Using *Mathematica* with Multivariable Calculus

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Abstract

The Department of Mathematics at the University of Oklahoma (OU) is developing technology-based materials for its engineering calculus sequence, both to enhance conceptual understanding and to prepare students for problem-solving with the computational power available. In this paper, we discuss the in-class use of *Mathematica* animations and sequences of overhead transparencies, and the out-of-class use of problem sets and the World Wide Web, with multivariable calculus. A goal of the ongoing project is to offer interested instructors a variety of materials that will enable them to incorporate technology at a level of integration that they deem appropriate.

Need For Technology

Technology has played an increasingly important role in our society over the past several decades. Perhaps nowhere is this more apparent than in engineering fields throughout industry, government, and academia. It is difficult to imagine any modern engineering work being performed without the aid of a computer! Thus, a necessary part of any undergraduate education in the engineering disciplines is exposure to and competency with technology. For many years, engineering departments at colleges and universities have made it a point to incorporate technology into their curricula, exposing students to the latest in computational, design, and modeling software. It is only fitting that calculus and other engineering mathematics courses, as necessary prerequisites to most studies in engineering, also incorporate the latest technological innovations, both to enhance conceptual understanding and to prepare students for problem-solving with the computational power available.

One of the most pressing challenges mathematics departments face is the *appropriate* use of technology — especially calculators and computers — in calculus courses. The Department of Mathematics at the University of Oklahoma (OU) is developing technology-based materials for its engineering calculus sequence. Ultimately the goal is to offer calculus instructors the option to blend such technology with relevant mathematics content and instructional strategies in a coherent and easily usable manner.

Although at OU there is some use of the graphing calculators in differential and integral calculus, this paper focuses on a project to incorporate *Mathematica* into the multivariable calculus course. This part of the calculus sequence introduces the analysis of functions of more than one variable (including partial derivatives and iterated integrals) and the analysis of vector fields (including curl, divergence, line integrals, and surface integrals). This content requires students to visualize the two- and three-dimensional objects being studied — an ability that often eludes many otherwise successful students[1]. Our project began with this multivariable calculus course in order to employ *Mathematica*'s powerful graphing capabilities where they are most immediately needed. Our assumption is that appropriate use of *Mathematica* will lead to easier analysis and deeper understanding of these often-complicated objects.

Using Computer Software to Facilitate Visualization

A goal of the ongoing project is to offer interested instructors a variety of materials that will enable them to incorporate *Mathematica* at a level of integration that they deem appropriate. The authors are charged with the task of designing in-class and out-of-class materials, and constructing an instructor's resource manual (hardcopy). In this section, we discuss the in-class use of animations and slides, and the out-of-class use of problem sets and the World Wide Web, and offer examples of items included in the resource manual.

<u>In-Class.</u> The authors embrace the use of tools that facilitate instruction and add to students' learning. Under this philosophy our activities use the computer not as an expensive overhead projector but rather to accomplish tasks that other instructional tools can not. One such task is the animation of graphics, presented in-class by way of a demonstration on a computer.

For example, to help students understand the relationship between the three-dimensional graph of a surface and its contour diagram, we use a computer animation. The example presented in Figure 1 shows a paraboloid. Marked on the paraboloid are the circles obtained by intersecting the paraboloid with horizontal planes; these circles are referred to here as "horizontal cross-sections." The animated migration of these cross-sections to the xy-plane is used in class to demonstrate the relationship between a three-dimensional surface and a contour diagram. Figure 1 shows a few frames of the animation along with the resulting contour diagram.

In addition to animations, the authors also use *Mathematica* to produce sets of overhead transparencies. Calculus textbooks usually include figures to go with explanations and examples; some textbooks even provide slides of selected graphics. Yet, these graphics tend to display only the complicated result of a set of mathematical actions on the object. Using *Mathematica*, we create sets of slides that depict sequences of mathematical actions. This database of slides, included in the instructor's resource manual, is intended to enable instructors to select a sequence of images that best illustrates their presentation of the topic.

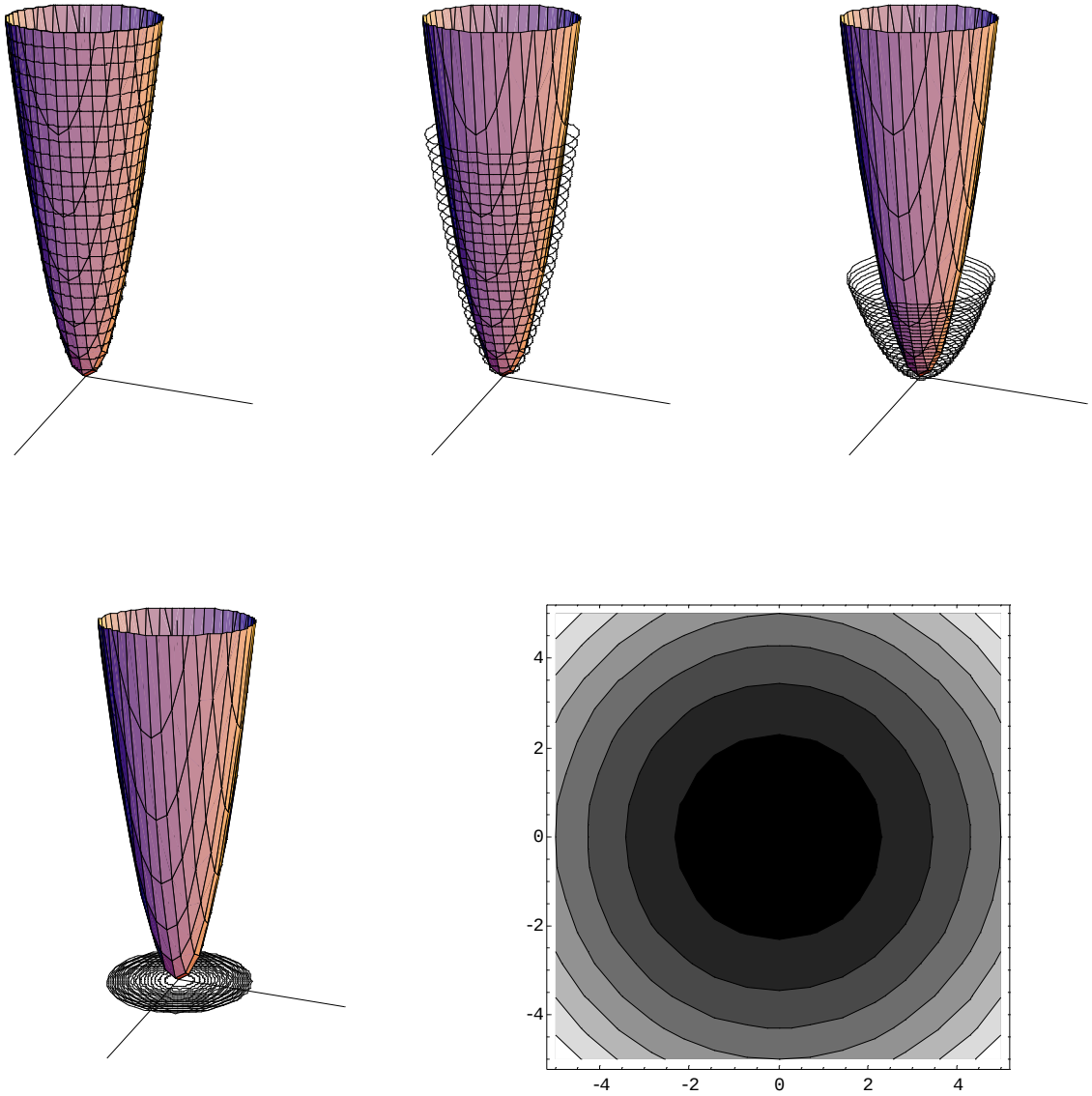

Figure 1. Key frames of the animation: Construction of a Contour Diagram.

For instance, the multivariable calculus course at OU introduces the use of Lagrange multipliers to maximize or minimize an objective function subject to one constraint condition. To illustrate the case involving a function of two variables and one constraint equation, the authors created a sequence of slides that offer an intuitive idea of why the method of Lagrange multipliers works. Figure 2 shows the surface $f(x,y) = 2x^2 + x + y^2 - 2$ and the constraint curve $x^2 + y^2 = 4$ in the xy-plane[2].

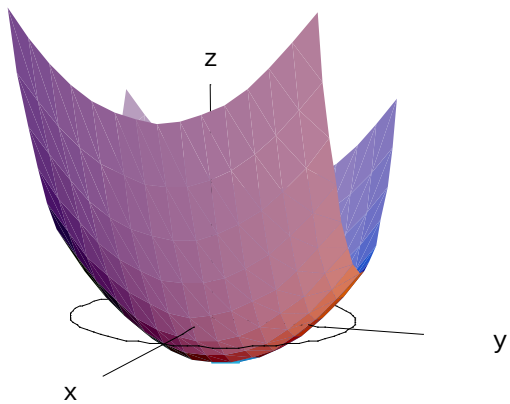

Figure 2. A surface and a constraint curve.

In 3-space the constraint equation $x^2 + y^2 = 4$ defines a cylinder. Intersecting this cylinder with the surface gives a curve in 3-space (Figure 3).

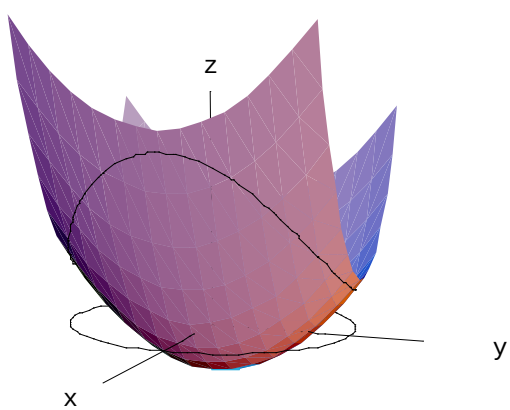

Figure 3. The surface, the constraint, and the intersection curve.

By plotting the intersection curve in 3-space (Figure 4), we are able to demonstrate visually that the function attains two local maxima and two local minima along the constraint curve.

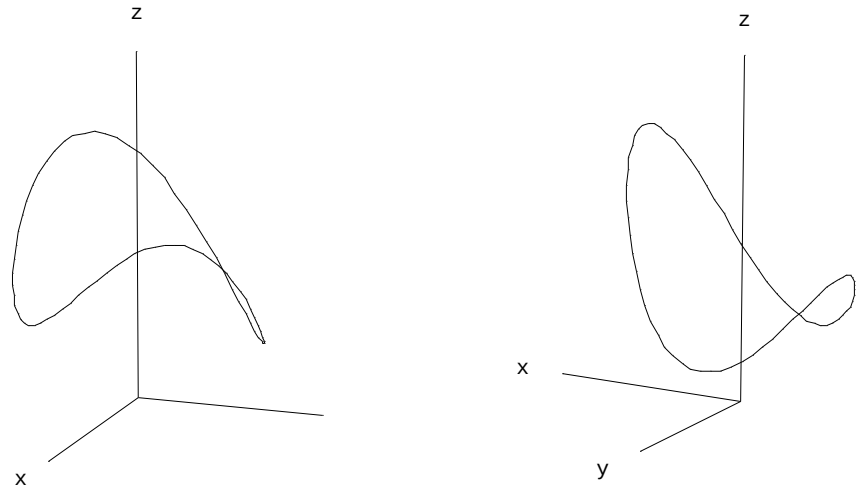

Figure 4. Two viewpoints of the intersection curve.

Another way to study the method of Lagrange multipliers is through the objective function's level curves (i.e., the curves in the contour diagram). This strategy assumes that the students have acquired some skills at translating between a three-dimensional graph and a two-dimensional contour diagram of the function. Using Figure 5 in class we demonstrate the relationship between the constraint curve —a two-dimensional object — and the level curves of the surface.

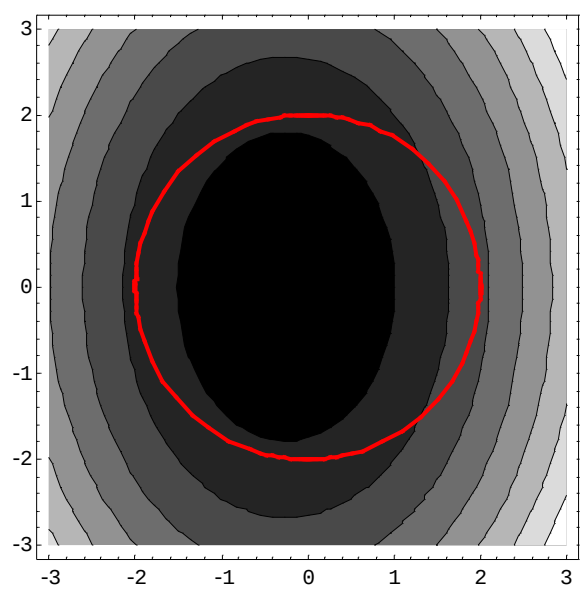

Figure 5. The level curves and the constraint curve.

We note how the objective function's values change as we traverse the constraint curve. For example, traveling along the constraint from the point (2,0), in either direction, the objective function's values decrease (in Figure 6, darker shading represents values farther down the z-axis). So (2,0) yields a local maximum with respect to travel on the constraint curve.

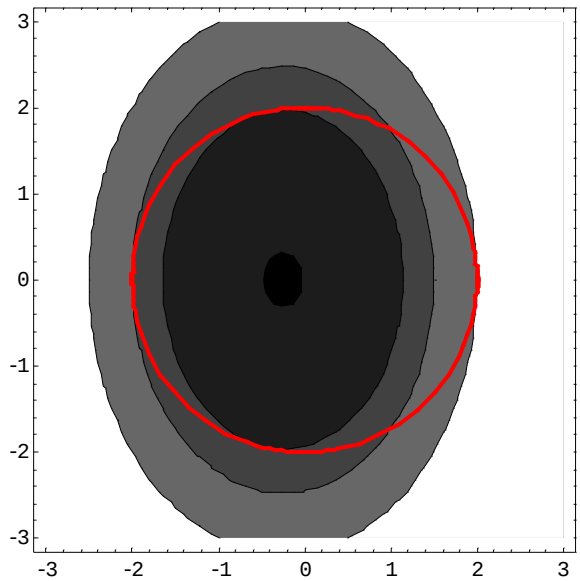

Figure 6. The constraint curve and the level curves to which it is tangent.

This observation leads to a discussion of the relationship between the level curves and the constraint curve, of the relationship between their gradient vectors, and finally of the algebraic system of equations to be solved.

Three-dimensional objects are difficult and time-consuming to draw well by hand. The availability of prepared sets of slides allows the judicious use of an overhead projector combined with the standard use of a chalkboard. In fact, certain sets of slides enable the instructors to superimpose one on top of another to emphasize relationships among objects. This combination of tools facilitates the instruction process that is so critical in helping students to visualize these objects.

Out-of-Class. In class we sometimes refer to the *Mathematica* commands used to create the in-class presentations but we do not emphasize the program itself during class time. Rather it is outside of class that students directly employ a computer algebra system to facilitate their learning of the calculus. We assume that students have not previously used *Mathematica*; students are not required to purchase *Mathematica*, or even to own a computer. Instead we anticipate that students will use the public computer labs on campus, so we hold "introduction to *Mathematica*" sessions and some office hours in the campus computer labs. In fact, we allow students to use any computer algebra system they prefer (e.g., Mathcad, Maple) — with the

understanding that the instructors do not guarantee answers to questions about any specific software except *Mathematica*. These initial assumptions bring the anticipated complications, including student resistance to and frustration with learning *Mathematica*, uncooperative software, crashing computers, and limited modem access to campus computer resources. Despite these barriers, we maintain an expectation that the benefits of appropriate computer-based activities outweigh the inconveniences.

The primary out-of-class activity for students is "problem sets." A problem set, for our purposes, contains items that are too involved for students to do in class. In some cases, these problems are intended to be worked by groups of students; also in some cases, the problems are designed to take advantage of *Mathematica* as a tool. After the "Generating a Contour Diagram" animation, for example, students are given sample syntax for *Mathematica* commands to graph three-dimensional objects and their contour diagrams. They then use these samples to do problems such as: "Consider the family of functions, $f(x,y) = Ax^2 + By^2$.

a) Investigate the family of functions by using the Plot3D to graph members of the family (be sure to include positive and negative values of A and B). Experiment with options, such as ViewPoint and PlotPoints, to produce (and submit) printouts that represent the surfaces nicely. How does the shape of the surface depend on the values of A and B?
b) Select a function from the family with A > 0 and B < 0. Produce (and submit) graphs of the cross-sectional views obtained by slicing the surfaces with the planes x = -2, x = 0, and x = 2.
c) Do the same as in Part b) with planes y = -2, y = 0, and y = 2.
d) Do the same as in Parts b) and c) with planes z = -2, z = 0, and z = 2.
e) Obtain a ContourPlot of the function you selected in Part b). How are the cross-sections in Part d) related to the Contour Plot? You may want to experiment with more cross-sections."

Students type their responses in a *Mathematica* notebook and printout their work to turn in. In the Lagrange multipliers example, the emphasis for students is the set up of the equations; they can then use *Mathematica* to solve the system. Again they apply the ContourPlot command, this time also using Plot to graph the constraint equation, to illustrate the solutions to the specific problem they worked. Samples of these problem sets are provided in the instructor's resource manual.

Along with the manual, the Department of Mathematics at OU offers a calculus web site[3] to make resources from this project readily available to instructors at OU and elsewhere. The web site includes:

- introductory documents for the use of *Mathematica* in calculus at OU;
- documents containing the *Mathematica* input and output used for in-class computer animations and overhead transparencies;
- links to the web pages of instructors currently teaching the course;
- links to web resources related to *Mathematica*.

Having the in-class animations and slides available on the web allows students to access the documents for review of the presentation and to print the graphics (in color!) to complete their notes. Some instructors also use the web to house problem sets, solution keys, and other resources for students.

Further Discussion and Future Directions

Our intention is to provide materials that facilitate the teaching and learning of visualization skills. The manual currently includes sample practice exams, problem set items, and sets of overhead transparency masters for selected topics. These materials and the animations are also available on the web[3].

Instructors need not know *Mathematica* to be able to use the animations or slides in class. The use of *Mathematica* outside of class, however, involves a considerable learning experience, and often considerable frustrations — both for the students and for the instructor. Course schedules are already tight with content, leaving little room for in-class discussion of how to use *Mathematica*. One open question is how to incorporate the use of a computer algebra system (or even a graphing calculator) with minimal complications to produce maximal learning.

Similar projects are occurring at other institutions and an expanding set of resources is available for such efforts[4]. The development of additional resources does not mean that we are "reinventing the wheel". Rather, we are adding to the set of options available to individual instructors, who decide which resources to use based on the goals of their courses, the needs of their students, and their own personal teaching philosophies and styles.

We have several additional goals for the multivariable calculus project, one of which involves the use of *Mathematica* for in-class animations and slides, and one of which involves the web site. First, we believe that each instructor brings to the classroom his or her own preferences in the form of content emphases, prepared explanations, and favorite examples. Therefore, we intend to re-structure our *Mathematica* documents so that an instructor can more easily input a chosen example for an animation or set of slides. Another goal is to institutionalize the "Calculus at OU" web site; that is, to designate a webmaster (probably a graduate student) who will maintain and update the site regularly. Furthermore, we hope to extend the project to the other courses in the calculus sequence, using *Mathematica* where appropriate.

Using technology to improve instruction, even in a specific course sequence such as calculus, is not monolithic. Our experience suggests that the key is availability of flexible, varied resources. Individual instructors often make differing personal decisions even for the same course. The availability of a menu of technological resources that can be used "off the shelf" may be the key to increasing the use of technology to facilitate teaching and learning.

References

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2. Stewart, J. (1995). Calculus (3rd ed.). Pacific Grove, CA: Brooks/Cole Publishing Company. p.820, exercise #30.
3. URL: http://www.math.ou.edu/~tjmurphy/MVCMMA.html
4. URL: http://archives.math.utk.edu/ has a partial listing of projects and resources.

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