{s}+2)\mathrm{X(s)}]|_{s=-2} = -1$$

$$\mathrm{X(s)} = \frac{1}{s+1} + \frac{-1}{s+2}$$

Taking the inverse Laplace $x(t) = (e^{-t} - e^{-2t})u(t)$

If the denominator of X(s) has linear factors with powers higher than one, we will have multiple terms in the partial fraction decomposition from this factor.

Example 3: Obtain the inverse Laplace Transform of X(s).

$$\mathrm{X(s)} = \frac{5s^2 + 1}{(s+1)(s+2)^2}$$

$$\mathrm{X(s)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

$$\mathrm{A} = [(\mathrm{s}+1)\mathrm{X(s)}]|_{s=-1} = 6$$

$$\mathrm{B} = \frac{d}{ds}[(s+2)^2\mathrm{X(s)}]|_{s=-2} = \text{-1}$$

$$\mathrm{C} = [(s+2)^2\mathrm{X(s)}]|_{s=-2} = -21$$

$$X(s) = \frac{6}{s+1} + \frac{-1}{s+2} + \frac{-21}{(s+2)^2}$$

$$x(t) = (6e^{-t} - e^{-2t} - 21te^{-2t})u(t)$$

Example 4: Multiplication of a 2×3 matrix by a 3×1 matrix.

$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \end{bmatrix}_{2\times3} \times \begin{bmatrix} t_{11} \\ t_{21} \\ t_{31} \end{bmatrix}_{3\times1} = \begin{bmatrix} r_{11} \times t_{11} + r_{12} \times t_{21} + r_{13} \times t_{31} \\ r_{21} \times t_{11} + r_{22} \times t_{21} + r_{23} \times t_{31} \end{bmatrix}_{2\times1}$$

Example 5: Write the following set of equations in the matrix format.

$$x + y + z = 6$$

$$2y + 5z = -4$$

$$2x + 5y - z = 27$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 5 \\ 2 & 5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ 27 \end{bmatrix}$$

If we consider the matrices above as A= $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 5 \\ 2 & 5 & -1 \end{bmatrix}$, X= $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and B=$\begin{bmatrix} 6 \\ -4 \\ 27 \end{bmatrix}$, then we can write AX=B.

By multiplying the A matrix by X vector, the left side of the set of three equations is obtained.

In the above example, to solve for X vector which includes the three unknowns, both sides of the matrix equation must be multiplied by the inverse of A.

Example 6: Calculate the inverse of A in the last example:

- Calculate minors: m_{11} = -2-25 = -27 m_{21} = -1-5 = -6 m_{31} = 5-2 = 3
  m_{12} = 0-10 = -10 m_{22} = -1-2 = -3 m_{32} = 5-0 = 5
  m_{13} = 0-4 = -4 m_{23} = 5-2 = 3 m_{33} = 2 -0 = 2
- Calculate cofactors: c_{11} = -27 c_{21} = 6 c_{31} = 3
  c_{12} = 10 c_{22} = -3 c_{32} = - 5
  c_{13} = -4 c_{23} = - 3 c_{33} = 2

$$adj\ A = \begin{bmatrix} -27 & 10 & -4 \\ 6 & -3 & -3 \\ 3 & -5 & 2 \end{bmatrix}^T = \begin{bmatrix} -27 & 6 & 3 \\ 10 & -3 & -5 \\ -4 & -3 & 2 \end{bmatrix}$$

$| A |$ = -21

$$A^{-1} = \frac{adj\ A}{|A|} = \begin{bmatrix} 1.29 & -0.29 & -0.14 \\ -0.48 & 0.14 & 0.24 \\ 0.19 & 0.14 & -0.09 \end{bmatrix}$$

Therefore, X can be calculated by $X = A^{-1}B = \begin{bmatrix} 1.29 & -0.29 & -0.14 \\ -0.48 & 0.14 & 0.24 \\ 0.19 & 0.14 & -0.09 \end{bmatrix} \begin{bmatrix} 6 \\ -4 \\ 27 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$.

Example 6: The state space model of a system is given by:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u$$

where x_1 and x_2 are the state variables, *u* is the input and *y* is the output (all scalars). If the initial condition of x is $x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and the input is a step function, find Y(s).

$$\begin{aligned} Y(s) &= C(sI - A)^{-1}[BU(s) + x(0)] + DU(s) \\ &= \begin{bmatrix} 2 & 1 \end{bmatrix} \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} \right)^{-1} \left[\begin{bmatrix} 0 \\ 1 \end{bmatrix} \times \frac{1}{s} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right] + 1 \times \frac{1}{s} \\ &= \begin{bmatrix} 2 & 1 \end{bmatrix} \left(\begin{bmatrix} s+1 & -1 \\ 0 & s+2 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ \frac{1}{s} \end{bmatrix} + \frac{1}{s} \\ &= \begin{bmatrix} 2 & 1 \end{bmatrix} \frac{1}{(s+1)(s+2) - 0} \begin{bmatrix} s+2 & 1 \\ 0 & s+1 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{s} \end{bmatrix} + \frac{1}{s} \\ &= \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{(s+1)} & \frac{1}{(s+1)(s+2)} \\ 0 & \frac{1}{(s+2)} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{s} \end{bmatrix} + \frac{1}{s} \\ &= \begin{bmatrix} \frac{2}{(s+1)} & \frac{s+3}{(s+1)(s+2)} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{s} \end{bmatrix} + \frac{1}{s} \\ &= \frac{2}{(s+1)} + \frac{s+3}{s(s+1)(s+2)} + \frac{1}{s} \end{aligned}$$

The transfer function obtained above can be decomposed by partial fraction, and the inverse Laplace can be used to obtain y(t) if necessary.

Example 7: Calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ when $z = x^3 + 5xy - 4y + 1$

$$\frac{\partial z}{\partial x} = 3x^2 + 5y$$
$$\frac{\partial z}{\partial y} = 5x - 4$$

Transient and Steady State Response Analyses:

Example 1: Calculate the steady state error of the feedback system shown in Figure 1 when the reference input R(s) is a step input, and $G(s) = \frac{s+2}{s(s+5)(s+8)}$.

Step input: $R(s) = {}^{1}/_{s}$

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)} = \lim_{s \to 0} \frac{s \times {}^{1}/_{s}}{1 + \frac{s+2}{s(s+5)(s+8)}} = \lim_{s \to 0} \frac{s(s+5)(s+8)}{s(s+5)(s+8) + s + 2}$$

$$e_{ss} = \frac{0 \times 5 \times 8}{0 + 0 + 2} = 0$$

The steady state error of the system to a step reference input is zero, and the output of the closed loop system can follow a step reference input.

Example 2: Calculate the steady state error of the feedback system shown in Figure 1 when the reference input R(s) is an acceleration input $r(t) = {}^{1}/_{2}\, t^2$, and $G(s) = \frac{s+2}{s(s+5)(s+8)}$.

Step input: $R(s) = {}^{1}/_{s^3}$

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)} = \lim_{s \to 0} \frac{s \times {}^{1}/_{s^3}}{1 + \frac{s+2}{s(s+5)(s+8)}} = \lim_{s \to 0} \frac{{}^{1}/_{s^2} \times s(s+5)(s+8)}{s(s+5)(s+8) + s + 2}$$

$$= \lim_{s \to 0} \frac{(s+5)(s+8)}{s(s(s+5)(s+8) + s + 2)} = \frac{5 \times 8}{0 \times 2} \to \infty$$

The above example indicates that the steady state error of a type one system (where type refers to the number of integrators in the open loop transfer function or the power N in s^N of the denominator of the open loop transfer function) to an acceleration input is infinite and the output of the system cannot follow the acceleration input and diverges from it.

This is the general method of calculating steady state errors. For more simplicity error constants can be defined for each input, and the steady state errors will be calculated based on them [5].

Control Systems Analysis and Design by the Root-Locus Method:

Example 1: Write the angle and magnitude condition for the open loop poles and zeros of the open-loop transfer function G(s)H(s) shown in the following s-plane. Assume the following for the poles and zeros in Fig. 3:

α_1 and α_2 are associated with the poles at $-3 \pm j1$.

α_3 is associated with the pole at 4.

α_4 is associated with the pole at −5−j5.

θ_1 and θ_3 are associated with the zeros at $-7 \pm j2$

θ_2 is associated with the zero at −6.

The test point is $p_J = -5 + j4$.

To obtain the angles from poles such as α_1 we can calculate the complement of it, by first taking the tangent inverse of the ratio of the opposite side to the adjacent side, and then subtract it from 180°.

$$\alpha_1 = 180° - tan^{-1}\left(\frac{4-1}{(-3-(-5))}\right) = 180° - 56.31° = 123.69°$$

$$\alpha_2 = 180° - tan^{-1}\left(\frac{4-(-1)}{(-3-(-5))}\right) = 180° - 68.2° = 111.8°$$

$$\alpha_3 = 180° - tan^{-1}\left(\frac{4-0}{(4-(-5))}\right) = 180° - 23.97° = 156.03°$$

$$\alpha_4 = 180° - tan^{-1}\left(\frac{(4-(-5))}{-5-(-5)}\right) = 180° - 90° = 90°$$

$$\theta_1 = tan^{-1}\left(\frac{4-2}{(-5-(-7))}\right) = 56.31°$$

$$\theta_2 = tan^{-1}\left(\frac{4-(0)}{(-3-(-5))}\right) = 45°$$

$$\theta_3 = tan^{-1}\left(\frac{4-(-2)}{(-5-(-7))}\right) = 71.57°$$

$\angle G(s)H(s) = \theta_1 + \theta_2 + \theta_3 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 = 56.31° + 45° + 71.57° - 123.69° - 111.8° - 156.03° - 90° = -308.64°$

As can be seen the angle of open loop transfer function $G(s)H(s)$ at $s = p_J$ is not a multiple of 180°, and the angle condition is not satisfied. Therefore, p_J is not on the root locus of this system.

This means that no gain can be found to have this point as one of the closed loop poles of the system.

If the angle condition was satisfied, the next criterion to check would be the magnitude condition. Here is how to check this condition:

$$|G(s)H(s)| = \frac{product\ of\ vector\ lengths\ from\ \ p_J\ to\ zeros}{product\ of\ vector\ lengths\ from\ \ p_J\ to\ poles}$$

Length of vector from p_J to first pole = $\sqrt{(-3-(-5))^2+(4-1)^2} = 3.61$

Length of vector from p_J to second pole = $\sqrt{(-3-(-5))^2+(4-(-1))^2} = 5.39$

Length of vector from p_J to third pole = $\sqrt{(4-(-5))^2+(4-0)^2} = 9.85$

Length of vector from p_J to fourth pole = $\sqrt{(-5-(-5))^2+(4-(-5))^2} = 9$

Length of vector from p_J to first zero = $\sqrt{(-7-(-5))^2+(4-2)^2} = 2.83$

Length of vector from p_J to second zero = $\sqrt{(-6-(-5))^2+(4-0)^2} = 4.12$

Length of vector from p_J to third zero = $\sqrt{(-7-(-5))^2+(4-(-2))^2} = 6.32$

The above calculated lengths will be employed to calculate $|G(s)H(s)|$.

$$|G(s)H(s)| = \frac{2.83 \times 4.12 \times 6.32}{3.61 \times 5.39 \times 9.85 \times 9} = 0.043 \neq 1$$

Therefore, the magnitude condition is not satisfied.

Control Systems Analysis and Design by the Frequency-Response Method:

Example 1: Find the frequency response of a system with transfer function $G(s) = \frac{s+0.5}{s+10}$

To calculate $|G(j\omega)|$, the magnitude of the numerator of $G(j\omega)$ must be divided by the magnitude of the denominator of the $|G(j\omega)|$.

$$G(j\omega) = \frac{j\omega + 0.5}{j\omega + 10} = \frac{\sqrt{\omega^2 + 0.5^2}}{\sqrt{\omega^2 + 10^2}} = \frac{\sqrt{\omega^2 + 0.25}}{\sqrt{\omega^2 + 100}}$$

The angle of $|G(j\omega)|$ is the angle of the numerator of $G(j\omega)$ minus the angle of the denominator of the $|G(j\omega)|$.

$$\angle G(j\omega) = \ tan^{-1}\left(\frac{\omega}{0.5}\right) - tan^{-1}(\frac{\omega}{10})$$

For the input *cos(3t)*, $G(j\omega)$ is

$$G(j\omega) = \frac{\sqrt{\omega^2+0.25}}{\sqrt{\omega^2+100}} \angle(tan^{-1}\left(\frac{\omega}{0.5}\right) - tan^{-1}(\frac{\omega}{10}))$$

Since $\omega = 3$ $$G(j3) = \frac{\sqrt{3^2+0.25}}{\sqrt{3^2+100}} \angle(tan^{-1}\left(\frac{3}{0.5}\right) - tan^{-1}(\frac{3}{10})) = 0.30\angle 63.84°$$

The output of the system can be calculated as below:

$$y(t) = 0.30\cos(3t + 63.84°)$$

In calculating the output from the frequency response, the amplitude of the output is calculated by multiplying the amplitude of $G(j3)$ by the amplitude of the input which was 1. The phase of the output is the phase of $G(j3)$ plus the phase of input.

Control Systems Analysis and Design in State Space:

Example 1: Find Eigenvalues and eigenvectors of $\begin{bmatrix} 0 & 2 \\ -4 & -6 \end{bmatrix}$.

$$\left|\begin{bmatrix} 0 & 2 \\ -4 & -6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right| = 0$$

$$\left|\begin{bmatrix} -\lambda & 2 \\ -4 & -6-\lambda \end{bmatrix}\right| = \lambda^2 + 6\lambda + 8 = 0 \Rightarrow \lambda_1 = -2 \quad , \lambda_2 = -4$$

$$\lambda_1 x_1 = Ax_1 \quad \Rightarrow \quad (A - \lambda_1 I)\, x_1 = 0$$

$$\begin{bmatrix} 2 & 2 \\ -4 & -4 \end{bmatrix}\begin{bmatrix} x_{1,1} \\ x_{1,2} \end{bmatrix} = 0$$

$$2x_{1,1} + 2x_{1,2} = 0$$

$$-4x_{1,1} - 4x_{1,2} = 0$$

$$x_{1,1} = -x_{1,2} \Rightarrow x_1 = \begin{bmatrix} x_{1,1} \\ x_{1,2} \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda_2 x_2 = Ax_2 \quad \Rightarrow \quad (A - \lambda_2 I)\, x_2 = 0$$

$$\begin{bmatrix} 4 & 2 \\ -4 & -2 \end{bmatrix}\begin{bmatrix} x_{2,1} \\ x_{2,2} \end{bmatrix} = 0$$

$$4x_{2,1} + 2x_{2,2} = 0$$

$$-4x_{2,1} - 2x_{2,2} = 0$$

$$-2x_{2,1} = x_{2,2} \Rightarrow x_2 = \begin{bmatrix} x_{2,1} \\ x_{2,2} \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Example 2: Are the following set of vectors linearly independent? [6]

[3 -2 0 4], [5 0 0 1], [-6 1 0 1], [2 0 0 3] The determinant of the matrix formed from the vectors is 0.

$$\det\left(\begin{bmatrix} 3 & -2 & 0 & 4 \\ 5 & 0 & 0 & 1 \\ -6 & 1 & 0 & 1 \\ 2 & 0 & 0 & 3 \end{bmatrix}\right) = 0$$

Therefore, the vectors are not linearly independent, or they are linearly dependent.

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