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Undergraduate STEM Students' Comprehension of Function Series and Related Calculus Concepts

Dr. Emre Tokgoz, Quinnipiac University

Emre Tokgoz is currently the Director and an Associate Professor of Industrial Engineering at Quinnipiac University. He completed a Ph.D. in Mathematics and another Ph.D. in Industrial and Systems Engineering at the University of Oklahoma. His pedagogical research interest includes technology and calculus education of STEM majors. He worked on several IRB approved pedagogical studies to observe undergraduate and graduate mathematics and engineering students' calculus and technology knowledge since 2010. His other research interests include nonlinear optimization, financial engineering, facility allocation problem, vehicle routing problem, solar energy systems, machine learning, system design, network analysis, inventory systems, and Riemannian geometry.

Elif Naz Tekalp

My name is Elif Naz Tekalp. I am a junior industrial engineering student at Quinnipiac University. I also have a mathematics and general business minor. I am interested in the role of mathematics in engineering education and professional life. I was very passionate about the research that I participated in with Dr. Emre Tokgoz.

Hasan Alp Tekalp

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[1]Emre Tokgöz, [1]Berrak S. Tekalp, [1]Elif. N. Tekalp, and [2]Hasan A. Tekalp

[1]Emre.Tokgoz@qu.edu, [1]Elif.Tekalp@qu.edu, [1]Berrak.Tekalp@qu.edu, [2]Hasan.Tekalp@qu.edu

[1]Industrial Engineering, School of Engineering, Quinnipiac University, Hamden, CT, 06518

[2]Mechanical Engineering, School of Engineering, Quinnipiac University, Hamden, CT, 06518

Action-Process-Object-Schema (APOS) is a constructivist methodology that relies on learners' ability to construct and reconstruct certain mental structures and organize them in schemas [1]. Taylor series expansion of functions have important applications in STEM including image processing and algorithm design; therefore, understanding and improving Taylor series mental construction of undergraduate engineering students are important in engineering and mathematics education. In this work, 20 undergraduate engineering students' responses to a set of power series questions are analyzed by using APOS theory. The research question used in [19] is used in this work for data collection from a different group of participants. Qualitative and quantitative analysis of the participants' video recorded oral interviews and written responses are executed. Similar to the results attained in [19], the analysis of the collected empirical data indicated a well-established knowledge of approximation, a poor center of Taylor series expansion of function conceptual knowledge, and a well-established knowledge of the meaning of infinity within infinite series concept. The results of this work can help mathematics and engineering educators to improve students' series knowledge and develop a successful teaching methodology.

1. Introduction

Series expansion of functions is frequently used as a part of teaching and research in STEM fields, therefore understanding and improving STEM students' series knowledge is important for improving STEM education. There is little attention given to pedagogical research on understanding STEM students' function series comprehension and their ability to solve related calculus problems [1,3,5,6,8-16]. The results of this work can help developing a successful teaching methodology of Taylor series after determining areas that can be used for improving learners ability to respond questions. The same research question is empirically evaluated in [19] to continue investigating undergraduate STEM students' ability to respond to the following set of power series questions:

Q. In a few sentences legibly answer each of the following questions (a) through (d).

a) Describe the difference, if any, that exists between e^x and $1 + \frac{x}{1!} + \frac{x^2}{2!}$

b) Describe the difference, if any, that exists between $e^1 + e^1 \frac{(x-1)}{1!} + e^1 \frac{(x-1)^2}{2!}$ and

$e^2 + e^2 \frac{(x-1)}{1!} + e^2 \frac{(x-1)^2}{2!}$

c) Describe the difference, if any that exists between the infinite series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ and the infinite series $\sum_{n=0}^{\infty} e^2 \frac{(x-2)^n}{n!}$

d) Describe the difference, if any that exists between finite series $\sum_{n=0}^{k} \frac{x^n}{n!}$ and the infinite series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

Empirical data is collected for evaluating STEM students' comprehension of the series concept that requires knowledge of several calculus sub-concepts. The data collection methodology of this research received Institutional Review Board (IRB) approval at a mid-sized university located at the Northeast side of the United States. Twenty research participants are compensated for their written and video recorded oral interview responses to the research question. The collected data is analyzed qualitatively and quantitatively by using the Action-Process-Object-Schema (APOS) theory that will be explained in the next section.

2. Collected Empirical Data & Nature of Research

APOS is a constructivist methodology that relies on learners' ability to construct and reconstruct certain mental structures and organizing them in schemas to solve mathematical problems. APOS theory is initiated with Piaget's theory of reflective abstraction [17] and got expanded to K16 mathematics education and RUME in recent years. It was applied in 1997 to mathematical topics for analyzing combined math knowledge of a student in a specific subject [1]. Action, process, object, and schema are the mental structures proposed as a part of the APOS theory to follow developmental stages of the learners. The main goal of this theory is to observe and categorize mental structures through observations of learners' mental mechanisms; it is important to understand the totality of knowledge and its' reflection in applications.
In the relevant APOS literature, learners' conceptual view of the function was studied in [3] by relying on Piaget's study of functions in 1977 in mathematics education used for forming action-process-object idea [18]. Conceptual construction of sub-concepts and schemas to be able to have a comprehensive understanding of the main concept was the main point of empirical APOS classification. Cooley, Trigueros and Baker derived results by using thematization of schema with the intent to expose those possible structures acquired at the most sophisticated stages of schema development [2,4]. Empirical data was collected in [2] to determine APOS classification of learners based on a calculus graphing problem. Please see [7] for a comprehensive coverage of the APOS theory.

The following APOS theory classification is used for analyzing the collected data in this work:

- Action: Participating students were expected to transform their mental knowledge of Taylor series and perform basic operations.
- Process: The individual was expected to reflect upon an action when the action was repeated and can make an internal mental construction called a process by which the individual can think of it as performing the same kind of action without an external support.
- Object: Realization of the process' totality and transformations can act on it as a result of individual's awareness...
- Schema: A linkage of collected actions, processes, objects, and other schemas help to form the participant to structure a framework by using general principles in individual's mind...

The following are the key elements of the APOS classification based on the detailed analysis of the research question:

- Demonstrate an ability to recognize the difference between the finite and infinite series and reflect this knowledge as the correct response to the research question's parts (a), (b) and (d).
- Recognize and reflect knowledge on the finite series approximation of the exponential function.

- Be able to explain the difference between different power series' centers for the same function's power series approximations.
- Recognize the connection between three interrelated concepts: power series, function corresponding to the power series, and the center of the power series expansion.

The next four sections are designed to analyze the following four parts of the research question:

- Taylor series comparison of finitely and infinitely many terms.
- Basic center concept of power series expansion on functions.
- Function comparison with Taylor series approximation.
- Center concept's association with approximation of Taylor series.

Seventh section contains the APOS classification of the participants' responses, and the last section is devoted to conclusion.

3. Taylor Series with Finitely and Infinitely many Terms

The difference between finitely and infinitely many Taylor series terms is one of the most important concepts that students might be expected to have a good mental construction by the educators due its simplistic structure. This section is designated to the analysis of the participants' comprehension of the difference between the finite and infinite series representation of the exponential function. 85% of the participants were able to explain the difference between the finite and infinite series that is provided them in part (d). These participants were able to "act" upon the basic operations and build mental structures at the Action level of conceptual understanding. The following two responses are basic representative examples of the participants that are classified at the Action level of APOS for the research findings.

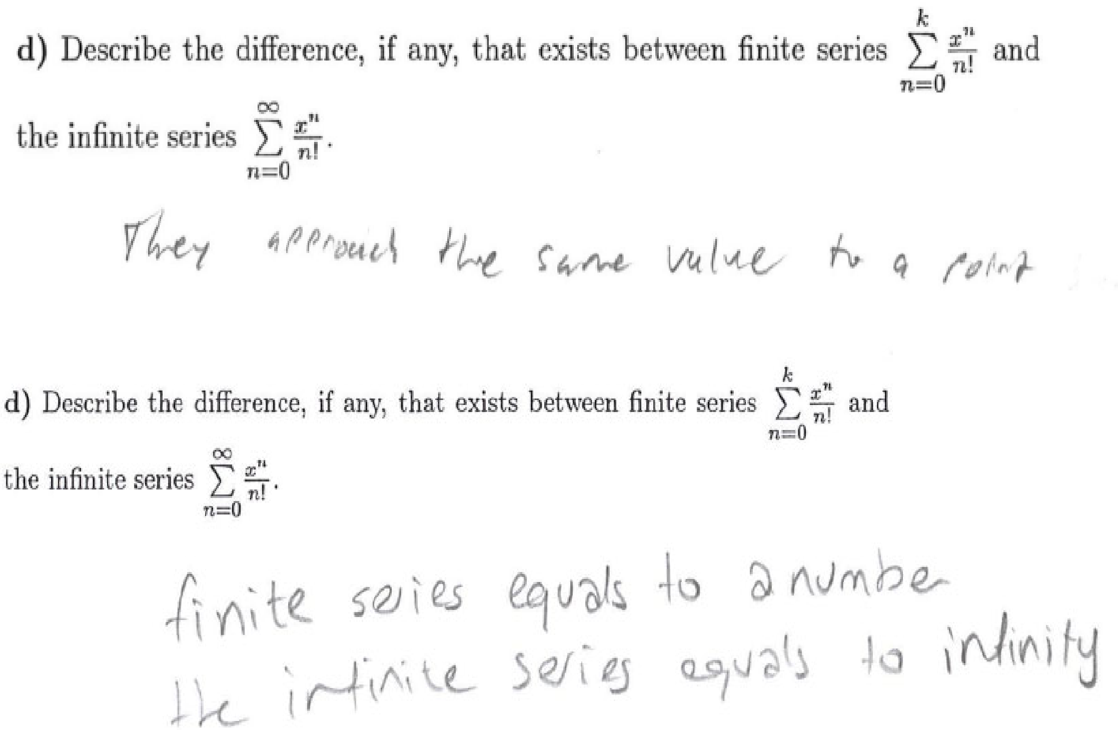
d) Describe the difference, if any, that exists between finite series $\sum_{n=0}^{k} \frac{x^n}{n!}$ and the infinite series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. The diffence is ~~an~~ a ~~function~~ series will carry out and never stop, the ~~first~~ finite series will stop when it carries out K values. The trends will look the same and follow the same path until one stops and the other continues

d) Describe the difference, if any, that exists between finite series $\sum_{n=0}^{k} \frac{x^n}{n!}$ and the infinite series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

There is, one will stop at a certain limit and other goes till infinity

Figure 1. Responses of participants 4 (on top) and 17 (at the bottom).

Only 15% of the participants had hard time to compare finite and infinite series. In Figure 2, two examples of such participants' responses that are not qualified to be in the Action level are provided.



d) Describe the difference, if any, that exists between finite series $\sum_{n=0}^{k} \frac{x^n}{n!}$ and the infinite series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

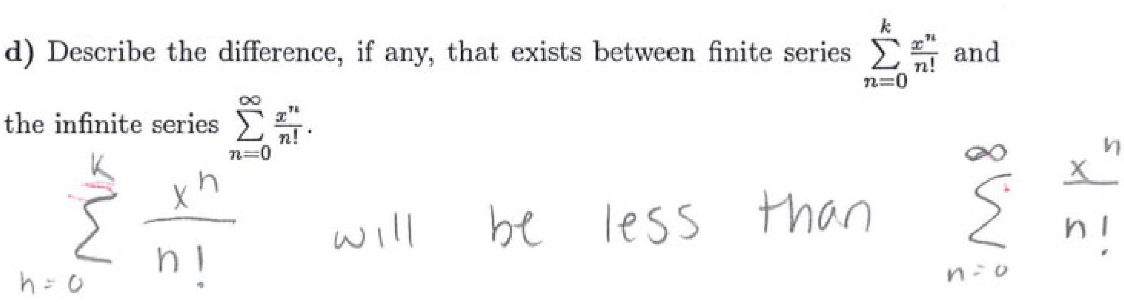
They approach the same value to a point

d) Describe the difference, if any, that exists between finite series $\sum_{n=0}^{k} \frac{x^n}{n!}$ and the infinite series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

finite seies equals to a numbe
the infinite series equals to infinity.

Figure 2. Responses of participants 15 (on top) and 18 (at the bottom) to part (d) of the question.

One of the research participants, Participant 2, with the response in the following figure, had a literal numerical explanation of the difference between the given finite and infinite series. Comparative answers of this nature are given by some of the other research participants for other parts of the question (such as Participant 6 explaining parts (a)-(c) of the research question with numerical calculations.)



d) Describe the difference, if any, that exists between finite series $\sum_{n=0}^{k} \frac{x^n}{n!}$ and the infinite series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

$\sum_{h=0}^{k} \frac{x^h}{h!}$ will be less than $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

Figure 3. Response of participant 2 to part (d) of the research question.

4. Function Comparison with Taylor Series Approximation

Comparison of a closed form function with its Taylor series expansion require a good understanding of the function and the corresponding Taylor series terms. The difference can be realized in several different ways including algebraically and geometrically, or a combination of these two representations. The analysis goal of the research question is to not only explore the levels of students' comprehension of the concept but also to observe justifications of their written responses. The rest of this section is reserved to display the qualitative and quantitative analysis of the participants' responses to part (a) of the research question.

Figure 4 displays the response of participant 3 to the research question with the justification of "no difference" between the exponential function and its' finite series approximation; 31.25% of the research participants who responded to this question had the same justification.

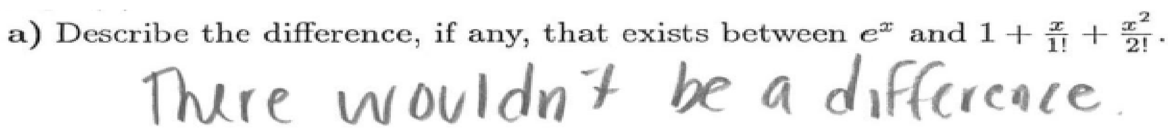


Figure 4. Response of participant 3 to part (a) of the question.

The answer of participant 5 to part (a) of the question is strictly numerical with function values calculated to show the difference between the function and the finite series approximation of the function. This participant is the only person (6.25%) who tried to provide a numerical justification by choosing numerical values.

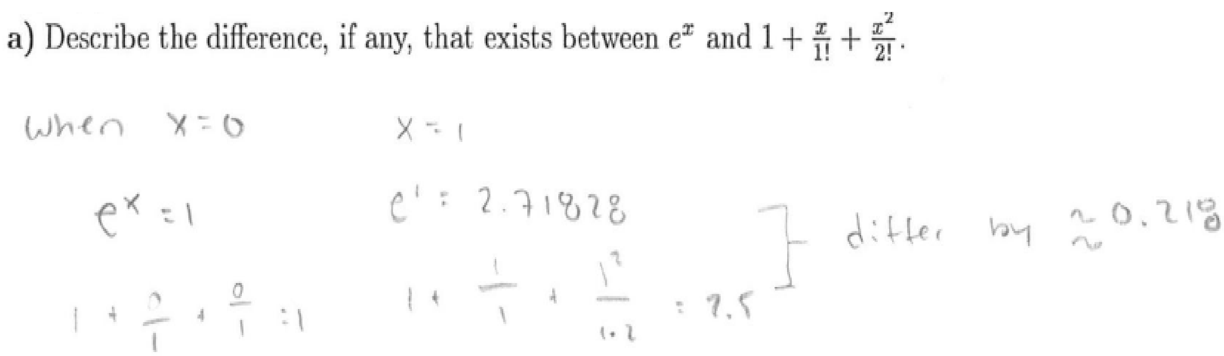


Figure 5. Numerical calculations of participant 5 to part (a).

Participant 12 stated "series would continue forever" to explain the difference between the finite approximation of the exponential function from series standpoint. Participant 20 also had a similar response to explain the difference.

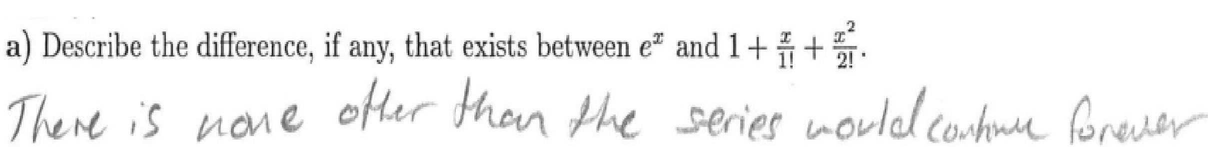
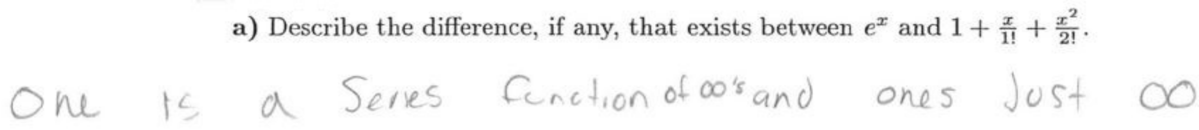


Figure 6. Response of participant 12 to part (a) of the question.

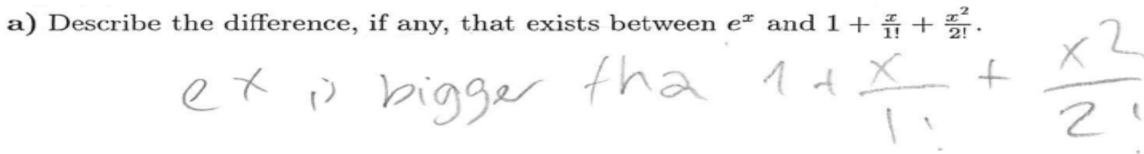
It is natural to expect engineering undergraduate students to use an appropriate mathematical language to be able to express their ideas. Participant 16 had the most confusing response among all the participants (displayed in Figure 7). The main idea behind this response was that e^x would have infinity as the value as x approaches infinity while the approximation function would have several infinity values due to having x and x^2 terms. This answer served as an outlier within the participant responses from a conceptual understanding perspective.


a) Describe the difference, if any, that exists between e^x and $1+\frac{x}{1!}+\frac{x^2}{2!}$.

One is a Series function of ∞'s and ones Just ∞

Figure 7. Response of participant 16 that displays improvement on the conceptual understanding.

Inequality comparison is done in Figure 8 below by only one participant by stating that the exponential function is "bigger than" its finite series approximation.


a) Describe the difference, if any, that exists between e^x and $1+\frac{x}{1!}+\frac{x^2}{2!}$.

e^x is bigger tha $1+\frac{x}{1!}+\frac{x^2}{2!}$

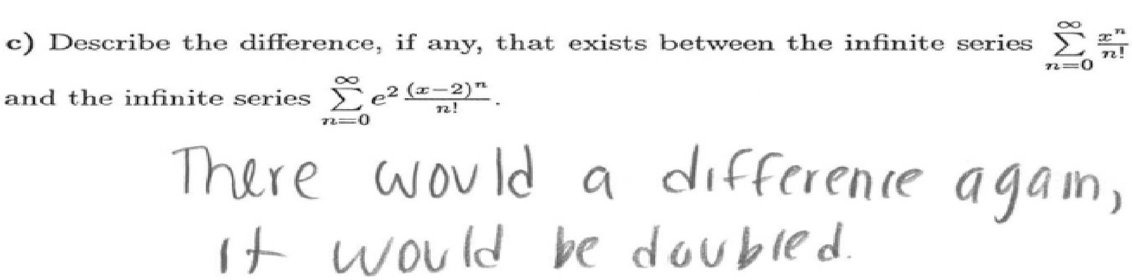
Figure 8. An inequality comparison of participant 18 to part (a) of the research question.

In conclusion, 62.5% of the students that responded to part (a) of the research question demonstrated a correct understanding in their responses. These participants' responses helped with further APOS classification as a part of the exponential function's approximation; particularly in observing whether the participants could improve their Action level classification to Process level classification by answering (a) correct. Further analysis and numerical results will be explained in Section 7.

5. Center concept for power series expansion of functions

Center concept taking place in the power series expansion of functions is one of the important and somehow complicated concepts that students have hard time to construct a good mental conceptualization. This concept is particularly helpful to observe students' advanced conceptualization of the power series expansion of functions and therefore APOS classification. Participants' ability to recognize the difference between the given two power series expansion of the exponential function is one of the key points of the schema classification. Analysis of the responses to part (c) of the question is presented in this section.

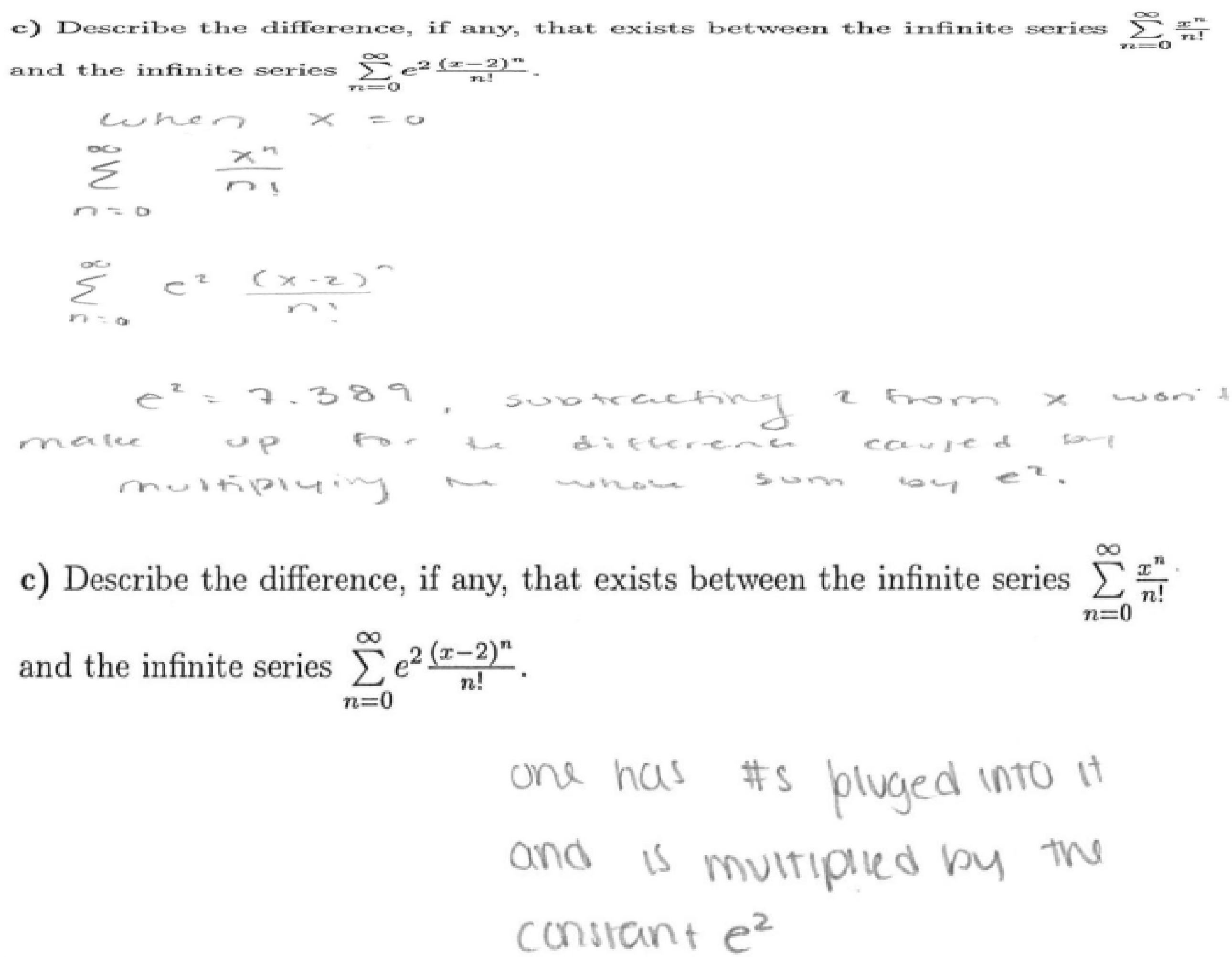
Only 33.33% of participants responded had the correct response to this part of the research question; some of the responses of the remaining participants are shared below. For instance, participant 3 explained the difference between the two infinite series as "doubled".


c) Describe the difference, if any, that exists between the infinite series $\sum_{n=0}^{\infty}\frac{x^n}{n!}$ and the infinite series $\sum_{n=0}^{\infty}e^2\frac{(x-2)^n}{n!}$.

There would a difference again, it would be doubled.

Figure 9. Response of participant 3 to part (c) of the research question.

Some of the participants, such as participants 5 and 6, with written responses shown in Figure 10 tried to justify the difference based on numerical value differences on the series.



c) Describe the difference, if any, that exists between the infinite series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ and the infinite series $\sum_{n=0}^{\infty} e^2 \frac{(x-2)^n}{n!}$.

when $x = 0$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sum_{n=0}^{\infty} e^2 \frac{(x-2)^n}{n!}$$

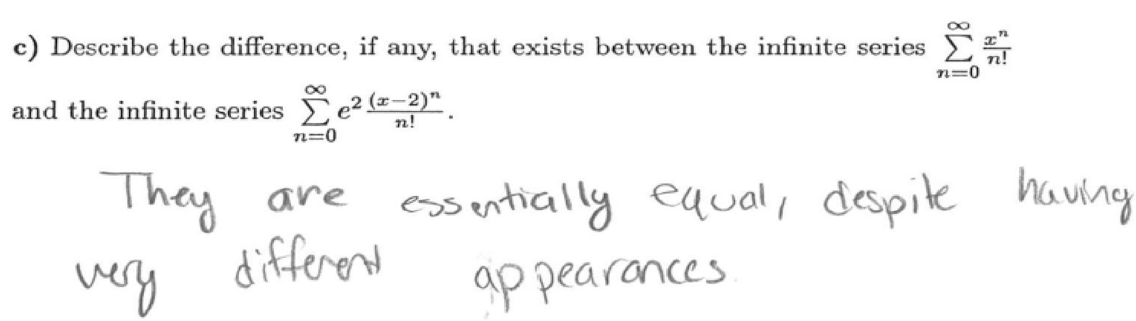
$e^2 = 7.389$, subtracting 2 from x won't make up for the difference caused by multiplying the whole sum by e^2.

c) Describe the difference, if any, that exists between the infinite series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ and the infinite series $\sum_{n=0}^{\infty} e^2 \frac{(x-2)^n}{n!}$.

one has #s pluged into it and is multiplied by the constant e^2

Figure 10. Responses of participants 5 (on top) and 6 (at the bottom) explaining difference numerically.

One of the participants, participant 10 with the response displayed in Figure 11, explained the similarity from "appearance" perspective without considering the actual meaning of the center concept as a part of the series.

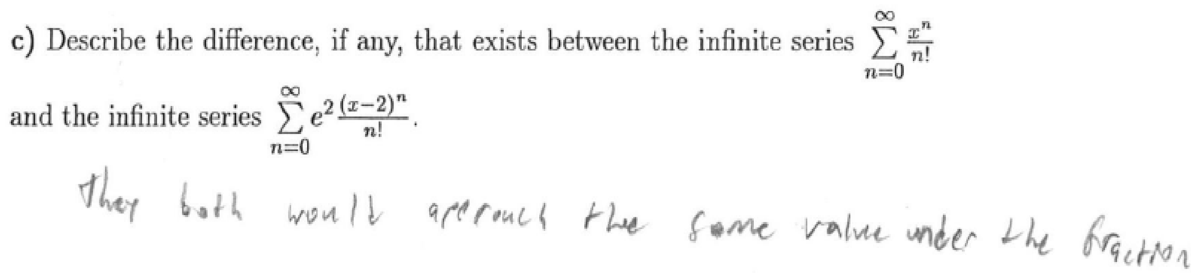


c) Describe the difference, if any, that exists between the infinite series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ and the infinite series $\sum_{n=0}^{\infty} e^2 \frac{(x-2)^n}{n!}$.

They are essentially equal, despite having very different appearances

Figure 11. Participant 10 stated that the series are the same "despite having different appearances".

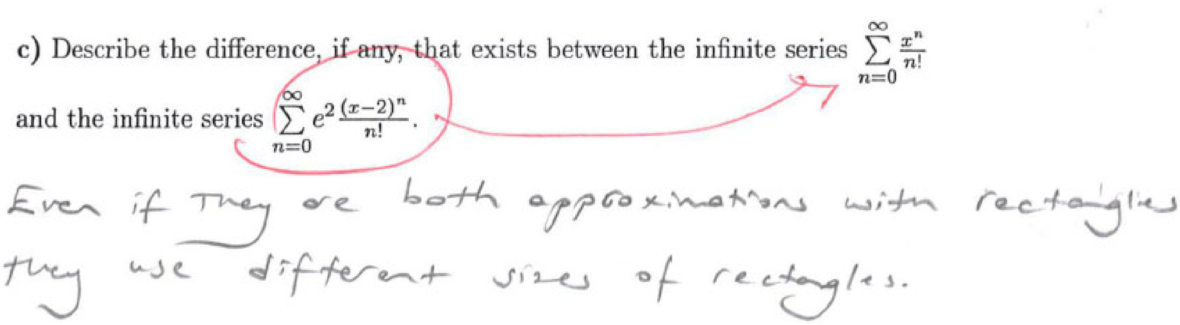
Participant 15 with the response displayed in Figure 12 tried to explain the similarity between the two series from a limiting value perspective without any attention paid on the center concept.


c) Describe the difference, if any, that exists between the infinite series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ and the infinite series $\sum_{n=0}^{\infty} e^2 \frac{(x-2)^n}{n!}$.

They both would approach the same value under the fraction

Figure 12. Participant 15 used the term “fraction” to explain the similarity between the series.

Only one of the participants, participant 19, tried to explain the difference between the given two series from “rectangle approximation” perspective in Figure 13 below. This is an outlier response in the set of responses of the participants to the research question.


c) Describe the difference, if any, that exists between the infinite series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ and the infinite series $\sum_{n=0}^{\infty} e^2 \frac{(x-2)^n}{n!}$.

Even if They are both approximations with rectangles they use different sizes of rectangles.

Figure 13. Answer of participant 19 part (c) of the research question.

In conclusion, one thirds of the participants’ responses to part (c) supported the correct answer to the research question that helped with further APOS classification on participants “center of series” understanding; the correct responses to this question are incorporated to other correct responses for APOS classification.

6. Center Concept & Approximation of Taylor Series

The schema classification of the participants is mainly driven by the responses given to part (b) of the research question. This question contained the combination of the center and approximation concepts of power series expansion of functions. The correct answers given to this question indicated the students’ Object level comprehension. Qualitative responses and quantitative analysis to part (b) are going to be presented throughout this section.

“The limits” of the two series are determined to be different by Participant 1 in the response presented in Figure 13 due to the differences on the exponents 1 and 2; these exponents are stated to give different results “most of the time.”

b) Describe the difference, if any, that exists between $e^1 + e^1\frac{(x-1)}{1!} + e^1\frac{(x-1)^2}{2!}$ and $e^2 + e^2\frac{(x-2)}{1!} + e^2\frac{(x-2)^2}{2!}$.

The limits are using different exponents which will give different answers most of the time

Figure 13. Participant 1 explains the difference as the difference between "exponents".

Participant 4 stated the difference as "one would be constantly ahead of the second" in Figure 14 from geometric perspective. This participant recognized the similarity between the two functions derivation from the exponential function's perspective and explained their similarity due to this commonality.

b) Describe the difference, if any, that exists between $e^1 + e^1\frac{(x-1)}{1!} + e^1\frac{(x-1)^2}{2!}$ and $e^2 + e^2\frac{(x-2)}{1!} + e^2\frac{(x-2)^2}{2!}$.

The difference is, the value the we get from the first one would be constantly ahead of the second but it would evidently show the same trend because they are both e^x.

Figure 14. Response of Participant 4 with the explanation linked to the exponential function.

Participants 7 and 20 with the responses displayed in Figure 15 are two of the participants among all that had a response to the question and justified the difference between the two-finite series approximation of the exponential function geometrically for different centers.

b) Describe the difference, if any, that exists between $e^1 + e^1\frac{(x-1)}{1!} + e^1\frac{(x-1)^2}{2!}$ and $e^2 + e^2\frac{(x-2)}{1!} + e^2\frac{(x-2)^2}{2!}$.

the first one is a parabola but the second one is a smaller parabola

b) Describe the difference, if any, that exists between $e^1 + e^1\frac{(x-1)}{1!} + e^1\frac{(x-1)^2}{2!}$ and $e^2 + e^2\frac{(x-2)}{1!} + e^2\frac{(x-2)^2}{2!}$.

This set is shifted one unit to the right

Figure 15. Responses of participants 7 (on top) and 20 (at the bottom) with geometric justification.

Figure 16 contains the response of participant 19. This is the only person who tried to derive one of the finite series from the other. The difference is explained as e^1 and the slope values of the functions are stated to change as the function changes.

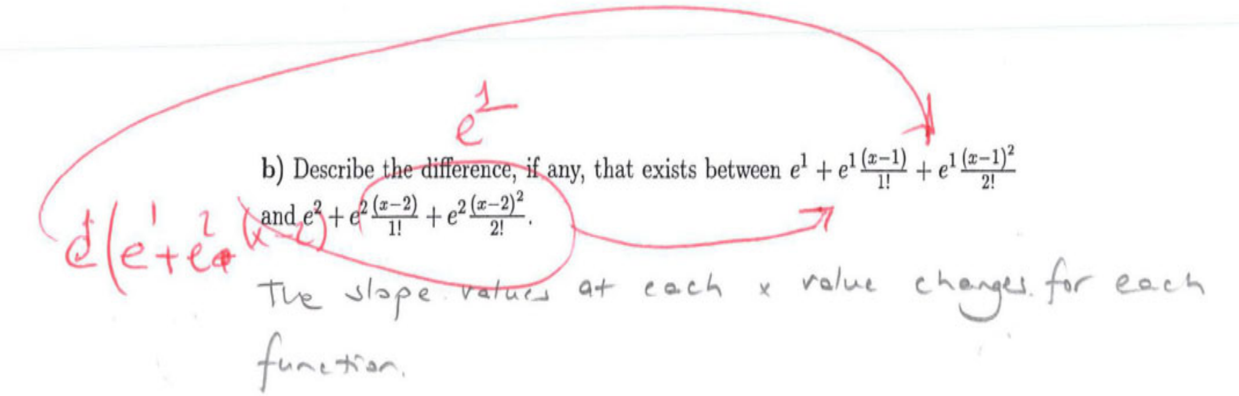


Figure 16. Response of participant 19 with explanation depending on slope values.

In conclusion, only 16.67% of the participants has the responses closest to the correct response to part (c) of the research question. The responses to part (b) are critical in schema classification for both “center” and “approximation” understanding of the participants; further analysis and numerical results are explained in the next section.

7. APOS Classification & Analysis of the Data

The participants’ APOS classification outlined in this section is based on the following APOS specifications introduced in Section 2.

- **Action** categorization is based on the participants’ ability to recognize the differences between finite and infinite series and be able to act on this knowledge to provide correct responses. 85% of the participating students had mental construction of basic Taylor series knowledge and performed basic operations related to power series approximation. Majority of the research participants are not able to act upon the center concept of series in their responses. Only 3 participants (15%) could make a connection between the center concept and power series in their responses.
- **Process** classification is 40% of the individuals that could reflect their knowledge on different parts of the research question for explaining the difference between the finite and infinite series. This percentage of success is lowered to 15% for the participants by making it a process for the “center” concept of the series. These students can repeat the process of solution and could make internal mental constructions by performing the same kind of action without an external support.
- **Object** classification is only 15% of the participants that could explain the difference between the finite and infinite series as well as different centers of the same series by realizing the totality of the corresponding

processes. The participants ability to respond to all parts of the research question is the key point of this classification.

- **Schema** classification is 5% of the participants who could link all collected actions, processes, objects, and schemas to form the correct framework by using general principles in the conceptualization of power series. Students that had the mental construction strength to relate power series, function correspondence to the power series, and the center concept of the function reflected to the power series expansion of the function are classified to be in this category.

Correlation analysis of participant responses to parts (a)-(d) are also performed. The following response methods (per responded displayed in each bullet point) depended on the mathematical keywords and the response method used by the participants depending on their cognitive approaches.

- Correlated responses with justification depending on the term "end" of the series.
- Use of the following terms: "doubled", "parabola", "same", "approach", "infinity".
- Numerical calculations for justification.
- Answers based on the algebraic appearance of the functions.
- Use of the term "bigger" and comparison of limiting values by using the term "infinity”.
- Comparison by using "rectangle" approximation for integrals or using the exponential function.
- Geometric perspective by using the term "shifted".

A correlation rate of 65% is determined for following a similar mathematical justification (i.e. specific mathematical keywords on their personal responses (not a comparison with other participants) to parts (a)-(d) of the research question.

In this work, STEM students’ mental construction of power series and its sub-concepts that are particularly important in real life applications are observed. The research question designed for empirical data collection allowed the research team to be able to analyze and interpret APOS classification of the students. The analysis of this classification showed students needing to recall concepts through external guidance. For instance, in an exam, a question with its solution given to students in a virtual environment can help students to recall the corresponding concept while the second question without any solution can be used for assessing students’ ability to demonstrate conceptual understanding on the same concept. This approach could be designed in a different way that would make the first question a bonus question: Students who can answer the first question without seeing the solution can get extra credit for the solution and those who do not know how to answer the first question can first see the solution before moving onto the second question and do not lose any points on the first question. Another method can be designing questions that the learners can apply their knowledge on multiple sub-concepts; A set of questions (similar to the research question covered in this work) can be used for teaching and evaluating learners that can include a sub-concept or multiple sub-concepts in a progressing fashion.

Additionally, from teaching perspective, assuming learners know basic concepts taking place in the power series expansion of functions (i.e., all the pre-requisite information) does not appear as a good strategy for teaching the corresponding concepts; Allowing the learners to build knowledge on their mentally constructed subjects by using pre-recorded videos on such concepts might be the way to teach them more advanced concepts. Time is limited for educators to cover all the concepts in the classroom if fundamental concepts taking place in the power series are also covered in a course, therefore flipping the course or a part of the course might be the way to re-develop and teach power series related courses. Other researchers are invited to design similar questions to the research question used in this work and evaluate their students in a similar structure used in this work.

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