AC 2007-324: USING NEURAL NETWORKS TO MOTIVATE THE TEACHING OF MATRIX ALGEBRA FOR K-12 AND COLLEGE ENGINEERING STUDENTS

Sharlene Katz, California State University-Northridge

Sharlene Katz is Professor in the Department of Electrical and Computer Engineering at California State University, Northridge (CSUN) where she has been for over 25 years. She graduated from the University of California, Los Angeles with B.S. (1975), M.S. (1976), and Ph.D. (1986) degrees in Electrical Engineering. Recently, her areas of research interest have been in engineering education techniques and neural networks. Dr. Katz is a licensed professional engineer in the state of California.

Bella Klass-Tsirulnikov, Sami Shamoon College of Engineering (formerly Negev Academic College of Engineering), Beer Sheva, Israel

Bella Klass-Tsirulnikov is a senior academic lecturer at Sami Shamoon College of Engineering, Beer Sheva, Israel (former Negev Academic College of Engineering). She accomplished mathematics studies at Lomonosov Moscow State University (1969), received Ph.D. degree in mathematics at Tel Aviv University (1980), and completed PostDoc studies at Technion - Israel Institute of Technology (1982). From 1995 she also holds a Professional Teaching Certificate for grades 7 – 12 of the Israeli Ministry of Education. Dr. Klass-Tsirulnikov participates actively in the research on functional analysis, specializing in topological vector spaces, as well as in the research on mathematics education at different levels.

Using Neural Networks to Motivate the Teaching of Matrix Algebra for K-12 and College Engineering Students

Abstract

Improving the retention of engineering students continues to be a topic of interest to engineering educators. Reference 1 indicates that seven sessions at the 2006 ASEE Annual Conference were devoted to this subject. In order to be successful in an engineering program, it is recognized that students must have a solid background in mathematics. Studies have shown that students will be more motivated to study and learn mathematics if abstract mathematical concepts are presented in the context of interesting examples and applications[2 – 5]. To become appealing and relevant, abstract mathematical concepts should be connected to engineering and real life issues, as suggested by the guidelines of ASEE Engineering K-12 Centre[6].

In two previous papers[7 - 8] the co-authors have presented methods for improving the teaching of important mathematical concepts to K-12 and college level engineering students. In reference 7, the co-authors provided a method of teaching the concept of infinity that combines a rigorous development of the concept of infinity in freshman level mathematics courses for engineering students and an intuitive approach to infinity with hands-on exercises for K-12 students. In reference 8, the co-authors developed materials on topics from number theory, essential to the field of data security and suitable for K-12 students, as well as for remedial or preparatory courses for engineering freshmen.

This paper represents the third part in this continuing project of developing methods for improving the teaching and learning of mathematical concepts for engineering students. It presents an interesting context in which to teach simple matrix algebra, developing practical applications that can be used for both K-12 and college level algebra courses. The main application demonstrated in this paper is the design of a character recognition system, using a simple neural network. It leads to a series of interesting exercises with practical applications. The authors contend that these applications will motivate students to practice matrix operations which otherwise may seem tedious and to further motivate them to focus on their mathematics courses.

I. Introduction

The teaching of matrix algebra takes place in a variety of courses. In K-12, students are typically introduced to matrix operations in their second year algebra courses and/or pre-calculus. In college, students come across matrix algebra in pre-calculus, linear algebra, differential equations and linear systems courses. In lower division college mathematics, the widely used application of matrix operations is typically the solution of simultaneous equations, based on the fundamental idea of viewing a system of linear equations as a product of a matrix and a vector (see for example, section 1.4 of reference 9).

It is not until the upper division engineering courses that students see interesting practical applications of linear algebra. The standard college textbooks on linear algebra provide application models of matrix algebra and linear systems used in economics, computer graphics and electrical circuit design[9-10]. Some of these models are relatively simple, such as the Cambridge Diet Formula (see Section 1.9 of reference 9). However, the models that are most important for engineering studies require specific knowledge. The mathematical models in economics widely use the notions of demand, consumption and equilibrium. The computer graphics requires understanding of 2D-3D topology. The mathematical model of electrical networks design is based on Kirchhoff's Laws [9-10].

Introducing advanced topics of specific disciplines in a basic linear algebra course is a complicated task. It raises the issues of what are the applications that generate common interest among freshmen, and how much time-consuming advanced engineering knowledge should be allowed in a basic mathematical course without hurting freshmen interest. It also brings up the fundamental question of what is the meaning and essence of teaching mathematics in a way that is attractive and relevant for engineering students. As teachers, we know that often students are unable to connect practical matrix applications in upper division engineering courses with the basics of linear algebra taught during the first year of engineering studies. There is no doubt that we should think hard about how to teach mathematics to engineering freshmen in the most appealing and useful way[6].

In this paper, we present a design of a simple artificial neural network for character recognition that exercises basic matrix operations, such as adding and multiplying matrices and finding determinants. Artificial neural networks are based on the model of biological neural system, and they can be relatively simply explained to non-professionals. Students will be surprised to learn that the simple neural network system they designed can correctly recognize a distorted character not recognizable by their own eyes. The process of designing the network provides useful exercises for practicing matrix manipulations in an appealing and relevant way.

Section II of this paper briefly presents the topics in matrix algebra that are needed for these exercises. Since our paper is intended to serve the scientific community of mathematicians, including K-12 teachers, we feel that some explanation is needed for the terminology and notions used in the field of artificial neural networks. These explanations are provided in the next two sections. Section III presents the concept of artificial neural networks, and Section IV overviews the use of artificial neural networks in designing character recognition systems. Section V details a method for designing and testing the character recognition system, suitable for both K-12 students and engineering freshmen. Section VI gives a number of meaningful exercises in basic matrix algebra that students can perform based on this design method. Section VII summarizes the paper.

II. Topics in Matrix Algebra

As mentioned earlier, the teaching of matrix algebra takes place in a variety of courses for K-12 students, and entry-level college students. These courses provide ideal opportunities to use the exercises presented in this paper. Before using these exercises students should be exposed to the following concepts related to matrix algebra.

Matrix Definition and Notation: A matrix is a rectangular array of elements of the form:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The first subscript of each element denotes the row and second subscript denotes the column. The matrix that has m rows and n columns is said to be an *m x n* matrix, or a matrix of size "*m* by *n*".

Matrix Addition and Subtraction: The sum (difference) of two matrices A and B is written A+B (A-B). Matrices must be the same size to be added or subtracted. The entries in the resulting sum A+B are $a_{ij} + b_{ij}$ for $i = 1 \ldots m$ and $j = 1 \ldots n$. The entries in the difference A-B are $a_{ij} - b_{ij}$ for $i = 1 \ldots m$ and $j = 1 \ldots n$.

Matrix Multiplication: Let A be an *m x n* matrix and B an *r x p* matrix. The product of these two matrices, AB, is only defined if $n = r$. The resulting product C = AB is an *m x p* matrix in which:

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}; \quad \text{i=1...m}; \quad \text{j=1...p}$$

Transpose of a Matrix: The transpose of an *m x n* matrix A is an *n x m* matrix, denoted A^T. The transpose matrix A^T has as its rows, the columns of A, or:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

Determinant of a Matrix: We use the recursive definition of the determinant. If A is a *2x2* matrix, its determinant D is given by:

$$D = \det(A) = a_{11}a_{22} - a_{12}a_{21}$$

For $m > 2$, the determinant, D, of an *m x m* (square) matrix A is given by:

$$D = \det(A) = a_{j1}C_{j1} + a_{j2}C_{j2} + \cdots + a_{jm}C_{jm} \quad j = 1, 2, \cdots, or \text{ m}$$

or

$$D = \det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{2j}C_{2j} \quad j = 1, 2, \cdots, or \text{ m}$$

where C_{ij} is the (i, j)-cofactor of A:

$$C_{ij} = (-1)^{i+j} M_{ij}$$

and M_{ij} is the determinant of the submatrix of A obtained from A by deleting the row and column of the entry a_{ij}.

Inverse of a Matrix: If D=det(A) is not equal to zero, the matrix A is invertible. The inverse of an *m x m* (square) matrix A is given by:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{m1} \\ C_{12} & C_{22} & \cdots & C_{m2} \\ \vdots & \vdots & & \vdots \\ C_{1m} & C_{2m} & \cdots & C_{mm} \end{bmatrix}$$

where C_{ij} is defined above.

Inner Product of Vectors: The inner product (or dot product) of two vectors, both of length n, is given by:

$$a \cdot b = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n$$

Orthogonal Vectors: Vectors a and b are said to be orthogonal if their inner product is equal to zero.

The reader is referred to any standard text[9-10] that covers matrix algebra for treatment of these topics.

III. Overview of Neural Networks

The human brain consists of clusters of neurons that are interconnected in a biological neural system. The strength of these connections is varied as people receive and process information, or in other words, learn. An artificial neural network, consisting of interconnected artificial neurons, is modeled after humans' neural system. The strength of artificial interconnections is varied as we train the artificial neural network to process the received information. In this way, the artificial neural network can learn[11-15]. Throughout this paper, the term "neural network" will refer to this artificial neural network.

Neural networks have applications in classification, pattern recognition, and function approximation problems. They are used in communications, aerospace, defense, financial, manufacturing, and medical applications. This paper will focus on an application of neural networks in the area of pattern recognition: a simple character recognition system.

The diagram for a single neuron in a neural network is shown in figure 1. A neuron usually receives many simultaneous inputs, that are represented by the input vector of information, **p**. We assume that **p** is a column vector of length, *R*, or an *R x 1* matrix. Some inputs, i.e. the coordinates of the input vector **p**, are more important or intensive than others. Each input, i.e. each coordinate of **p,** has its own weight that determines the intensity of the input signal and its interconnection strength. A vector of weights, W, is assumed to be a row vector of length *R*, or a *1 x R* matrix. As its human prototype, the artificial neuron starts to process this input vector of information. It multiplies the weight vector **W** by the input vector **p**, obtains a scalar **Wp**, and adds to it a *bias term*, a scalar b. The result, a scalar, n = **Wp** + b, is referred to as the net input for a transfer function, f, which then produces the final output scalar, a.

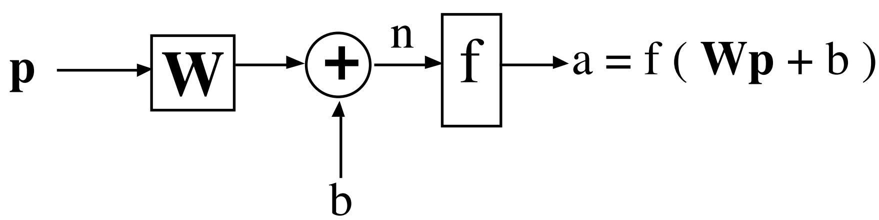


Figure 1. Single Neuron in an Artificial Neural Network

When we design a neural network for character recognition in section V, we shall use a scheme with no bias term b (b = 0), simplifying the explanations of the design process needed for freshmen and high school students who do not have any experience or knowledge with neural networks.

A variety of functions, f**,** are used in neural networks depending on the application. In this paper we will utilize the so-called Symmetrical Hard Limit function. This function is denoted *hardlims* and defined as follows:

$$\text{hardlims}(x) = \begin{cases} -1, & x < 0 \\ +1, & x \geq 0 \end{cases}$$

Thus the input of the single neuron is an *R x 1* matrix **p**, and its final output is a scalar a = hardlims(**Wp** + b), depending upon whether the result n = **Wp** + b is positive or negative.

A neural network can contain multiple neurons. Each neuron receives the same input vector, **p**, but produces a separate output. A network of *S* neurons has *S* outputs and can be represented in a manner similar to the single neuron network shown in figure 1. However, the weights are now the rows of a weight matrix **W** of size *S x R*. Accordingly, **b, n**, and **a** become column vectors of length *S*, or *S x 1* matrices. Thus, for an *S* neuron neural network with input **p**, we obtain *S* outputs, which are contained in the *S x 1* matrix:

$$\mathbf{a} = \text{f}(\mathbf{Wp} + \mathbf{b}) = \text{hardlims}(\mathbf{Wp} + \mathbf{b})$$

The design of the neural network refers to the process by which the values of **W** and **b** are determined to optimize the network performance. The values of **W** and **b** that will provide the appropriate output for a given input are determined by training the network. There are many methods for training neural networks. The network in this example is trained using supervised learning. In supervised learning we use a set of input vectors for which the desired output vector is known. The desired output for a particular input is referred to as the target output, **t**. Based on these given input/output pairs (**p, t**) we will find **W** and **b**.

IV. Character Recognition Systems

A character recognition system takes an input that corresponds to a particular character (alphabetic, numeric, symbol, etc.). The input characters may be printed or in cursive writing. For example, optical character recognition systems are able to identify the characters in handwritten or typed images and are used by post offices to read the zip codes on mail. Character recognition problems are particularly difficult because of the large number of variations that can appear on the input for a particular character.

In this example we want to design a neural network that will recognize the numerical input that is applied and output the appropriate character. Assume that the input characters are formed by pixel patterns on a 6 x 5 grid as the numbers 0 – 6 are shown in figure 2.

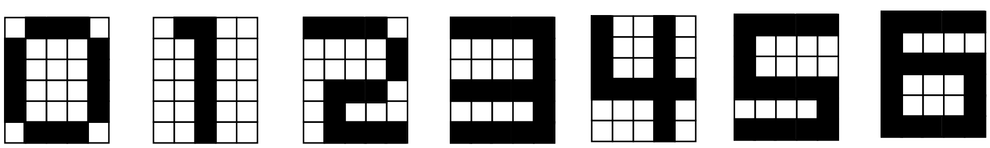


Figure 2. Numerical Inputs to a Character Recognition System

If the numeral is input to the network with the correct pixel pattern, a properly designed system will output that character. However, if a numeral is input to the system with a deviated pattern (say with noise) we would also like for the system to recognize that numeral and for the correct numeral pattern to be output. Figure 3 shows an example of a neural network that correctly identifies a "2" with noise at the input and outputs the correct pattern.

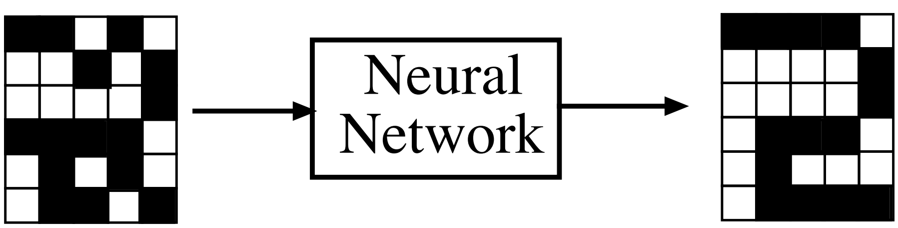


Figure 3. A Neural Network that Correctly Recognizes a Noisy "2"

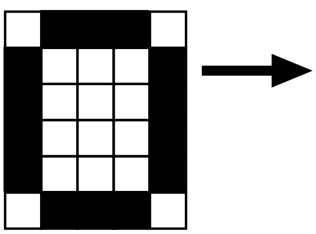
A robust character recognition system will be able to recognize characters with many variations, or a significant noise component. For a given system, it is generally true, that if the number of characters to be recognized increases then its ability to recognize characters with noise decreases.

V. Design of a Neural Network Character Recognition System

In this section we develop a simple procedure for designing a neural network to recognize numerical inputs described in Figure 2. The method we use can be applied to the design of neural networks to recognize other character sets. We start by presenting the problem.

<u>The Problem</u>: A neural network receives numerals that are presented on a 6 x 5 pixel grid as shown in figure 2. These inputs may be corrupted by noise. Design a neural network to recognize the numeral at the input in the presence of noise and output that numeral with the correct pixel pattern on the same 6 x 5 grid.

<u>Input Vectors</u>: As stated in section III, the input to a neural network is a column vector. In order to describe the numeral inputs we need to form this column vector. In the case of the 6 x 5 patterns, since there are a total of 30 pixels in each numeral we will use a vector of length 30. A 1 in the vector will represent a black grid square and a -1 in the vector will represent a white grid square. The 6 x 5 grids will be traversed from left to right and top to bottom to form a column of 1s and -1s to apply to the neural network. For example, the numeral "0" is represented by the vector $\mathbf{p_0}$ as shown in figure 4. Note that throughout this paper many of the vectors and matrices are shown as transposes to save space.



$$p_0 = [\,-1, 1, 1, 1, -1, 1, -1, -1, -1, 1, 1, -1, -1, -1, 1, 1, -1, -1, -1, 1, 1, -1, -1, -1, 1, -1, 1, 1, 1, -1\,]^T$$

Figure 4. Representation of a "0" as an Input Vector (shown as a transpose)

<u>Creation of Training Set</u>: The training set refers to the set of known input vectors paired with their target output vectors (**p**, **t**). In this problem if a "0" is input we want a "0" to be output, or we have a training pair ($\mathbf{p_0}$, $\mathbf{t_0}$) where $\mathbf{p_0}$ and $\mathbf{t_0}$ are both defined as shown in figure 4. Similarly we can create a training pair for each input that we want the network to recognize.

<u>Hebb Rule</u>: The heart of the design of neural network is to find the values of the **W** and **b** matrix/vectors. There are many methods for determining these values, or training the network. The training method selected is based on the particular application of neural networks. In the case of pattern recognition Hebbian Learning is often used. Within the learning rules that are categorized as Hebbian Learning, the best known is the Hebb Rule, introduced by Donald Hebb in 1949. The Hebb Rule is surprisingly simple: if a neuron receives an input from another neuron, and both are highly active (i.e. have the same sign), the interconnection between the neurons (i.e. the weight) should be strengthened.

The simplicity of Hebb Rule makes it attractive for working with high school and other students who do not have any experience or knowledge of neural networks. The Hebb Rule uses no bias vector, which means that in figure 1, $\mathbf{b} = \mathbf{0}$, and the final output **a** is given by:

$$\mathbf{a} = f(\mathbf{Wp}) = \text{hardlims}(\mathbf{Wp})$$

The weight matrix is found using the formula[11]:

$$\mathbf{W} = \mathbf{TP}^{\mathbf{T}}$$

where

> **T** = matrix with target vectors as columns
> **P** = matrix with input vectors as columns

We begin with a system that will recognize the numerals “0” and “1” only. Thus, we have two input/target pairs. Corresponding to the “0” we have the value of $\mathbf{p_0}$ shown in figure 4. The target vector, $\mathbf{t_0} = \mathbf{p_0}$ since we want the output to display the letter that is input. Corresponding to the “1” shown in figure 2 we have:

$$\mathbf{p}_1 = \mathbf{t}_1 = [-1,1,1,-1,-1,-1,-1,1,-1,-1,-1,-1,1,-1,-1,-1,-1,1,-1,-1,-1,-1,1,-1,-1,-1,-1,1,-1,-1]^T$$

Thus, the P and T matrices are given by:

$$\mathbf{P} = \mathbf{T} = \begin{bmatrix} -1 & 1 & 1 & 1 & -1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & 1 & -1 & -1 & -1 & 1 & -1 & 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 & -1 & -1 & 1 & -1 & -1 \end{bmatrix}^T$$

Now that **P** and **T** have been determined we can find the weight matrix **W** using the formula shown above. Note that **T** is a 30 x 2 matrix and $\mathbf{P^T}$ is a 2 x 30 matrix. Thus, the multiplication is defined and the resulting **W** matrix will be 30 x 30, or:

$$\mathbf{W} = \mathbf{TP}^{\mathbf{T}} = \begin{bmatrix} 2 & -2 & -2 & 0 & \cdots & \cdots & 0 & -2 & 0 & 2 \\ -2 & 2 & 2 & 0 & \cdots & \cdots & 0 & 2 & 0 & -2 \\ -2 & 2 & 2 & 0 & \cdots & \cdots & 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 2 & \cdots & \cdots & 2 & 0 & 2 & 0 \\ \vdots & \vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots & \vdots \\ & & & & & & & & & \\ 0 & 0 & 0 & 2 & \cdots & \cdots & 2 & 0 & 2 & 0 \\ -2 & 2 & 2 & 0 & \cdots & \cdots & 0 & 2 & 0 & -2 \\ 0 & 0 & 0 & 2 & \cdots & \cdots & 2 & 0 & 2 & 0 \\ 2 & -2 & -2 & 0 & \cdots & \cdots & 0 & -2 & 0 & 2 \end{bmatrix}$$

In addition to providing practice with matrix multiplication, this provides students with an opportunity to think about the sizes of the matrix and verify that the multiplication is defined.

<u>Testing the Neural Network Performance</u>: We begin testing the neural network by applying the 0 and 1 without noise. Applying the input vector $\mathbf{p_0}$ we obtain an output:

$$\mathbf{a} = \text{hardlims}(\mathbf{Wp_0})$$

which we find to be equal to the target vector $\mathbf{t_0}$. Similarly, we find that if we apply the $\mathbf{p_1}$ vector, as an input the $\mathbf{t_1}$ target vector is output. Again the student will have a chance to practice matrix multiplication.

These calculations confirm that the neural network recognizes the inputs without noise. Next, we want to see how well it recognizes the two inputs in the presence of noise. For example, say we apply the "0" with five pixels that have been reversed as shown in figure 5.

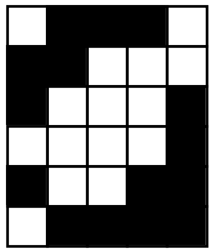


Figure 5. A "0" with 5 Reversed Pixels

If we form an input vector, $\mathbf{p_{05}}$ corresponding to this noisy "0" and compute the output we find that the output vector is exactly equal to the vector representation of the noiseless "0", or $\mathbf{a} = \mathbf{t_0}$. In other words, the neural network is able to correctly detect an input in the presence of five pixels of noise and output the target vector, $\mathbf{t_0}$ correctly.

As we continue to increase the number of pixel errors and test the network we find that for up to eight pixel errors, or over 25% of the pixels in error, in either the "0" or the "1" the neural network correctly outputs the target vector. For example, the noisy "1" shown in figure 6 has 8 pixel errors. It is barely recognizable as a "1" to the human eye yet the neural network is able to correctly detect it.

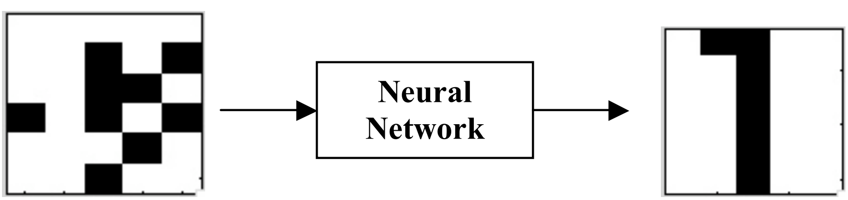


Figure 6. A "1" inputs with 8 Pixel Errors

If the number of pixel errors is increased to nine or more the neural network begins to produce incorrect outputs. As the number of pixel errors increases the probability of error increases. Testing the neural network can provide many matrix multiplication problems for a class of students.

<u>Testing the Capacity of the Neural Network</u>: Thus far our neural network has only been designed to detect two numerals. We would like to increase the capacity of the network so that it can detect a larger number of numerals. We can repeat the procedure described above for the complete set of seven numerals (0 – 6) shown in figure 2 and repeated in figure 7. The procedure is the same. All that changes is the sizes of the vectors and matrices. Now the **P** and **T** matrices each have seven columns (30 x 7). The resulting **W** matrix is still 30 x 30. Once **W** has been determined we test the system by applying each of the seven digits without noise. The

outputs shown in figure 7 are found in this case. Note that only three of the numerals are output correctly (1, 3, and 4).

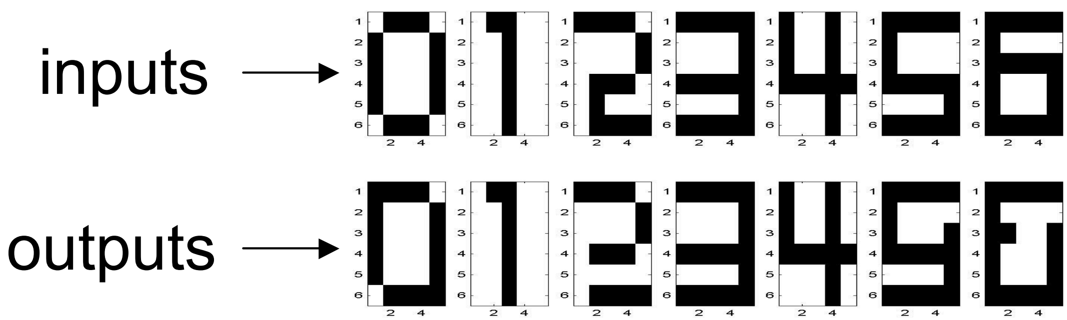


Figure 7. Inputs/Outputs for Neural network with Seven Numerals

An analysis of the Hebb Rule[11] shows that it only works perfectly for input vectors, **p**, that are orthogonal. This gives students an opportunity to check and verify that the seven input vectors used here are not orthogonal. For example:

$$\mathbf{p}_0 \cdot \mathbf{p}_1 = 0$$

Thus, the vectors represented "0" and "1" are orthogonal. This is expected since the network recognized the "0" and "1" inputs in the two numeral case. However,

$$\mathbf{p}_0 \cdot \mathbf{p}_2 = 6$$

Thus, the "0" and "2" vectors are not orthogonal. This (and other examples) may explain the error in the expanded system.

Pseudoinverse Rule: A modification of the Hebb Rule has been found that improves the performance of the character recognition system in the case of non-orthogonal input vectors. This method is called the Pseudoinverse Rule. In addition to improving system performance this new rule provides another excellent opportunity for students to practice matrix algebra. Using the Pseudoinverse Rule, the weight matrix **W** is found using:

$$\mathbf{W} = \mathbf{T}^*\mathbf{P}^+$$

where

$$\mathbf{P}^+ = \left(\mathbf{P}^T\mathbf{P}\right)^{-1}\mathbf{P}^T$$

P+ is known as the Pseudoinverse matrix[11]. For the character recognition system with numerals 0 – 6, the **P+** matrix is found to be:

$$
\mathbf{P}^{+} = \left(\mathbf{P}^{T}\mathbf{P}\right)^{-1}\mathbf{P}^{T} = \begin{bmatrix}
-.0811 & .0108 & .0108 & .0137 & \cdots & .0252 & .0108 & .0137 & -.0695 \\
-.0097 & .0323 & .0323 & -.0241 & \cdots & -.0526 & .0323 & -.0241 & -.0381 \\
.0556 & .0252 & .0252 & .0411 & \cdots & .0446 & .0252 & .0411 & .0592 \\
.0134 & -.0131 & -.0131 & -.0006 & \cdots & -.0500 & -.0131 & -.0006 & -.0360 \\
.0484 & -.0326 & -.0326 & .0368 & \cdots & -.0611 & -.0326 & .0368 & -.0496 \\
-.0216 & .0332 & .0332 & -.0137 & \cdots & .0603 & .0332 & -.0137 & .0524 \\
.0844 & .0231 & .0231 & .0348 & \cdots & -.0091 & .0231 & .0348 & .0405
\end{bmatrix}
$$

and

$$
\mathbf{W} = \mathbf{T}\mathbf{P}^{+} = \begin{bmatrix}
.2710 & -.0073 & -.0073 & .1088 & \cdots & \cdots & .0120 & -.0073 & .1088 & .1742 \\
-.0073 & .1441 & .1441 & .0143 & \cdots & \cdots & .0796 & .1441 & .0143 & .0579 \\
-.0073 & .1441 & .1441 & .0143 & \cdots & \cdots & .0796 & .1441 & .0143 & .0579 \\
.1088 & .0143 & .0143 & .1361 & \cdots & \cdots & .0624 & .0143 & .1361 & .0351 \\
\vdots & \vdots & \vdots & \vdots & & & \vdots & \vdots & \vdots & \vdots \\
 & & & & & & & & & \\
.0120 & .0796 & .0796 & .0624 & \cdots & \cdots & .1847 & .0796 & .0624 & .1342 \\
-.0073 & .1441 & .1441 & .0143 & \cdots & \cdots & .0796 & .1441 & .0143 & .0579 \\
.1088 & .0143 & .0143 & .1361 & \cdots & \cdots & .0624 & .0143 & .1361 & .0351 \\
.1742 & .0579 & .0579 & .0351 & \cdots & \cdots & .1342 & .0579 & .0351 & .2733
\end{bmatrix}
$$

Testing the neural network with the $\mathbf{p_0} - \mathbf{p_6}$ input vectors into this new **W** matrix we find that the outputs correspond correctly to the target vectors as shown in figure 8.

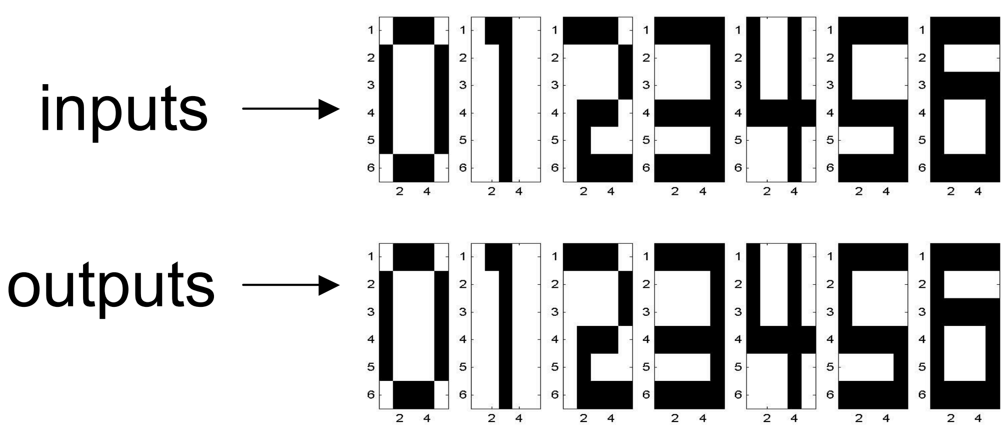


Figure 8. Neural Network Inputs and Outputs Using Pseudoinverse Matrix

This neural network can now be tested to see how well it responds to inputs with noise. Figure 9 shows an example of inputs with five pixel errors in each numeral input. The outputs corresponding to the 0 – 3 numeral inputs have no errors. The outputs corresponding to the 4 - 6 numeral inputs have only one pixel error each.

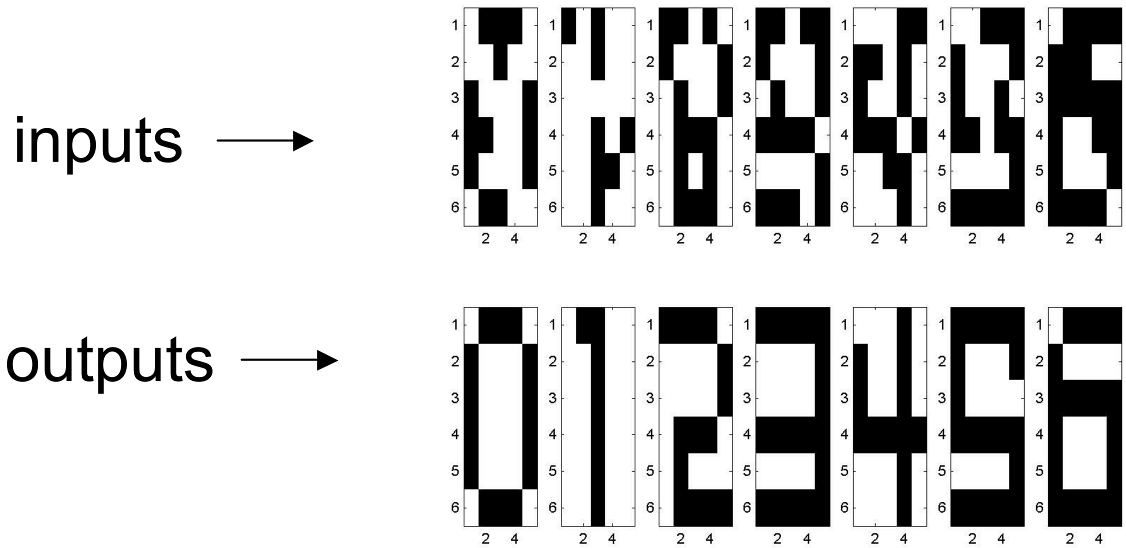


Figure 9. Outputs Corresponding to Inputs with Five Pixel Errors Each

Further testing will show that as with the first case the probability of error increases as the number of pixel errors increases and/or the number of numerals being stored by the network increases.

VI. Exercises for High School and College Students

Repeating the calculations performed in the design example of Section V will provide students with practice in matrix multiplication. This section contains other exercises that can be used to further motivate students to practice matrix manipulations. Simple exercises such as those found in 1 – 3 below may be more appropriate for K – 12 students, while exercises 4 – 5 provide additional challenges for more motivated college students.

<u>Exercise 1 - Testing the System</u>: For the system designed in section V we would like to check the occurrence of error for each character as the number of pixel errors is varied from zero to 30. Begin with a system that is designed to recognize only two numerals (0 and 1). Apply each of the two inputs with varying numbers of pixel errors and observe the output. For each case vary the position of the pixel error. Students can use a calculator or existing software, or even write a program to repeat these calculations. Repeat the tests for systems that recognize more than two numerals.

<u>Exercise 2 - Increasing the Character Set</u>: Repeat the design performed in section V for more than seven numerals. How many characters can the system recognize?

Exercise 3 - ABC Character Recognition: Design a neural network to recognize the letters A, B, and C.

Exercise 4 – Alternate Character Sets: Have students create their own character set by writing on a grid. Design a character recognition system for this new character set.

Exercise 5 - Create an orthogonal character set: Modify the character set in exercise 3 so that the characters are orthogonal. Compare the performance of this system to the one in exercise 3.

Many similar exercises can be created. Once a student understands the concepts they may invent their own exercises.

VII. Summary and Conclusions

This paper represents the third part in a continuing project of developing methods for improving the teaching and learning of mathematical concepts for engineering students. The focus of this paper is on teaching basic matrix operations and applying them to design a simple neural network for character recognition. The procedure for carrying out this design is presented, and a number of exercises that can be performed by college freshmen and advanced K-12 students are given. The authors contend that these applications will motivate students to practice matrix operations which otherwise may seem tedious, and to further motivate them to focus on their mathematics courses. The educational philosophy behind the methods and applications presented in this paper is based on the "computation-to-abstraction" sequencing of teaching linear algebra to engineering students, as opposed to the "abstraction-to-computation" approach[16].

Regarding K-12 students, the authors suggest that a collaborative effort of experienced teachers in biology, physics and mathematics is required to produce good results and to benefit the students.

The authors are now testing the practice of teaching the concept of infinity, as described in reference 7. In the upcoming years, the authors will be testing the ideas of teaching the topics of number theory, presented in reference 8 and relevant to data security. In the sequel, the authors will test the methods and exercises described in this paper, to assess their effect on student learning.

References

1. Proceedings of the 2006 ASEE Annual Conference, Chicago, Illinois, June 2006.
2. van Alphen, Deborah K., and Katz Sharlene, A Study of Predictive Factors for Success in Electrical Engineering, Proceedings of the 2001 ASEE Annual Conference.
3. Buechler, Dale. "Mathematical Background versus Success in Electrical Engineering," Proceedings of the 2004 ASEE Annual Conference.

4. Carpenter, Jenna; and Schroeder, Bernd S. W. "Mathematical Support for an Integrated Engineering Curriculum", Proceedings of the 1999 ASEE Annual Conference.
5. Buechler, Dale, and Papadopoulus, Chris. "Initial Results from a Math Centered Engineering Applications Course", Proceedings of the 2006 ASEE Annual Conference.
6. Douglas Josh, Iversen Eric, and Kaliyandurg Chitra. "Engineering in the K-12 Classroom: an Analysis of Current Practices and Guidelines for Future," ASEE Engineering K-12 Centre, November 2004, http://www.engineeringk12.org/Engineering_in_the_K-12_Classroom.pdf.
7. Klass-Tsirulnikov Bella and Katz Sharlene, "The Concept of Infinity from K-12 to Undergraduate Courses", Proceedings of the 2006 ASEE Annual Conference, Chicago, Illinois, June 2006.
8. Klass-Tsirulnikov Bella and Katz Sharlene, "Prime Numbers and the Totient Function: the First Step to Cryptography I the K-12 Classroom," Proceedings of the 5th Annual ASEE Global Colloquium on Engineering Education, Rio de Janeiro, Brazil, October 2006.
9. Lay, David C., "Linear Algebra and Its Applications," 2nd edition .Addison Wesley, New York,1997.
10. Kolman, B. and D. Hill, "Introductory Linear Algebra with Applications," Prentice-Hall, Englewood Cliffs, New Jersey, 2001.
11. Hagan, Martin T. , Howard B. Demuth, and Mark Beale, "Neural Network Design," PWS Publishing Company, Boston, 1996.
12. Picton, Phil, "Neural Networks," 2nd ed., PALGRAVE, New York, 2000.
13. Bishop, Christopher M., "Neural Networks for Pattern Recognition," Oxford University Press, Oxford, 1996.
14. Fausett, Laurene, "Fundamentals of Neural Networks," Prentice Hall, Upper Saddle River, New Jersey, 1994.
15. Haykin, Simon, "Neural Networks," Prentice Hall, Upper Saddle River, New Jersey, 1999.
16. Klapsinou, Alkistis, and Gray, Eddie, "The Intricate Balance between Abstract and Concrete in Linear Algebra", Proceedings of the 23rd International Conference of the Psychology of Mathematical Education (Ed. O. Zaslavsky), Vol. 3, pp. 153-160, Haifa, Israel, 1999.