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Using Modeling Activities to Engage Students in Learning

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Abstract

Mathematical models using differential equations are among the most difficult topics for the engineering majors at our institute. Most of them are required to take an introductory differential equations course during their sophomore years, and some of them take a mathematical modeling course as an elective, afterwards. We address how mathematical modeling activities can be used to motivate and engage students in learning difficult topics in this paper. We use the modeling activities of a falling column of water as an example that includes data collection, parameter estimation and model validation.

Keywords

Differential equations, mathematical modeling, Torricelli's law, data collection, parameter estimation

Introduction

Mathematical models using differential equations are taught in an introductory differential equations course, which is required for most of the engineering majors at our college which is primarily a four-year undergraduate school. Afterwards they can study additional mathematical models using differential equations in a mathematical modeling course as an elective. Many students find these topics either difficult or uninteresting. Studies show that engaging students in modeling activities can pique students interest and improve their learning experience [1], [2]. Also, SIMIODE [3] provides a rich resource of suggestions and ideas for modeling activities and scenarios.

Using Torricelli's law to model the height of falling water using a first-order differential equation (as in the next section) is frequently taught in the mathematical modeling course at our college. In the introductory differential equations course, students are sometimes given the first-order differential equation directly due to the time constraint and the focus of the course is on the techniques of solving the equations. However, since the objective of the modeling activities is to estimate the parameter in the model, instructors who teach an introductory differential equations course can also incorporate similar activities in the introductory course. The modeling activities include data collection and estimating the parameter in the model using the data collected. Data can be collected by conducting the experiment directly if a container or bottle with a small hole at the bottom and measuring tools are available. Alternatively, students can collect data by recording the height of the water and corresponding time from a video of the experiment that is available online [4]. Engaging students in such activities may increase their interest and attention in a course that they consider difficult or boring. Also, students learn from the activities that a constant given in a mathematical model is actually not easy to obtain directly.

Modeling the Height of Falling Water in a Cylindrical Tank

To find the mathematical model of the height of falling water in a cylindrical tank using differential equation, we review Torricelli's law, or the law of conservation of energy. Let $h(t), V(t), v(t)$ be the height, volume and exit speed of the water at time t, respectively, in a cylindrical tank with a small hole at the bottom. Also, let the height of the water at the bottom be 0. Assume that $h(0) = h_0 > 0$ and $v(0) = v_0$. Torricelli's law states that fluid in a tank will escape through a sharp-edged hole at the bottom of the tank with the same speed v that it would acquire falling from the same height h to the level of the hole, i.e., $v = \sqrt{2gh}$ where g is the gravitational constant. Torricelli's law may be obtained by using the law of conservation of energy.

Since a cylindrical tank has the same cross-sectional area, say A, using the chain-rule, the rate of change of the volume of the water in the tank can be written as

$$\begin{aligned} \frac{dV}{dt} &= \frac{dV}{dh} \times \frac{dh}{dt} \\ &= \frac{d(Ah)}{dh} \times \frac{dh}{dt} \\ &= A\frac{dh}{dt} \end{aligned}$$

Also, considering how water exits the tank, the volume of the water in the tank changes at the rate of

$$\begin{aligned} \frac{dV}{dt} &= -av \\ &= -a\sqrt{2gh} \end{aligned}$$

where a is the cross-sectional area of the hole, and the negative sign indicates the volume of the water decreases, and v is the speed of the water exiting the tank when the water is at height $h(t)$, and v is then replaced by $\sqrt{2gh}$ using Torricelli's law.

Putting together the above two ways of expressing the rate of change of the volume of the water in the cylindrical tank, we have

$$\begin{aligned} \frac{dV}{dt} &= A\frac{dh}{dt} \\ &= -a\sqrt{2gh} \end{aligned}$$

that is,

$$\frac{dh}{dt} = -b\sqrt{2gh}, h(0) = h_0 \tag{1}$$

If we consider other factors such as friction caused by water going through the hole [5], b is used as a broad constant not just $\frac{a}{A}$. Using the method of separation of variables, we have the following

$$h(t) = (\sqrt{h_0} - \frac{b\sqrt{g}}{2}t)^2 \tag{2}$$

Although b is supposed to be a constant and its value is usually given (or the cross-sectional areas A and a are given) in textbook problems, in reality its value is difficult to obtain directly. Students learn about this from the data collection activity addressed in Section and the parameter estimation activity addressed in Section . We believe that these activities may pique students' interest and engage them in the classroom.

Data Collection Activity

Since the height of falling water in the cylindrical container is a function of time, a list of height values and the corresponding time values should be collected. Conducting an experiment to collect data requires a cylindrical container such as a plastic soft drink bottle, a tool to drill a small hole near the bottom of the bottle, a measuring tape or ruler, and a timer. To avoid recording both the height of the water and the time, we can mark the heights on the bottle first from 0 centimeter (corresponding to the top of the hole) to 12 centimeters with increment of 1 centimeter. If at time $t = 0$ second, the height of the water is 12 centimeters, then we record the corresponding time when the height of the water drops to 11 centimeters, 10 centimeters, etc. as shown in Table 1. To promote interest and collaboration among students, divide the students into groups of which consist of three students: a student announces the height to which the water drops, another student calls the time, and another student writes down the time corresponding to the specific height. Alternatively, by taping the measuring tape or ruler on the bottle, starting at 0 seconds, for every 4 seconds, record the height of the water corresponding to a specific time as shown in Table 2, and students work in groups of three as suggested above.

If conducting an experiment is inconvenient, data can also be collected using the online videos provided by SIMIODE [4]. These videos allow the measurements to be more accurate since they have pause buttons, and they make the data collection process much easier. Also, these videos are demonstration examples of how to conduct an experiment.

Data in Table 3 were collected from the online video [6]. Note that since each student might pause at different time to record the time and the height of the water as well as possible measuring errors and recording errors, data collected by each student (or each group of students) would be different even if the same online video was used. Also, since the time in the video started at 8 seconds, each observed time was subtracted by 8, which was called the zeroed time in Table 3.

Parameter Estimation and Model Validation Activity

To estimate parameter b in Equation (2), we ask the students to use the data they collected (similar to the data shown in Table 3), where the zeroed times are used and let the gravitational constant be $g = 980cm/s^2$. In [5], Excel Solver was used to estimate b since using "Add Trendline" cannot

Table 1: Record the time for specific heights of the water during an experiment

Time (s)	Height (cm)
	12
	11
	10
	9
	8
	7
	6
	5
	4
	3
	2
	1
	0

Table 2: Record the height of the water for specific time during an experiment

Time (s)	Height (cm)
0	
4	
8	
12	
16	
20	
24	
28	
32	
36	
40	

find b directly (b is inside the square term). Excel Solver is an add-in tool that requires a few steps to set up before the students can use it. Also, the Excel sheet containing the data collected has to be set up in a way (such as having an initial guess of b and SSE (sum of squares of errors) must be calculated) before the Solver is used. Some of our students struggled with using Excel Solver. To avoid using Excel Solver, we use a different way to estimate b by applying the data transformation technique first.

Data transformation is a useful technique for parameter estimation if obtaining the parameter(s) the usual way (taking derivatives or partial derivatives and solving a system of equations for the parameters) becomes difficult. This technique is taught in our mathematical modeling class using the textbook [7]. For example, suppose x and y data are collected for a model

$$y = Ae^{Bx^2}$$

Table 3: Record the time and the corresponding height of the water from an online video

Observed time (s)	Zeroed time (s)	Height (cm)
8	0	11.2
12	4	10
16	8	9
20	12	8
24	16	7.2
28	20	6.3
32	24	5.4
36	28	4.8
40	32	4.1
44	36	3.4
48	40	2.9
52	44	2.3

It's difficult to estimate the parameters A and B the usual way (or by using tools like "Add Trendline" in Excel directly). If we take the natural logarithm of both sides of the equation to get

$$ln(y) = ln(A) + Bx^2$$

first, then the relationship of $ln(y)$ and x is quadratic (or the relationship of $ln(y)$ and x^2 is linear), so we can find B and $ln(A)$ using the usual method. One downside of the data transformation technique is that it may distort data. Another downside is that the optimal parameters for the transformed model may not be the optimal parameters for the original model. The later may not necessarily be a disadvantage; for example, algorithms used in Excel Solver for nonlinear optimization also do not guarantee optimal solutions.

If we take the square root of both sides of Equation (2) and combine terms, we have Equation (3), and the quantity $\frac{2}{\sqrt{g}}[\sqrt{h_0} - \sqrt{h(t)}]$ is proportional to t where b is the proportionality constant. We can use the "Add Trendline" tool as shown in Figure 1 on the transformed data set and choose linear and set y-intercept to be 0. Also, Figure 2 shows this process.

$$\frac{2}{\sqrt{g}}[\sqrt{h_0} - \sqrt{h(t)}] = bt \tag{3}$$

The estimate obtained by using the data in Table 3 and the transformation technique is $b = 0.0027$. Also, the coefficient of determination $R^2 = 0.9995$ is shown if it's selected to be displayed on the chart. If R^2 obtained by a student was not close to 1, the student was told to check his or her activities starting from the data collection process to see if any errors were made earlier such as data collection error, measurement error, recording error, or computation error. Although the data collected by the students are different from the data in Table 3, their estimate of b is close to the estimate in Figure 2. Also, students are allowed to use any other software such as

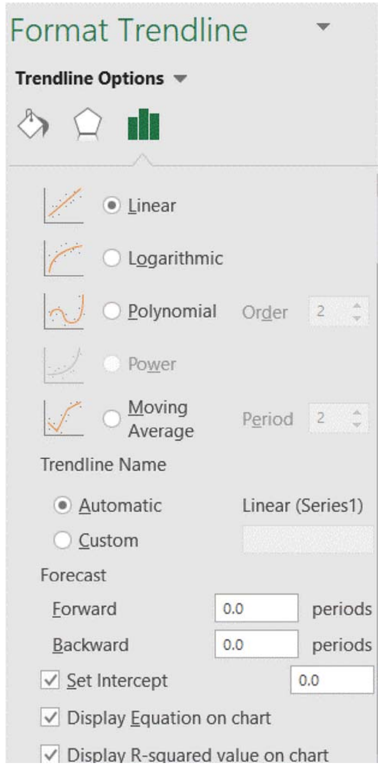

Figure 1: Add Trendline tool in Excel

MATLAB or Python to estimate b. However, most of our students seem to like using the "Add Trendline" tool in Excel.

Next, we ask the students to compare the observed height of the falling water to the height predicted from the mathematical model. Figure 3 shows the model validation process by comparing the data from the model to the original data where squares of errors and sum of squares of errors are computed, and a scatter plot showing the comparison is illustrated. This validation process provides students confidence in how good their models are.

We find that by doing activities like the ones demonstrated in this paper can pique students interest in studying topics that they consider difficult or boring. Also, similar activities can improve students engagement and collaborations in classroom.

Conclusion

We addressed using modeling activities as an approach to engage students in a mathematical modeling course and enhance their learning experience. As an example, we discussed the activities of modeling a falling column of water which include data collection, parameter estimation and model validation. The feedback from our students was generally positive. Although some of our students were reluctant to study the mathematical models using differential equations, most of them, especially the engineering students, enjoyed the modeling activities and the learning experience which were meaningful and memorable to them. As a result, students seemed to be more interested and attentive in the course.

Using data transformation technique, we have y = b*x, and b can be obtianed using the regression analysis to find the best fit line going through origin (using "Add Trendline" tool if Excel is used).

Observed Time (s)	Observed Height h (cm)	Zeroed Time x (s)	y = 2/sqrt(g) *[sqrt(h0) - sqrt(h)]
8	11.2	0	0
12	10	4	0.011778485
16	9	8	0.022146024
20	8	12	0.033107413
24	7.2	16	0.042380422
28	6.3	20	0.053452248
32	5.4	24	0.065347496
36	4.8	28	0.073838151
40	4.1	32	0.084446349
44	3.4	36	0.096005976
48	2.9	40	0.105012235
52	2.3	44	0.116918565

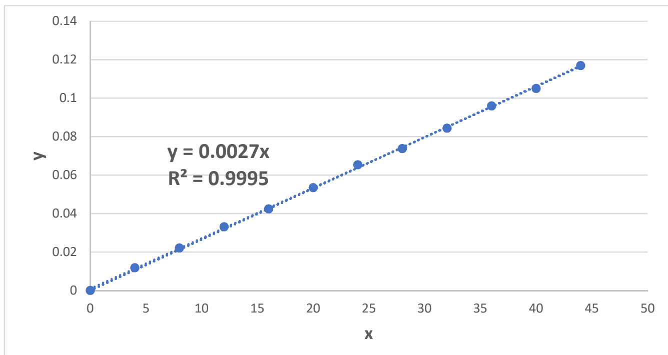

Figure 2: Estimating b using data transformation

Observed Time(s)	Zeroed Time (s)	Observed Height (cm)	Modeled Height (cm)	Square of Error
8	0	11.2	11.2000000	3.15544E-30
12	4	10	10.1449038	0.020997102
16	8	9	9.1419937	0.020162215
20	12	8	8.1912698	0.03658415
24	16	7.2	7.2927321	0.008599249
28	20	6.3	6.4463806	0.021427282
32	24	5.4	5.6522153	0.063612536
36	28	4.8	4.9102361	0.012151994
40	32	4.1	4.2204431	0.014506537
44	36	3.4	3.5828363	0.198041065
48	40	2.9	2.9974156	0.39608213
52	44	2.3	2.4641811	0.771167158
			Sum of Squares of Errors	**1.563331419**

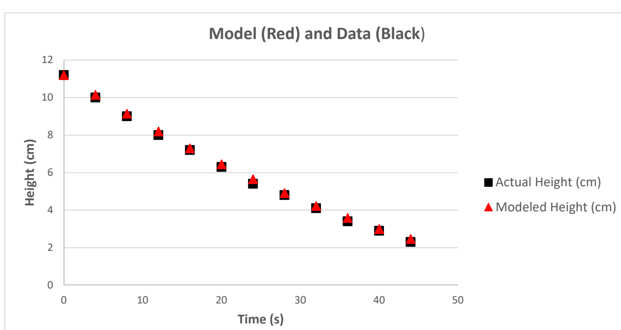

Figure 3: Compare the data from the model to the original data

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