Use of *Mathcad* in Computing Beam Deflection by Conjugate Beam Method

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Abstract

The four-year, ABET-accredited Civil Engineering Technology curriculum at Georgia Southern University includes a required, junior-level course in Structural Analysis. One of the topics covered is the conjugate beam method for computing slope and deflection at various points in a beam. The conjugate beam method is a geometric method and it relies only on the principles of statics. The usefulness of this method lies in its simplicity. The students can utilize their already acquired knowledge of shearing force and bending moment to determine a beam's slope and deflection.

An approach to teaching this important method of structural analysis that complements the traditional lecturing through inclusion of a powerful, versatile and user-friendly computational tool, is discussed in this paper. Students will learn how to utilize *Mathcad* to perform a variety of calculations in a sequence and to verify the accuracy of their manual solutions. A *Mathcad* program is developed for this purpose and examples to illustrate the computer program are also included in this paper. The integration of *Mathcad* will enhance students' problem-solving skills, as it will allow them to focus on analysis while the software performs routine calculations. Thus it will promote learning by discovery, instead of leaving the student in the role of a passive observer.

Introduction

With the objective of enhanced student learning, adoption of various instructional technology and inclusion of computer-aided problem-solving modules into the curriculum has been a trend for civil engineering and civil engineering technology programs. More specifically, the effective incorporation of a variety of software packages for the teaching-learning process related to the structural analysis course has been addressed in several articles[1,2,3,4,5,6,7,8] in recent years. Analysis of both statically determinate and statically indeterminate structures, by classical methods (slope-deflection and moment distribution) and stiffness method, using EXCEL, MATLAB and *Mathcad*[9], have been covered in those articles. However, one very important and useful method, the conjugate beam method, was not addressed. The purpose of this paper is to present a simple and effective approach used by the author to teach this important topic of structural analysis incorporating the use of Mathcad software.

Conjugate Beam Method

Structures deform when subjected to loads, and a vast majority of structures undergo elastic deformations only, under service loads. For linear elastic behavior, the Principle of Superposition remains valid. Thus load effects (slope, deflection etc.) due to different types of loads can be combined to obtain the final results.

The conjugate beam method is based on consideration of the geometry of the deflected shape of a beam. A conjugate beam is a fictitious beam of the same length as the actual beam, but its supports (as well as internal connections) are such that if the conjugate beam is loaded with the *M/EI* diagram of the real beam, the shearing force and bending moment at any point on the conjugate beam are equal, respectively, to the slope (θ) and deflection (Δ) at that point of the real beam. *M* is the bending moment and *EI* represents the flexural rigidity of the beam, where *E* is the modulus of elasticity of beam material and *I* is the moment of inertia of beam cross-section. The basis of the method is that the relations among load, shear and bending moment in a beam are *similar* to the corresponding relations among *M/EI,* slope and deflection of the beam. The application of the laws of equilibrium on a differential element of a beam leads to a pair of equations relating the load, shear and moment. Likewise, integration of the governing differential equation of elastic beam theory, expressing the moment-curvature relationship at a point, leads to a pair of equations relating *M/EI,* slope and deflection. These derivations can be found in any standard textbook on structural analysis[10,11,12].

Advantages of *Mathcad*

Mathcad, an industry-standard calculation software, is used because it is as versatile and powerful as programming languages, yet it is as easy to learn as a spreadsheet. Additionally, it is linked to the Internet and other applications one uses everyday.

In *Mathcad*, an expression or an equation looks the same way as one would see it in a textbook, and there is no difficult syntax to learn. Aside from looking the usual way, the expressions can be evaluated or the equations can be used to solve just about any mathematics problem one can think of. Text can be placed anywhere around the equations to document one's work. *Mathcad*'s two- and three-dimensional plots can be used to represent equations graphically. In addition, graphics taken from another Windows application can also be used for illustration purpose. *Mathcad* incorporates Microsoft's OLE 2 object linking and embedding standard to work with other applications. Through a combination of equations, text, and graphics in a single worksheet, keeping track of the most complex calculations becomes easy. An actual record of one's work is obtained by printing the worksheet exactly as it appears on the screen.

Program Features

The program developed by the author will require input data pertaining to the geometry of the problem, material property and the loading. More specifically, the following information is required as input data: beam type (simply-supported, simply-supported with overhang and cantilever), length, moment of inertia and modulus of elasticity, magnitudes and lengths of distributed loads, and magnitudes and locations of concentrated loads. Based on the input data, calculations are carried out in the following steps for *each* concentrated load and *each* distributed load:

1. Calculate the support reactions for the real beam.
2. Calculate the moment at equal intervals along the length of the beam.
3. Calculate the *M/EI* at those points.
4. Choose the appropriate supports and internal connections, if any, for the conjugate beam
5. Calculate the support reactions for the conjugate beam.
6. Calculate the conjugate beam shear (i.e. slope of the real beam) at equal intervals throughout the length of the beam.
7. Calculate the conjugate beam moment (i.e. deflection of the real beam) at equal intervals throughout the length of the beam.

Final results of slopes and deflections are obtained by superposing the results of steps 6 and 7, respectively**.**

Student Assignment and Assessment of Performance

The author provided an abridged version of the program (limited to only simply-supported beam with no overhang, i.e., Case 1) to his class. A list of variables used in the program and the program code are given in the Appendix. The students were asked to modify the program such that problems on simply-supported beam with overhang (Case 2) and cantilever beam (Case 3) can be solved. Since it was a relatively small class (only 11 students), the class was divided into two groups – 6 in one and 5 in the other. Each group was further subdivided into two subgroups (of 2 or 3 students) to work on Case 2 and Case 3. After the modifications were done, each group had to validate their program using two problems of known solutions.

The group performance was assessed on the basis of three parameters: clarity, efficiency and length of program codes, and each group was assigned a grade based on performance. In addition, a quiz was given to the class to test their knowledge of Mathcad programming and this provided a measure of individual accountability. Thus the grade of each student for this exercise was an weighted average of group performance grade and an individual quiz grade.

Example Problems

Three example problems (representing Case 1, Case 2 and Case 3) with their solutions obtained by using the program are given below. For any of these problems, one or more input data change would translate to change in the slope and deflection of the beam. Any number of combinations of input data is possible and students can see the effects of these changes instantaneously. Moreover, with further additions to the program, it would be feasible to include other types of loads (e.g., ramp load).

Example 1: Determine the slopes at ends A and D and the deflections at points B and C of the beam shown in Figure 1. Use E = 1,800 ksi and I = 46,000 in^4. (Reference11, Example 6.7)

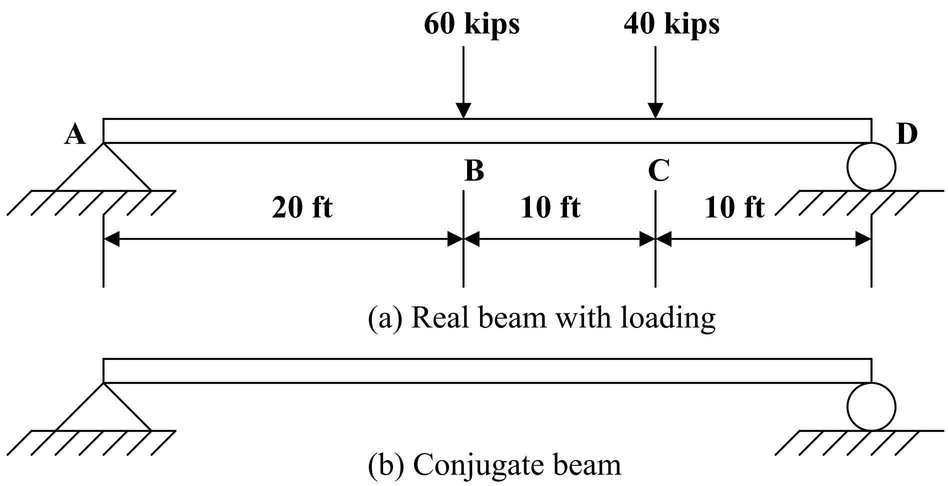


Figure 1. Beam of Example 1

Input Data (Example 1):

Beam support type: Case := 1

Beam span: $L := 40 \cdot ft$ Length of overhang : $L' := 0 \cdot ft$

Modulus of elasticity: $E := 1800000 \cdot \frac{lb}{in^2}$ Moment of inertia: $I := 46000 \cdot in^4$

Concentrated loads:

Number of concentrated loads: $n := 2$

Magnitudes: $P := (60000\ \ 40000\ \ 0\ \ 0)^T \cdot lb$ Locations: $a := (20\ \ 30\ \ 0\ \ 0)^T \cdot ft$

Uniformly distributed load:

Magnitude: $w := 0 \cdot \frac{lb}{ft}$

Location: $d1 := 0 \cdot ft$ $d2 := 0 \cdot ft$

Number of divisions: $div := 80$

Solution (Example 1):

$x_j =$ (ft)

0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6
6.5
7
7.5
8

$\theta =$

	1
1	-0.015
2	-0.015
3	-0.015
4	-0.015
5	-0.015
6	-0.015
7	-0.014
8	-0.014
9	-0.014
10	-0.014
11	-0.014
12	-0.014
13	-0.014
14	-0.013
15	-0.013
16	-0.013
17	-0.013

$\Delta =$

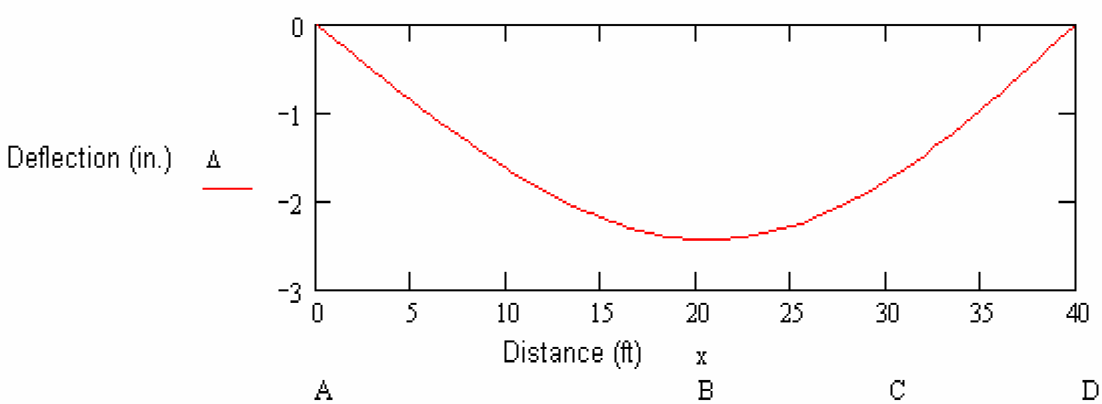
	1
41	-2.435
42	-2.438
43	-2.437
44	-2.432
45	-2.423
46	-2.41
47	-2.393
48	-2.372
49	-2.347
50	-2.319
51	-2.287
52	-2.251
53	-2.212
54	-2.169
55	-2.123
56	-2.073
57	-2.02

$\theta_A = -0.015$ rad

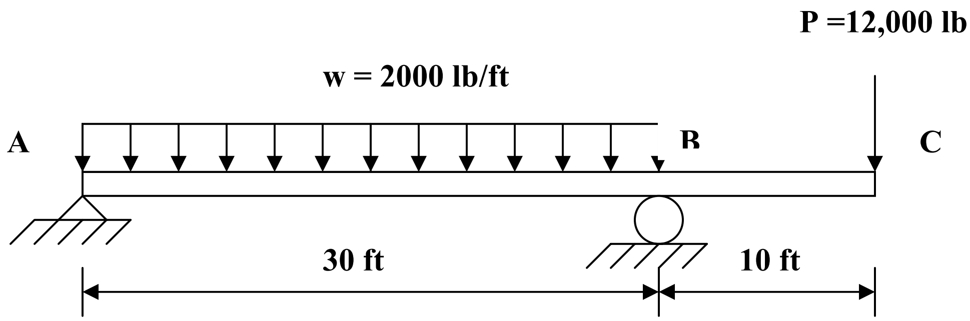
$\theta_D = 0.017$ rad.

$\Delta_B = -2.435$ in.

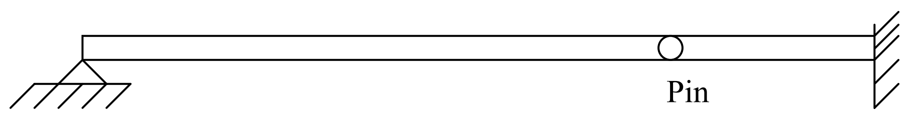
$\Delta_C = -1.774$ in.



Example 2: Determine the deflection at point C of the beam shown in Figure3. Use E = 29,000 ksi and I = 2000 in^4.(Reference 11, Example 6.10)



(a) Real Beam with Loading



(b) Conjugate Beam

Figure 2. Beam of Example 2

Input Data (Example 2):

Beam support type: Case := 2

Beam span: $L := 30 \cdot ft$ Length of overhang : $L' := 10 \cdot ft$

Modulus of elasticity: $E := 29000000 \frac{lb}{in^2}$ Moment of inertia: $I := 2000 \cdot in^4$

Concentrated loads:

Number of concentrated loads: $n := 1$

Magnitudes: $P := (12000\ \ 0\ \ 0\ \ 0)^T \cdot lb$ Locations: $a := (40\ \ 0\ \ 0\ \ 0)^T \cdot ft$

Uniformly distributed load:

Magnitude: $w := 2000 \cdot \frac{lb}{ft}$

Location: $d1 := 0 \cdot ft$ $d2 := 30 \cdot ft$

Number of divisions: $div := 80$

Solution (Example 2):

x_j = (ft)

32
32.5
33
33.5
34
34.5
35
35.5
36
36.5
37
37.5
38
38.5
39
39.5
40

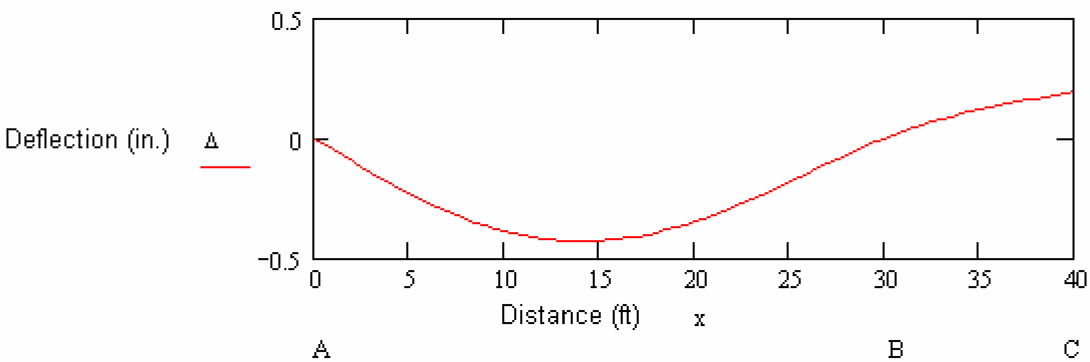
θ =

	1
65	$2.071 \cdot 10^{-3}$
66	$1.955 \cdot 10^{-3}$
67	$1.847 \cdot 10^{-3}$
68	$1.747 \cdot 10^{-3}$
69	$1.654 \cdot 10^{-3}$
70	$1.568 \cdot 10^{-3}$
71	$1.49 \cdot 10^{-3}$
72	$1.419 \cdot 10^{-3}$
73	$1.356 \cdot 10^{-3}$
74	$1.3 \cdot 10^{-3}$
75	$1.251 \cdot 10^{-3}$
76	$1.21 \cdot 10^{-3}$
77	$1.177 \cdot 10^{-3}$
78	$1.151 \cdot 10^{-3}$
79	$1.132 \cdot 10^{-3}$
80	$1.121 \cdot 10^{-3}$
81	$1.117 \cdot 10^{-3}$

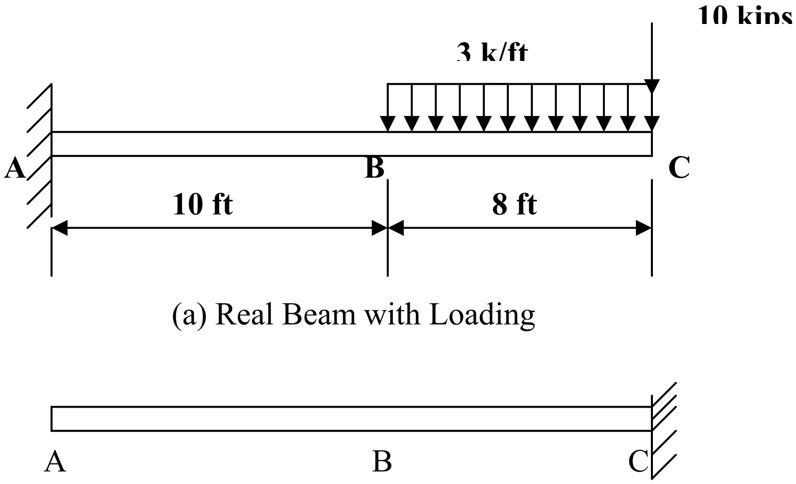
Δ =

	1
65	0.056
66	0.068
67	0.079
68	0.09
69	0.1
70	0.11
71	0.119
72	0.128
73	0.136
74	0.144
75	0.152
76	0.159
77	0.166
78	0.173
79	0.18
80	0.187
81	0.194

$\Delta_C = 0.194$ in.



Example 3: Determine the slope and deflection at B and C of the cantilever beam shown in Figure 4. Use E = 29,000,000 psi, and I = 4000 in^4. (Reference 12, Problem 10.6)



(a) Real Beam with Loading

(b) Conjugate Beam

Figure 3. Beam of Example 3

Input Data (Example 3):

Beam support type: $Case := 3$

Beam span: $L := 18 \cdot ft$ Length of overhang : $L' := 0 \cdot ft$

Modulus of elasticity: $E := 29000000 \cdot \frac{lb}{in^2}$ Moment of inertia: $I := 4000 \cdot in^4$

Concentrated loads:

Number of concentrated loads: $n := 1$

Magnitudes: $P := (10000 \ \ 0 \ \ 0 \ \ 0)^T \cdot lb$ Locations: $a := (18 \ \ 0 \ \ 0 \ \ 0)^T \cdot ft$

Uniformly distributed load:

Magnitude: $w := 3000 \cdot \frac{lb}{ft}$

Location: $d1 := 10 \cdot ft$ $d2 := 18 \cdot ft$

Number of divisions: $div := 36$

Solution (Example 3):

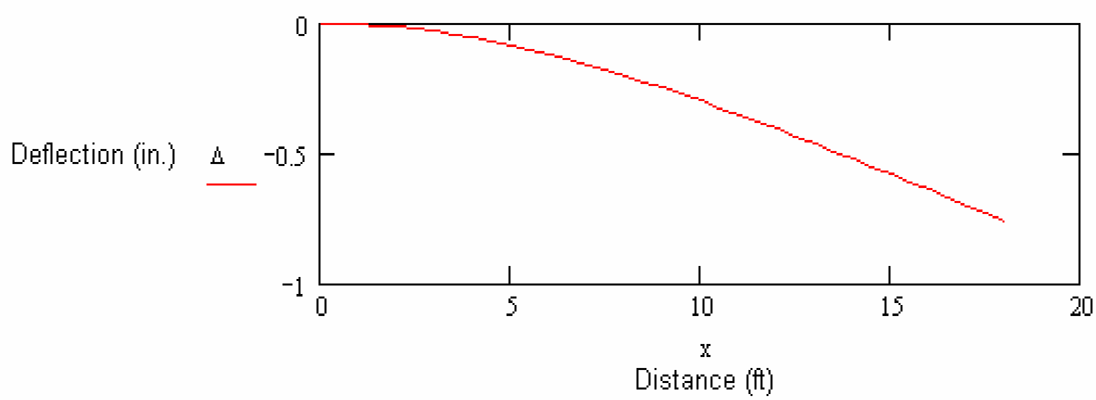
x_j = (ft)		θ =		Δ =
10	21	$-4.295\cdot10^{-3}$	21	-0.3
10.5	22	$-4.399\cdot10^{-3}$	22	-0.326
11	23	$-4.493\cdot10^{-3}$	23	-0.353
11.5	24	$-4.578\cdot10^{-3}$	24	-0.38
12	25	$-4.653\cdot10^{-3}$	25	-0.408
12.5	26	$-4.719\cdot10^{-3}$	26	-0.436
13	27	$-4.777\cdot10^{-3}$	27	-0.464
13.5	28	$-4.828\cdot10^{-3}$	28	-0.493
14	29	$-4.871\cdot10^{-3}$	29	-0.522
14.5	30	$-4.908\cdot10^{-3}$	30	-0.551
15	31	$-4.938\cdot10^{-3}$	31	-0.581
15.5	32	$-4.962\cdot10^{-3}$	32	-0.611
16	33	$-4.98\cdot10^{-3}$	33	-0.641
16.5	34	$-4.994\cdot10^{-3}$	34	-0.67
17	35	$-5.003\cdot10^{-3}$	35	-0.7
17.5	36	$-5.009\cdot10^{-3}$	36	-0.731
18	37	$-5.01\cdot10^{-3}$	37	-0.761

$\theta_B = -4.295 \times 10^{-3}$ rad

$\Delta_B = -0.3$ in.

$\theta_C = -5.01 \times 10^{-3}$ rad.

$\Delta_C = -0.761$ in.



Student Response

As mentioned before, the student assignments were group activities. The intent was to encourage cooperative learning. In general, students were quite receptive to the use of *Mathcad*, although they had no prior exposure to the software. The author had to familiarize the students with the essential features of *Mathcad*, before they were given the assignment. As part of the course, a two-hour-per-week computational laboratory makes it possible for the author to teach the basics of this software. Eleven students answered a survey which is summarized in Table 1.

Table 1. Summary of Student Surveys

Statement	Strongly Disagree	Disagree	Not Sure	Agree	Strongly Agree
The use of *Mathcad* for Conjugate Beam Method was worthwhile and should be continued.	1	2	2	4	2
The use of *Mathcad* helped me learn the topic and increased my problem-solving skills.	1	2	2	3	3
Learning to use *Mathcad* was difficult, time- consuming and/or frustrating.	1	3	1	3	3
The programming part made me think more about the concept behind the topic.	1	2	1	4	3
Mathcad should be incorporated into Structural Analysis course for other topics as well.	1	2	2	3	3

Also, responses to two open-ended questions are summarized below.

Question 1: What did you like the *most* about using *Mathcad* for this topic?

Answers: "Means to verify my solution right away," "Instant table and graph of solution," "Verifying principle of superposition," "Immediate solution for multiple loadings," "Pretty neat software, although learning the stuff took me a while," "Working with the program gave me a better understanding of the method," "That I could solve a problem with several loads immediately, which would take me for ever to solve by hand."

Question 2: What did you like the *least* about using *Mathcad* for this topic?

Answers: "Learning Mathcad," "Using different types of variables," "Too many rules," "Remembering different toolbars; often cause frustration," "Using different tools."

From the survey, it appears that majority of the students are in favor of using the software for this topic (as well as others), despite the learning curve associated with new software. They also have acknowledged enhanced learning. The higher test scores on this particular topic bear testimony of enhanced learning.

Although no specific feedback information as to teamwork experience was asked in the survey, the author plans to include a specific question on this matter next time. Informal inquiry with the students has, however, revealed a positive response from the students.

The author also plans to do the following as future measures, in order to make the idea of inclusion of *Mathcad* appeal to most students:

1. Introduce *Mathcad* to students early on, possibly in a basic mechanics course in their sophomore year, or even in their freshman year.
2. Use the software in the Structural Analysis course more extensively.
3. Use it in other Civil Engineering Technology courses as well.

Conclusions

For most part, the suggested approach to complement the traditional lecturing provides a better insight in the subject matter, in addition to making a convenient checking procedure readily available. The students can instantaneously solve complex problems involving multiple loading conditions and different support types, and also examine what-if scenarios by changing one or more parameters as input data (a manual solution for such a problem would be very tedious and time consuming). Also, the students acquire enhanced problem-solving skills, as they are engaged in, not just using the *Mathcad* software, but also in writing the programming code.

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Appendix

Mathcad Program for Beam Deflection by Conjugate Beam Method
Nirmal K. Das, Ph.D., P.E.

Sign Convention: Counterclockwise slopes are positive and upward deflections are positive. (NOTE: All applied loads are considered to be acting vertically downward.)

Input Variables:

Case	support types: 1 = simply-supported 2 = simply-supported with overhang 3 = cantilever
L	beam span
L'	length of overhang
E	modulus of elasticity of the beam material
I	moment of inertia of beam cross-section
n	number of concentrated loads
P_i	i-th concentrated load
a_i	distance to i-th concentrated load, from the left end
w	uniformly distributed load
d1	distance to the beginning of u.d.l., from the left end
d2	distance to the end of u.d.l., from the left end
div	number of divisions in the beam length for plots of slope/deflection

Output Variables:

θ	slope at a distance x from the left end of the beam
Δ	deflection at a distance x from the left end of the beam

1. Provide plotting information

Number of points: $pts := div + 1$

Interval between points: $int := \frac{L + L'}{div}$

2. Determine slope and deflection due to concentrated loads:

$k := E \cdot I$

$k = 5.75 \times 10^{8}\ lb\,ft^{2}$

$i := 1..n$

$b_i := L - a_i$

Left support reaction of real beam: $Ay_i := \frac{P_i \cdot b_i}{L}$

Right support reaction of real beam: $By_i := \frac{P_i \cdot a_i}{L}$

Maximum load on conjugate beam: $c_i := \frac{Ay_i \cdot a_i}{k}$

Resultant of left triangular load on conjugate beam: $R1p_i := \frac{c_i \cdot a_i}{2}$

Resultant of right triangular load on conjugate beam: $R2p_i := \frac{c_i \cdot b_i}{2}$

Left support reaction of conjugate beam: $Apy_i := \frac{-1}{L} \cdot \left[R1p_i \cdot \left(\frac{a_i}{3} + b_i \right) + R2p_i \cdot \left(\frac{2}{3} \cdot b_i \right) \right]$

$j := 1..pts$

$x_j := (j - 1) \cdot int$

Slope at a distance x due to *individual* concentrated loads:

$$Vxp_{i,j} := \begin{cases} Apy_i + \dfrac{c_i \cdot \left(x_j\right)^2}{2 \cdot a_i} & \text{if } \left(x_j \le a_i\right) \\ Apy_i + R1p_i + \dfrac{c_i}{2 \cdot b_i} \cdot \left(x_j - a_i\right) \cdot \left(L + b_i - x_j\right) & \text{if } \left(x_j > a_i\right) \end{cases}$$

Deflection at a distance x due to *individual* concentrated loads:

$$Mxp_{i,j} := \left[\begin{array}{l} Apy_i \cdot x_j + \dfrac{c_i \cdot (x_j)^3}{6 \cdot a_i} \text{ if } (x_j \le a_i) \\ Apy_i \cdot x_j + R1p_i \cdot \left(x_j - \dfrac{2 \cdot a_i}{3} \right) + \dfrac{c_i}{6 \cdot b_i} \cdot (x_j - a_i)^2 \cdot (3 \cdot L - 2 \cdot a_i - x_j) \text{ if } (x_j > a_i) \end{array} \right]$$

$$j := 1 .. pts$$

Slope at a distance x due to **all** concentrated loads:

$$VxpP_j := \sum_{i=1}^{n} Vxp_{i,j}$$

Deflection at a distance x due to **all** concentrated loads:

$$MxpP_j := \sum_{i=1}^{n} Mxp_{i,j}$$

3. Determine slope and deflection due to uniformly distributed load:

Support reactions of ***real*** beam: $w = 0 \frac{lb}{ft}$

Length of distributed load: $d := d2 - d1$

$d = 0\,ft$

Left support reaction: $R_L := \frac{w \cdot d}{L} \cdot (0.5 \cdot d + L - d2)$

Right support reaction: $R_R := w \cdot d - R_L$

Distance to maximum moment beyond d1: $dM_2 := \frac{R_L}{w}$

Moments on real beam:

$M_0 := 0$ $M_1 := R_L \cdot d1$

$M_2 := R_L \cdot \left(d1 + \frac{dM_2}{2} \right)$ $M_3 := R_R \cdot (L - d2)$

Support reactions for **conjugate beam** :

Conjugate beam loading:

Resultants of distributed loads: $A'_1 := \frac{d1 \cdot M_1}{2 \cdot k}$

$$M(z) := \frac{R_L}{\text{UnitsOf}(R_L)} \cdot z - \frac{1}{2} \cdot \frac{w}{\text{UnitsOf}(w)} \cdot \left(z - \frac{d1}{\text{UnitsOf}(d1)}\right)^2$$

$$d1p := \frac{d1}{\text{UnitsOf}(d1)} \qquad d1p = 0 \qquad d2p := \frac{d2}{\text{UnitsOf}(d2)} \qquad d2p = 0$$

$$f := \left(\int_{d1p}^{d2p} M(z)\,dz\right) \qquad fp := f \cdot lb \cdot ft^2 \qquad f2 := \int_{d1p}^{d2p} M(z) \cdot z\,dz \qquad f2p := f2 \cdot lb \cdot ft^3$$

$$A'_2 := \frac{fp}{k} \qquad A'_3 := \frac{(L - d2) \cdot M_3}{2 \cdot k}$$

Locations of centroids of conjugate beam loading areas:

$$xc1 := \frac{2}{3} \cdot d1 \qquad xc2 := \frac{f2p}{fp} \qquad xc3 := \frac{(L + 2 \cdot d2)}{3}$$

Left support reaction of conjugate beam:

$$R'_A := \frac{-1}{L} \cdot \left[A'_1 \cdot (L - xc1) + A'_2 \cdot (L - xc2) + A'_3 \cdot (L - xc3)\right]$$

$$M'_0 := \frac{M_0}{k} \qquad M'_1 := \frac{M_1}{k} \qquad M'_2 := \frac{M_2}{k} \qquad M'_3 := \frac{M_3}{k}$$

$j := 1 .. pts$

$x_j := (j - 1) \cdot int$

$$xp_j := \frac{x_j}{\text{UnitsOf}(x_j)}$$

$$M(z) := \frac{R_L}{\text{UnitsOf}(R_L)} \cdot z - \frac{1}{2} \cdot \frac{w}{\text{UnitsOf}(w)} \cdot \left(z - \frac{d1}{\text{UnitsOf}(d1)}\right)^2$$

$$Av_j := \begin{cases} \int_{d1p}^{xp_j} M(z)\,dz & \text{if } (xp_j \geq d1p) \wedge (xp_j \leq d2p) \\ 0 & \text{otherwise} \end{cases}$$

$$Av2_j := \begin{cases} \int_{d1p}^{xp_j} zM(z)\,dz & \text{if } (xp_j \geq d1p) \wedge (xp_j \leq d2p) \\ 0 & \text{otherwise} \end{cases}$$

$$Avp_j := Av_j \cdot lb \cdot ft^2 \qquad Av2p_j := Av2_j \cdot lb \cdot ft^3 \qquad xc_j := \frac{Av2p_j}{Avp_j} \qquad A_j := \frac{Avp_j}{k}$$

Slope at a distance x due to uniformly distributed load:

$$Vxw_j := \begin{cases} R'_A + \dfrac{M'_1 \cdot (x_j)^2}{2 \cdot d1} & \text{if } (x_j \leq d1) \\ R'_A + A'_1 + A_j & \text{if } x_j > d1 \wedge x_j \leq d2 \\ R'_A + A'_1 + A'_2 + \dfrac{1}{2} \cdot (x_j - d2) \cdot \left[M'_3 \cdot \left(2 - \dfrac{x_j - d2}{L - d2} \right) \right] & \text{if } (x_j > d2) \end{cases}$$

Deflection at a distance x due to uniformly distributed load:

$$Mxw_j := \begin{cases} R'_A \cdot x_j + \dfrac{M'_1 \cdot (x_j)^3}{6 \cdot d1} & \text{if } (x_j \leq d1) \\ R'_A \cdot x_j + A'_1 \cdot (x_j - xc1) + A_j \cdot (x_j - xc_j) & \text{if } x_j > d1 \wedge x_j \leq d2 \\ \left[R'_A \cdot x_j + A'_1 \cdot (x_j - xc1) + A'_2 \cdot (x_j - xc2) + \dfrac{M'_3}{6} \cdot \dfrac{(x_j - d2)^2}{L - d2} \cdot (3 \cdot L - 2 \cdot d2 - x_j) \right] & \text{if } x_j > d2 \end{cases}$$

4. Determine slope and deflection at a distance x due to *all* loads:

$$j := 1 .. pts$$

$$x_j := (j - 1)int$$

Slope at x: $\theta_j := VxpP_j + Vxw_j$

$$\Delta p_j := MxpP_j + Mxw_j$$

Deflection at x: $\Delta_j := \dfrac{\Delta p_j \cdot 12}{UnitsOf(\Delta p_j)}$