AC 2009-568: USING LOGARITHMS TO TEST THE SOLUTION OF A DIFFERENTIAL EQUATION IN THE LAB

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Abstract

The mathematical modeling of the damping force on a spring-mass system oscillating in a fluid as a linear function of velocity (linear damping) is a simplifying assumption that leads to an ordinary and linear differential equation with constant coefficients. This model provides a simple means to account for the experimental fact that energy is dissipated during oscillations as the moving mass pushes against and displaces the surrounding fluid. The analytical solution to this differential equation is compared with experimental data collected from testing a spring-mass system in the open air of a laboratory. Collected data are analyzed using the concept of the logarithms. It is shown that the model is reliable under special conditions.

Introduction

One way to bring excitement in the use of mathematics in the engineering classroom is to show that it can be applied to model physical reality accurately. This paper presents work conceived and implemented to test the extent to which an ordinary differential equation and its solution are valid for use in actual applications. The equation chosen is commonly used in mathematics, physics, and engineering courses[1-3].

We consider the ordinary differential equation given by

$$\ddot{x} + 2\varsigma\dot{x} + \omega_n^2 x = 0, \qquad \textbf{(1)}$$

with the following initial conditions

$$\begin{aligned} x(t=0) &= x_0 \\ \dot{x}(t=0) &= v_0 \end{aligned} \qquad \textbf{(2)}$$

where x is a function of time, the dots indicate derivatives of x with respect to time, and ω_n^2 and ς are constants that characterize the system.

Picture A. Photo of a smooth sphere on a linear spring oscillating in the open air of a laboratory.

Eq. (1) can represent the motion of a mass, m, suspended onto a linear spring of stiffness k that is oscillating in a viscous medium that has a constant damping coefficient c, as shown in Picture A. In these circumstances, ω_n is the natural frequency of oscillation of the system defined as

$$\omega_n = \sqrt{\frac{k}{m}}.$$

And ς is the damping ratio for the motion defined as $\varsigma = \dfrac{c}{2m\omega_n}$. Then, the general solution to Eq. (1), subject to initial conditions shown in Eq. (2), is given by[4]

$$x(t) = X_0 e^{-\varsigma\omega_n t}\cos(\omega_d t - \phi_0) \tag{3}$$

where

$$\omega_d = \sqrt{1-\varsigma^2}\,\omega_n,$$

$$X_0 = \sqrt{x_0^2 + \left(\frac{v_0 + \varsigma\omega_n x_0}{\omega_d}\right)^2}, and$$

$$\phi_0 = \tan^{-1}\left(-\frac{v_0 + \varsigma\omega_n x_0}{x_0\omega_d}\right). \tag{3a}$$

Eq. (3) indicates that, when the damping force is proportional to the velocity, as is the case in viscous damping, the displacement of the mass is a product of two functions of time: a sinusoidal function that is very similar to that found in undamped motion, and an exponential function that decreases with time. The latter indicates that the amplitude attained by the mass decreases (decays) exponentially with time while it is oscillating. In geometric terms, the exponential function represents the envelope of the plot of the motion of the mass with time.

When damping could be neglected, it was shown in earlier experimental work that the oscillatory motion of a sphere in air was almost perfectly sinusoidal. Indeed, in this case, the results of analysis and those of experiments agreed with a maximum discrepancy of 2.5% [5]. Therefore, the focus in this work is on the effect of damping on the oscillatory motion.

Analysis suggests that linear damping, as used here, has two effects on the motion of the suspended mass: it slows the system down by reducing its natural frequency from the undamped value, ω_n, to the damped value, ω_d. The ratio between the two is shown in Eq.(3b), where $\varsigma < 1$.

$$\frac{\omega_d}{\omega_n} = \sqrt{1-\varsigma^2}, \varsigma < 1. \tag{3b}$$

Damping also causes the amplitude of oscillation to decrease exponentially with time. The focus of the comparison between analysis and experiment will be on the exponential term, which amounts to comparing the mathematical nature of the envelope of the motion obtained in the lab with that indicated by the solution to the differential equation, Eq (3).

This paper discusses an experiment that was designed to test the validity of Eq. (3) in the laboratory. The experiment demonstrates the extent to which Eq.(3) is a reliable model for the oscillations of a mass that is suspended to a linear spring in a laboratory setting when air resistance is taken into account.

The remainder of this paper is organized in the following manner: first, we discuss how logarithms will be used to test Eq. (3) in the laboratory. Then, the design of the experiment is presented. Next, experimental data are presented and analyzed using Logarithms in two different ways. Finally, these experimental results are compared to the solution of the differential equation itself.

Use of logarithms in analysis of data

An important property of logarithms that is often exploited in analyzing nonlinear data is that the logarithm of a product AB equals the logarithm of A plus the logarithm of B. Thus, one can write

$$\ln(A \times B) = \ln(A) + \ln(B) \qquad \textbf{(4)}$$

This property is particularly useful when one expects data to vary exponentially, with say, time, as is the case in damped harmonic motion. By taking the logarithms of both sides of the equation and changing variables, one transforms the exponential relationship into a linear one, which is much easier to analyze. For example, consider the equation

$$y = ae^{bt} \qquad \textbf{(5)}$$

After taking the logarithms of both sides of this equation, one gets

$$\ln(y) = \ln(a) + bt \qquad \textbf{(6)}$$

Letting z = ln(y), one can then think of this as z vs. t, which makes the relationship linear.

$$z = \ln(a) + bt \qquad \textbf{(6a)}$$

A plot of z vs.t alloys one to identify **b** as the slope of the resulting straight line and the ln(a) as its z-intercept. One solves for **a** by using the inverse relationship between logarithms and exponentials. This leads to

$$a = e^{\ln(a)}. \qquad \textbf{(7)}$$

This type of operation is used routinely in Microsoft Excel to fit an exponential curve to experimental data. This will be illustrated below by using data from oscillating spheres. We will fit an exponential function to experimental data two different ways: by using Microsoft Excel and by doing it manually. We will then compare the results to show that they are identical. Comparing the term that multiplies $\cos\left(\omega_d t - \phi_0\right)$ in Eq.(3) with Eq.(5), it can be deduced that

$a = X_0$ and $b = -\varsigma\omega_n$.

The experiment and collected data

We designed a load cell and used it with computer data-acquisition equipment to collect data on a variety of spheres. The test setup, equipment, and procedures pertaining to this experiment were detailed in earlier work [6]. The two spheres that were used for this paper are described in Table 1. A sample set of data is shown in Table 2, where the instantaneous position of each sphere is tabulated versus time. Only data for amplitudes achieved in the upper half plane are shown in Table 2. Data in the lower half plane have the same magnitude but opposite algebraic signs. This is because the envelope for the motion of the mass is symmetrical about the time axis.

Table1. Data on the tested spheres

Type of sphere	Diameter (cm)	Mass (kg)	Natural Frequency Radians/second
Metal ball	37.1	0.245	12.87
Golf ball	40.1	0.055	27.19

In Table 1, we show data for two different spheres: a metal sphere and a golf ball. They were suspended to the same linear spring of stiffness k = 40.6 N/m separately and tested one after the other. The metal sphere was suspended from this spring and the resulting motion was studied earlier. It was shown that the ensuing oscillations were essentially viscously damped, with a maximum discrepancy between theory and experiment of 5% [6]. The motion of that sphere is being used here as a convenient reference with which that of the golf ball can be compared.

Table 2. Sample experimental data for two spheres

Time(s)	Metal x(cm)	Metal Log(x)	Golf x(cm)	Golf Log(x)
0	1.94	0.662688	1.875	0.628609
25	1.645	0.49774	1.525	0.421994
50	1.4	0.336472	1.3	0.262364
75	1.3	0.262364	1.1	0.09531
100	1.15	0.139762	0.93	-0.07257
125	1.01	0.00995	0.775	-0.25489
150	0.905	-0.09982	0.675	-0.39304
175	0.875	-0.13353	0.6	-0.51083
200	0.775	-0.25489	0.55	-0.59784
225	0.7	-0.35667	0.5	-0.69315
250	0.65	-0.43078	0.41	-0.8916
275	0.59	-0.52763	0.375	-0.98083

300	0.49	-0.71335	0.35	-1.04982
325	0.425	-0.85567	0.325	-1.12393
350	0.375	-0.98083	0.325	-1.12393
375	0.3	-1.20397	0.3	-1.20397
400	0.3	-1.20397	0.3	-1.20397
425	0.275	-1.29098	0.275	-1.29098
450	0.25	-1.38629	0.275	-1.29098
475	0.24	-1.42712	0.275	-1.29098
500	0.24	-1.42712	0.275	-1.29098

In Table 2, the first column represents the time elapsed since the oscillation started. In both cases, data were collected continuously for 500 seconds. The second column represents the instantaneous positions of the metal ball at the times shown in the first column. These values are measured from the position of static equilibrium of the mass while suspended on the spring. The third column is the natural logarithms of the positions of the spheres. The fourth and fifth columns represent similar data for the golf ball, respectively.

The raw data, position vs. time, from Table 2 are plotted in Fig.1. It can be seen that, in both cases, the amplitude of oscillation of each sphere decreases with time, although in different manners.

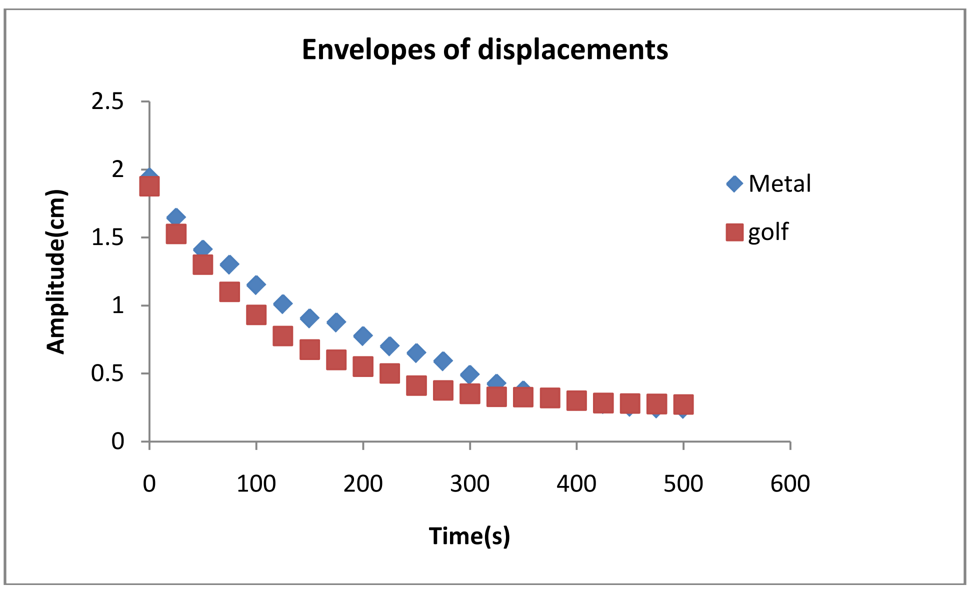

Fig. 1. Envelopes of the plots of position vs. time for two oscillating spheres

Curvefitting Using Microsoft Excel

An important task that remains to be done is to characterize the nature of these envelopes mathematically. To do that, we used Microsoft Excel to fit an exponential function to each set of data shown in Fig.1. The choice of exponential functions was indicated by the solution of the differential equation shown in Eq. (3). The plots with the fitted curves are shown in Fig. 2, where the continuous curves are the predictions of analysis and the discontinuous lines represent experimental data.

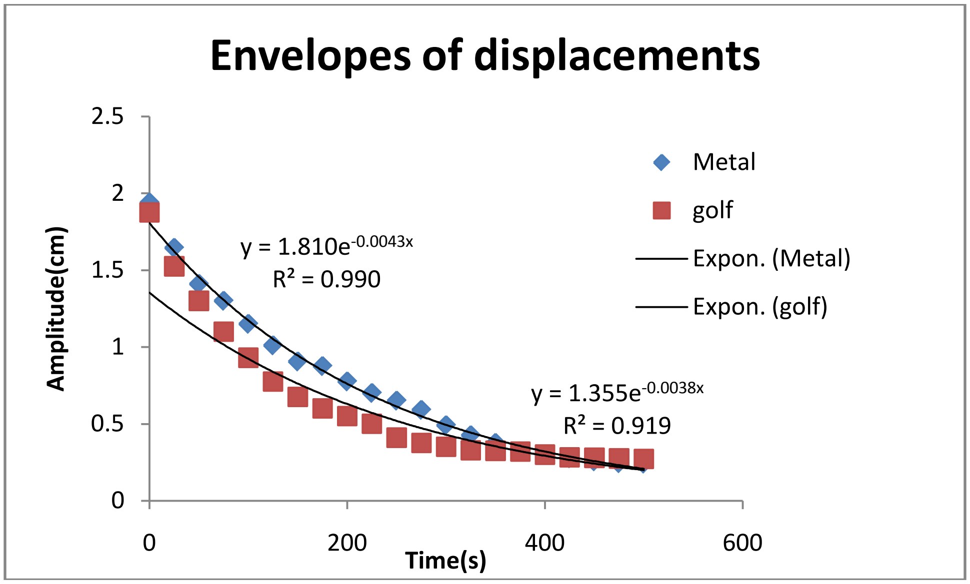

Fig. 2. Exponential curves fitted to the raw data shown in Fig. 2.

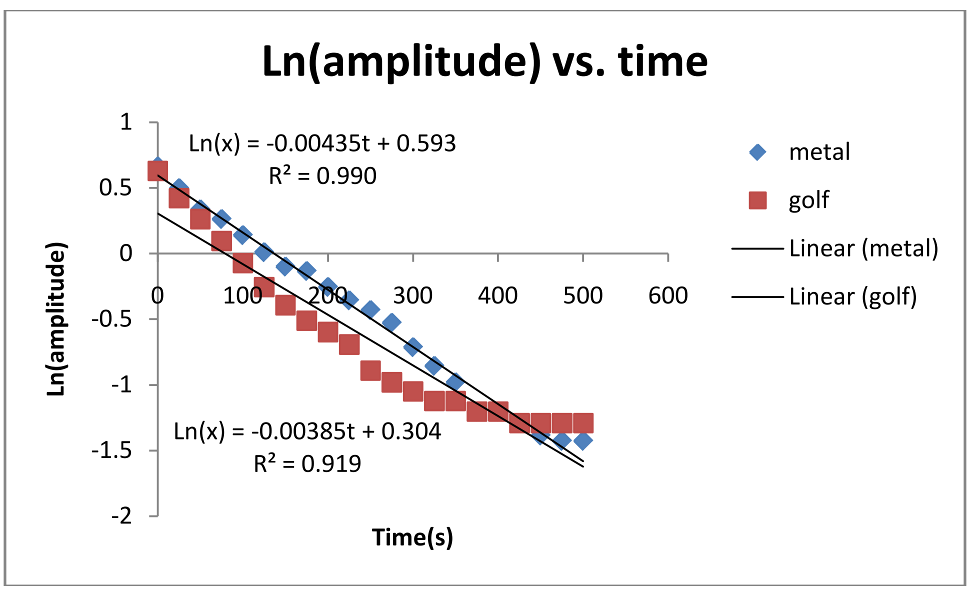

Fig. 3. The ln (amplitude) vs. time with straight lines fitted to the data

Curvefitting Using Logarithms

Processed data, log (position) vs. time, from Table 2 are plotted in Fig.3. It can be seen that, in both cases, the logarithms of the amplitudes of oscillation of each sphere decrease with time, although in different manners. We will fit a straight line to each set of data; then determine the slope and z-intercept of each line. To do that, we used Microsoft Excel. The choice of straight lines was indicated by Eq. (6). The plots with the fitted lines are shown in Fig. 3, where the continuous curves are the predictions of analysis and the discontinuous lines represent experimental data.

Table 3. Quantities determined using curvefitting of the raw data in Fig. 2. in Microsoft Excel.

Sphere	$\omega_n (rad / s)$	$\omega_d (rad / s)$	$a = X_0 (cm)$	$b = \varsigma\omega_n (rad / s)$	R^2
Metal ball	12.912	12.874	1.810	-0.00435	0.990
Golf ball	27.19	26.982	1.355	-0.00385	0.919

Table 4. Quantities determined using curvefitting and logarithms of data in Fig. 3.

Sphere	$\omega_n (rad / s)$	$\omega_d (rad / s)$	$a = X_0 (cm)$	$b = \varsigma\omega_n (rad / s)$	R^2
Metal ball	12.912	12.874	$e^{0.593} = 1.810$	-0.00435	0.990
Golf ball	27.19	26.982	$e^{0.304} = 1.355$	-0.00385	0.919

It can be seen by inspection of the graphs in Fig. 2 and Fig. 3 that the data from the metal sphere generated better fits than those from the golf ball. The experimental values of the coefficients found in Eq.(3) that were extracted from those graphs were tabulated in Tables 3 and 4 , respectively. It can be verified, by comparisons, that the two methods yielded the same exact values. Therefore, finding curves to fit data with exponential trends in Microsoft Excel amounts to using natural logarithms.

Comparisons: Analysis vs. experiment

We examined the goodness of the curve fits to our experimental data using the quantity R^2, the coefficient of determination.

If at each time t_i, a set of experimental data has a measured value y_i , we let $\bar{y}$ be the arithmetic average of the measured values y_i. After fitting a curve to the data, at each time t_i, there will be a corresponding value f_i computed using the mathematical model on which the fitted curve was based. A common measure of the departure of the data from the fitted curve is called the coefficient of determination, R^2, defined by

$$R^2 = 1 - \frac{\sum_i^N (y_i - f_i)^2}{\sum_i^N (y_i - \bar{y})^2} \qquad \textbf{(8)}$$

In statistics, it measures the match between the data being analyzed and the trend line used to represent them; indeed, when $R^2 = 1$, it is evident that $y_i = f_i$, for each i, and the measured data fit the curve perfectly. The smaller the value of R^2, the worse the match. Thus, in this study, R^2 helps determine the extent to which experimental data conform themselves to what is supposed to happen when the damping force is viscous, that is, a linear function of velocity of the sphere.

Using this coefficient as a measure, it can be seen from Tables 3 and 4 that the data for the metal ball match the fitted curve 99%, while for those from the golf ball, such a match is 92% perfect. If one considers this coefficient as a statistical measure of accuracy, then, it can be concluded that the oscillations of the metal sphere differed from the prediction of analysis by 1%, whereas those of the golf ball did so by 8%.

The major parameter in the mathematical model that distinguished one sphere from the other was the magnitude of the frequency of oscillations. The frequency of the golf ball was a little more than twice that of the metal ball. We attribute the differences in the behaviors of the two spheres to their frequencies of oscillation. Accordingly, we concluded that our data suggested that, when testing spheres in air, the natural frequency of their oscillations is a very important parameter, if one expects to observe damping behavior that is viscous. Furthermore, the data presented in this study indicate that, to achieve viscous damping in air, the natural frequency frequencies of the oscillations of spheres should be below 13 radians/s, or 2 Hz.

Conclusions

It is very common to model the free oscillation of a mass that is suspended to the end of a linear spring using an ordinary differential equation of second order with constant coefficients, the solution to which consists of circular functions. When damping needs to be accounted for, it is routine to assume that the damping force is viscous, that is, directly proportional to the first power of the instantaneous velocity of the mass.

An experiment was designed and carried out to test the validity of this model for practical applications. The chief purpose was to see the extent to which the mathematical solution is a reliable model for actual oscillations of a mass in the laboratory. The emphasis in this project was to examine the effect of the damping term on the solution, because the model that neglects damping was tested in previous studies and found to be in excellent agreement with analysis[5].

Instantaneous positions of a mass suspended onto a spring were collected versus time using a sensor that was connected to computer data-acquisition equipment; the envelopes of the plots of position vs. time of two different spheres were analyzed using logarithms and compared with what was expected from analysis. Results indicated that the frequency of oscillation has an important effect on the extent to which the model matches the actual behavior. This suggests that the damping coefficient is a dynamic parameter that depends on the dynamics of motion in addition to the expected dependence on the viscosity of the fluid and the geometry of the oscillating object. Indeed, the research literature in fluid mechanics has shown that the motion of a sphere in viscous fluid is a problem that presents researchers with considerable theoretical and experimental challenges [7-16]. The purpose of this work is to show that, under carefully chosen circumstances, the simplified concept of viscous damping that is used to model the vibration of a mass in a viscous fluid gives results that are in excellent agreement with experiments and that logarithms can be useful in that assessment.

When the frequency of oscillations was relatively low, the differential equation being investigated represented the motion of the mass accurately. Indeed, at a frequency of 13 rad/s, experimental data and the analytical solution to the differential equation itself differed by about 1%, on average. However, at a frequency of 27 rad/s, experimental data and the analytical solution to the differential equation differed by about 8%, on average. The amplitude of motion decreased more abruptly with time during most of the motion at high frequencies, such decreases were particularly abrupt at the beginning.

The key finding, however, is that the accuracy of the model varies with the natural frequency of the motion at hand. Our experiments suggest that this is due to the fact that the coefficient of viscous damping is a dynamic variable rather than a fixed parameter that solely reflects the interaction among the viscosity of the fluid, the geometry of the oscillating mass and its surface condition.

Finally, in addition to their use to check the validity of the solution of the differential equation shown in Eq.(1), the data included in this paper (Table 2) could also be used in class by instructors to demonstrate the process of fitting an exponential function to experimental data as a practical use of natural logarithms in physics and engineering.

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